Non-commutative locally convex measures

José Bonet

Instituto Universitario de Matemática Pura y Aplicada IUMPA

Universidad Politécnica de Valencia

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Introduction

Part of a program on **non-commutative measure theory.**

Started by **John D. Maitland Wright in 1980.**

Continued together with his coauthors **J.K. Brooks, A. Peralta, K. Saitô, I. Villanueva, K. Ylinen**

and Pepe Bonet (the locally convex touch)

Introduction

Classical measure theory

In classical theory, measures and integrals correspond to functionals or vector valued operators defined on a function algebra, like $C(K)$, K compact; or L^∞.

Non-commutative measure theory

Replacing these commutative algebras by non-commutative C^*-algebras, like $L(H)$, H Hilbert space, gave birth to non-commutative measure theory. But replacing C^*-algebras by more general classes of Banach spaces give rise to fruitful new insights.

Weak compactness and measure theory

Question 1

Let K be a compact Hausdorff space and let E be a complete locally convex space. Let $T \in L(C(K), E)$ a continuous operator. When does there exist an E-valued Baire measure μ on K such that

$$T(f) = \int_K f(t)\mu(dt) \quad \text{for each } f \in C(K)?$$

Question 2

Let $B(K)$ be the algebra of bounded Borel measurable functions on K. When does there exist an operator $T^\infty \in L(C(K), E)$ extending T and, whenever $(f_n)_n$ is a bounded monotone increasing sequence in $B(K)$ with pointwise limit f, it follows $T^\infty(f_n) \to T^\infty f$ in E?

Weak compactness and measure theory

When E is one dimensional, the answer to Question 1 is "always". This is the classical **Riesz Representation Theorem**.

When E is a Banach space, the answer is: precisely when T is a weakly compact operator. This was proved by **Grothendieck (1953) and Bartle, Dunford and Schwartz (1955)**.

Theorem (Lewis, 1970)

Let K be a compact Hausdorff space, let E be a complete locally convex space and let $T \in L(C(K), E)$ a continuous operator. T is weakly compact if and only if there is a regular measure $\mu : \Sigma \to E$ on the Borel subsets of K such that

$$T(f) = \int_K f(t)\mu(dt) \quad \text{for each } f \in C(K).$$

Weakly compact operators

Definition

If X is a Banach space and E is a locally convex space an operator $T \in L(X, E)$ is weakly compact if $T(X_1)$ is relatively $\sigma(E, E')$-compact in E. Here X_1 stands for the closed unit ball of X.

Grothendieck's extension of Gantmacher's Theorem

Let X be a Banach space and let E be a complete locally convex space. The following are equivalent for $T \in L(X, E)$

(1) T is weakly compact.

(2) $T''(X'') \subset E$.

(3) For each $C \subset E'$ which is equicontinuous, $T'(C)$ is relatively $\sigma(X', X'')$-compact.

C^*-algebras

- A is a C^*-algebra.
- A' is the dual of A.
- The bidual A'' can be identified with the von Neumann envelope of A in its universal representation.
- A' is the predual of a von Neumann algebra, hence $(A', \sigma(A', A''))$ is sequentially complete.

Weakly compact operators

Theorem

Let A be a Banach space such that $(A', \sigma(A', A''))$ is sequentially complete and let E be a complete locally convex space. Let $(T_n)_n$ be a sequence of weakly compact operators from A into E.

If $(T_n'' z)_n$ is a Cauchy sequence in E for each $z \in A''$, then there is a weakly compact operator $T : A \longrightarrow E$ such that $T_n'' z \longrightarrow T'' z$ in E for each $z \in A''$.

Theorem (Dieudonné, 1951)

Let K be a compact metric space, let $(\mu_n)_n$ be a sequence of Borel measures such that $\lim_{n \to \infty} \mu_n(U)$ exists for each open subset U of K. Then there exists a Borel measure μ such that $\mu_n(f) \to \mu(f)$ for each bounded Borel measurable function $f \in B(K)$.

Let A be a C^*-algebra. A projection $p \in A''$ is said to be a **range projection** for A if there exists $b \in A$, $0 \leq b \leq 1$, such that the monotone increasing sequence $(b^{1/n})_n$ converges to p in the topology $\sigma(A'', A')$. We write $p = RP(b)$.

Theorem (Brooks, Saitô, Wright (2003), Bonet, Wright (2009))

Let A be a C^*-algebra, let E be a complete locally convex space and let $(T_n)_n$ be a sequence of weakly compact operators $T_n : A \to E$. Suppose that, whenever $p \in A''$ is a range projection, $\lim_{n\to\infty} T_n''(p)$ exists in E. Then there is a unique weakly compact operator $T : A \to E$ such that $T''(x) = \lim_{n\to\infty} T_n''(x)$ for each $x \in A''$.

