Toeplitz operators on the space of analytic functions with logarithmic growth

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- We study Toeplitz operators T_φ in the space H_V^∞ of analytic functions on the unit disc which grow logarithmically at the boundary.

- Though the space H_V^∞ is quite far from the Bergman Hilbert space A^2, our results will be surprisingly similar to that basic case.

- The **space** H_V^∞ and the related **space** L_V^∞ were introduced and studied by Taskinen in 2003-04 as **substitutes of** the usual sup–normed spaces H^∞ and L^∞, **to avoid the well–known discontinuity problem of the Bergman projection** P with respect to the sup–norm.

- These spaces are **topological algebras** and they have the properties that H_V^∞ is a **closed subspace** of L_V^∞ and that P is a **continuous projection** from L_V^∞ onto H_V^∞. They are the smallest extension of H^∞ and L^∞ defined by weights with this property.

Basic notations and definitions

All function spaces are defined on the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. We **denote:**

- dA as the normalized two–dimensional Lebesgue measure on $\mathbb{D}$.
- $L^p := L^p(dA)$, the space of p–integrable functions on the disc $\mathbb{D}$ with respect to the measure dA.
- Here $1 \leq p \leq \infty$.
- A^p stands for the Bergman space, which is the closed subspace of L^p consisting of analytic functions.

The Bergman projection P is the integral operator

$$Pf(z) := \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta), \qquad , \quad z \in \mathbb{D}, \tag{1}$$

defined at least for all $f \in L^1(dA)$.

Toeplitz–operators

Given a function (symbol) $\varphi \in L^1_{\mathrm{loc}}(dA)$ we denote by M_φ the pointwise multiplication with φ, and by T_φ we **denote**

$$T_\varphi f(z) := PM_\varphi f(z) := \int_{\mathbb{D}} \frac{\varphi(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta). \tag{2}$$

The Berezin transform

$$\tilde{\varphi}(z) = \int_{\mathbb{D}} \frac{\varphi(\zeta)(1 - |z|^2)^2}{|1 - z\bar{\zeta}|^4} dA(\zeta), \tag{3}$$

defined for a function $\varphi \in L^1$.

Weight function

$$w(z) := 1 + |\log(1 - z)| \tag{4}$$

- This function is mainly used to define the radial weight $w(|z|)$.
- Sometimes it is however necessary to define the logarithm as an analytic function on $\mathbb{D}$: then the argument of $1 - z$ is understood to belong to the interval $]-\pi, \pi[$ for all $z \in \mathbb{D}$, and the definition becomes unambiguous.

The spaces H_V^∞ (and L_V^∞)

This space consists of analytic (resp. measurable) functions $f : \mathbb{D} \to \mathbb{C}$ such that for some $n \in \mathbb{N}$ and constant $C_n > 0$

$$|f(z)| \leq C_n w(|z|)^n \tag{5}$$

for (almost) all $z \in \mathbb{D}$.

- The **space** H_V^∞ is an (LB)–space, i.e., countable inductive limits of Banach spaces. In fact, it **is a complete** (LB)**-space**.

- A convenient way **to describe the topology of** H_V^∞ is obtained **with a projective description** of the inductive limit topology. This can be done as a consequence of the work of Bierstedt, Meise and Summers. More precisely, it allows us to define the topology by means of the following **family of weighted sup-seminorms:**

$$\|f\|_v := \sup_{z\in\mathbb{D}} |f(z)|v(z) \quad , \quad v \in V, \tag{6}$$

where V consists of all continuous, positive, radial functions $v : \mathbb{D} \to \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$v(z) \leq C_n w(z)^{-n} \tag{7}$$

- We still denote

$$B_n := \{f \in H_V^\infty \mid \sup_{z \in \mathbb{D}} |f(z)| \leq w(|z|)^n\}; \tag{8}$$

in the same way, using essential supremum, we define the subsets B_n^L of L_V^∞.

- The sets B_n are bounded and even precompact in H_V^∞. Every bounded subset of H_V^∞ is contained in a multiple of some B_n. The same holds for the sets B_n^L in L_V^∞, except that bounded sets need not be precompact in this case.

Proposition (Taskinen)

(1) *If* $v \in V$*, then the pointwise product* $w^k v$ *also belongs to* V*, for every* k*.*

(2) *The mapping* P *is a continuous projection from* L_V^∞ *onto* H_V^∞*.*

(3) *In addition,* $P(B_n^L) \subset C_n B_{n+1}$ *for all* n*.*

A linear operator from a complete locally convex space into itself is called compact (respectively, bounded), if it maps a neighbourhood of zero into a precompact (resp. bounded) set.

Remark

(1) A linear operator T between two (LB)–spaces is continuous, if and only if it maps bounded sets into bounded sets. In our case this means that $T_\varphi : H_V^\infty \to H_V^\infty$ is continuous if and only if for every $n \in \mathbb{N}$ one can find $C_n > 0$ and $m \in \mathbb{N}$ such that $T_\varphi(B_n) \subset C_n B_m$.

(2) One can also show that $T_\varphi : H_V^\infty \to H_V^\infty$ is compact, if and only if there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ one can find a constant C_n with $T_\varphi(B_n) \subset C_n B_m$.