Extending a result of Ryan on weakly compact operators

- E is a Fréchet space or a complete (DF)-space.
- $c_0(E)$ and $c(E)$.
- $\ell_1(E)$ and $\ell_\infty(E)$.
- $c(E)'_b \simeq \ell_1(E'_b)$.
- $\ell_1(E'_b)'_b \simeq \ell_\infty(E''_b)$.

Extending a result of Ryan on weakly compact operators

This is our extension of a result due to **Ryan (1979)**.

Theorem

Let A be a Banach space and let E be a Fréchet space or a complete (DF)-space.

Let T_n be the operators from A into E such that, for each $a \in A$, $\lim_n T_n a = 0$ in E.

Then the operator

$$\tilde{T} : A \to c_0(E),\ \tilde{T}a := (T_n a)_n,\ a \in A,$$

is weakly compact if and only if each T_n is weakly compact and $\lim_n T_n'' z = 0$ for each $z \in A''$.

Theorem

Let A be a Banach space such that $(A', \sigma(A', A''))$ is sequentially complete and let E be a Fréchet space or a complete (DF)-space.

Let $(T_n)_n$ be a sequence of weakly compact operators from A into E, such that $(T_n'' z)_n$ is a Cauchy sequence in E for each $z \in A''$.

Then $\tilde{T} : A \to c(E)$, $\tilde{T} a := (T_n a)_n, a \in A$, is a weakly compact operator.

If the assumption that A has a weakly sequentially complete dual is removed in this Theorem, the conclusion is no longer valid, as the example constructed by Ylinen in 2005 shows.

A continuity characterization of weakly compact operators

The **Right topology** $\rho(E)$ is the topology induced in E by the Mackey topology $\mu(E'', E')$ of the dual pair (E'', E'). Recall that $\mu(E'', E')$ is the topology on E'' of the uniform convergence on the absolutely convex $\sigma(E', E'')$-compact subsets of E'.

Theorem (Ruess (1982), Peralta, Villanueva, Ylinen, Wright (2007))

Let F and E be complete, barrelled locally convex spaces. Let $T \in L(F, E)$ be a continuous linear operator. The following conditions are equivalent:

(1) T maps bounded subsets in F into relatively weakly compact subsets of E.

(2) $T : (F, \rho(F)) \to E$ is continuous.

(3) There is an absolutely convex neighbourhood $V \in \mathcal{U}_0(F)$ such that the restriction $T|_V$ of T from V, equipped with the topology induced by the Right topology $\rho(F)$, into E is continuous.

Generalizing Nikodym Theorems

A classical **theorem of Nikodym** asserts that if a sequence of countably additive measures converges pointwise, then the limit is also a countably additive measure and, furthermore, countable additivity is uniform for the sequence of measures. As a consequence of our previous results, we get

Theorem

Let A be a Banach space and let E be a Fréchet space or a complete (DF)-space. Let $(T_n)_n$ be a sequence of weakly compact operators from A into E such $(T_n''z)_n$ converges to 0 in E for each $z \in A''$.

If $(a_j)_j \subset A$ converges to 0 in the Right topology of A, then $sup_{n\in\mathbb{N}} p(T_n(a_j))$ converges to 0 as $j \to \infty$ for each continuous seminorm p on E.

Generalizing Nikodym Theorems

Theorem

Let A be a Banach space such that $(A', \sigma(A', A''))$ is sequentially complete and let E be a Fréchet space or a complete (DF)-space. Let $(T_n)_n$ be a sequence of weakly compact operators from A into E such that $(T_n'' z)_n$ is a Cauchy sequence in E for each $z \in A''$. Let $Sa := \lim T_n a$ for each $a \in A$. Then

(1) If $(a_j)_j \subset A$ converges to 0 in the Right topology of A, then $sup_{n \in \mathbb{N}} p((T_n - S)(a_j))$ converges to 0 as $j \to \infty$ for each continuous seminorm p on E.

(2) If $(z_j)_j$ is a sequence in A'' converging to 0 in the topology $\mu(A'', A')$, then $sup_{n \in \mathbb{N}} q((T_n'' - S'')(z_j))$ converges to 0 as $j \to \infty$ for each continuous seminorm q on $(E'', \beta(E'', E'))$.

Weakly compact operators on a C^*-algebra

- Let $\psi \in A'$ be a positive functional on a C^*-algebra A, and define

$$p_\psi(a) := \psi((aa^* + a^*a)^{1/2}), \; a \in A.$$

  Then p_ψ is a seminorm on A.

- A positive functional $\psi \in A'$ satisfying $\|\psi\| = 1$ is called a **state**.

- The **universal σ-strong* topology of a C^*-algebra** A is the topology induced by all seminorms p_ψ, where ψ is a positive functional on A.

Weakly compact operators on a C^*-algebra

- It follows from a fundamental result of Akemann, 1967, that the restriction of the σ-strong* topology to the unit ball A_1 of A coincides with the restriction of the Right topology $\rho(A)$ to A_1.

- An **orthogonal sequence** in the C^*-algebra A is a sequence $(a_n)_n$ of self adjoint elements of the closed unit ball of A such that $a_n a_m = 0$ whenever $n \neq m$.

The omnibus theorem

Theorem

Let A be a C^*-algebra, E a complete locally convex space and $T : A \to E$ a continuous linear operator. The following conditions are equivalent:

(1) T is a weakly compact operator.

(2) $T : (A, \rho(A)) \to E$ is sequentially continuous.

(3) If $(a_n)_n$ is an orthogonal sequence in A, then $(Ta_n)_n$ converges to 0 in E.

(4) For every bounded universal strong $*$-null net $(a_\lambda)_\lambda$ in A we have $(T(a_\lambda))_\lambda$ converges to 0 in E.

(5) If $(a_n)_n$ is a sequence in A which is convergent in the universal σ-strong* topology, then $(Ta_n)_n$ converges in E.

The omnibus theorem continued

Theorem continued

(6) $T : (A, \rho(A)) \to E$ is continuous.

(7) For each $q \in \mathrm{cs}(E)$ there exist a state $\phi_q \in A'$ and $N_q :]0, \infty[\to]0, \infty[$ such that

$$q(Ta) \leq N_q(\varepsilon) p_{\phi_q}(a) + \varepsilon \|a\|.$$

Lemma, Akemann (1967)

A subset $K \subset A'$ is relatively $\sigma(A', A'')$-compact if and only if K is bounded and there exists a state $\phi \in A'$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $a \in A_1$ for which $p_\phi(a) < \delta$ then $|\psi(a)| < \varepsilon$ for all $\psi \in K$.

The omnibus theorem continued

- The characterization in the lemma is a deep theorem due to **Akemann (1967)**. It is very important in the proof.
- The implication (3) $\Rightarrow$ (1) is based on a very deep theorem due to **Pfitzner (1994)**.
- Conditions of type (7) in this context were introduced by **Jarchow (1986)**.

Weak E-valued integral. About Question 2.

- A_{sa} the set of self adjoint elements in A.
- A^σ stands for the smallest subspace of A'' containing A with the property that whenever $(b_n)_n$ is a monotonic sequence in $(A^\sigma)_{sa}$ with limit b in the weak operator topology of A'' (or in the topology $\sigma(A'', A')$), then $b \in A^\sigma$.
- By a fundamental theorem of Pedersen 1979, A^σ is a C^*-subalgebra of A'' and it is called the **Baire $*$-envelope of A or the Pedersen envelope of A.**

Weak E-valued integral. About Question 2.

Definition

$\tilde{T}: A^{\sigma} \longrightarrow E$ is a **weak E-valued integral for** A if it is continuous and for all $(b_n)_n$ monotonic sequence of self adjoint elements in A^{σ} such that b_n tends to b for the $\sigma(A^{\sigma}, A')$-topology then $\tilde{T}(b_n)$ tends to $\tilde{T}(b)$ for the $\sigma(E, E')$-topology.

Theorem. It extends a result of Wright (1980)

A continuous operator $T : A \longrightarrow E$ is weakly compact if and only if there is a weak E-valued integral $\tilde{T} : A^{\sigma} \longrightarrow E$ for A whose restriction to A coincides with T.

Weak E-valued integral. About Question 2.

Corollary

If A be a C^*-algebra and E is a complete locally convex space which contains no copy of c_0, then every $T \in L(A, E)$ is weakly compact.

- The commutative case of this result is due to **Panchapagesan, 1998**. The non-commutative case for a Banach space E is due to **Akemann, Dodds and Gamlen (1972)**.
- It is based on a theorem of **Tumarkin, 1970,** extending a result of **Bessaga and Pelczynski**, on the unconditonal convergence of weakly unconditionally Cauchy series in locally complete spaces not containing c_0.

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