Theorem

Assume $\varphi \geq 0$. *The Toeplitz operator* T_φ *is continuous* $H_V^\infty \to H_V^\infty$ *if and only if there exist* $k \in \mathbb{N}$ *and* $C > 0$ *such that the Berezin transform* $\tilde{\varphi}$ *satisfies*

$$\tilde{\varphi}(z) \leq Cw(|z|)^k = C\Big(1 + |\log(1 - |z|)|\Big)^k, \qquad z \in \mathbb{D}. \tag{9}$$

Proof of Necessity. We assume T_φ is continuous and find $k \in \mathbb{N}$ such that T_φ maps B_1 into CB_k (see the remark after (8) and last Remark, (1)). Considering for a moment the number $z \in \mathbb{D}$ as a parameter, we define the analytic function

$$K_z(\zeta) := \frac{1}{(1-\zeta\bar{z})^2} \qquad , \quad \zeta \in \mathbb{D}. \tag{10}$$

Trivially the estimate

$$\|(1-|z|^2)^2 K_z\|_\infty \leq 4 \tag{11}$$

holds for every z. Hence $T_\varphi((1-|z|^2)^2 K_z) \in 4CB_k$ for every z.

We have

$$|T_\varphi((1-|z|^2)^2 K_z)(\omega)| \leq 4Cw(|\omega|)^k \qquad (12)$$

for all $\omega \in \mathbb{D}$. But taking $\omega = z$, this means that

$$\begin{aligned} |\tilde{\varphi}(z)| &= \Big| \int_{\mathbb{D}} \frac{\varphi(\zeta)(1-|z|^2)^2}{|1-z\bar{\zeta}|^4} dA(\zeta) \Big| \\ &= \Big| \int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1-z\bar{\zeta})^2} \frac{(1-|z|^2)^2}{(1-\zeta\bar{z})^2} dA(\zeta) \Big| \\ &= |T_\varphi\big((1-|z|^2)^2 K_z\big)(z)| \\ &\leq Cw(|z|)^k. \qquad (13) \end{aligned}$$

This is the condition (9).

Theorem

Assume $\varphi \geq 0$. *The operator* $T_\varphi : H_V^\infty \to H_V^\infty$ *is compact, if and only there exist* $k \in \mathbb{N}$ *such that for every* $q \in \mathbb{N}$ *there exists* $C_q > 0$ *with*

$$\int_{\mathbb{D}} \frac{\varphi(\zeta)(1-|z|^2)^2}{|1-z\zeta|^4} w(\zeta\bar{z})^q dA(\zeta) \leq C_q w(|z|)^k, \quad z \in \mathbb{D} \qquad (14)$$

In the case of the reflexive Bergman spaces A^p it is clear that a Toeplitz operator is bounded, if its symbol φ is bounded on $\mathbb{D}$. This follows from the boundedness of the Bergman projection $P : L^p(dA) \to A^p$ for $1 < p < \infty$.

In case of H_V^∞ there is an analogous, quite straightforward sufficient condition based on a pointwise estimate of φ. Many unbounded symbols satisfy this condition. Moreover, there is an similar condition for the compactness of T_φ.

Proposition

If there exists a $k \in \mathbb{N}$ such that

$$|\varphi(z)| \leq Cw(|z|)^k \qquad \text{for all } z \in \mathbb{D}, \tag{15}$$

then $T_\varphi : H_V^\infty \to H_V^\infty$ is continuous.

Proof. If $v \in V$, then the weight $\omega := w^k v$ still belongs to V. Moreover

$$\|f\varphi\|_v \leq C\|f\|_\omega \tag{16}$$

for all $f \in H_V^\infty$, i.e., the multiplication operator M_φ is continuous $H_V^\infty \to L_V^\infty$.
The Bergman projection is continuous $L_V^\infty \to H_V^\infty$, hence, $T_\varphi = PM_\varphi$ is continuous on H_V^∞.

Proposition

Assume that for all $q \in \mathbb{N}$ there exist $C_q \in \mathbb{N}$ such that

$$|\varphi(z)| \leq C_q w(|z|)^{-q} \qquad \text{for all } z \in \mathbb{D}. \tag{17}$$

Then $T_\varphi : H_V^\infty \to H_V^\infty$ is compact.

It is not difficult to see that there are positive symbols which do not satisfy (15) but nevertheless determine continuous and even compact Toeplitz operators on H_V^∞.

Proposition

For any non negative $\varphi \in L^1(\mathbb{D})$ *such that the support of* φ *is a compact subset of* $\mathbb{D}$, *the operator* $T_\varphi : H_V^\infty \to H_V^\infty$ *is compact.*

Proof. Let $0 < r < 1$ be such that $\mathrm{supp}(\varphi)$ is contained in the closed disc $D(0,r)$. There exists a constant $C > 0$ such that $|1 - z\bar{\zeta}| = |1 - \zeta\bar{z}| \geq C$ for all $\zeta \in D(0,r)$ and $z \in \mathbb{D}$. Hence, for every q we have $w(\zeta\bar{z})^q \leq C_q$ for such ζ and z, and we can estimate

$$\int_{\mathbb{D}} \frac{\varphi(\zeta)(1-|z|^2)^2}{|1-z\bar{\zeta}|^4} w(\zeta\bar{z})^q dA(\zeta) = \int_{D(0,r)} \frac{\varphi(\zeta)(1-|z|^2)^2}{|1-z\bar{\zeta}|^4} w(\zeta\bar{z})^q dA(\zeta)$$

$$\leq C_q \int_{D(0,r)} \varphi(\zeta) dA(\zeta)(1-|z|^2)^2 \leq C'_q. \quad (18)$$

The result follows from our Theorem above.

This lecture is based in our joint article

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