Parameter dependence of solutions of differential equations on spaces of distributions

José Bonet

Universidad Politécnica de Valencia

II Iberian Mathematical Meeting, Badajoz, 2008

Joint work with P. Domański, Poznań (Poland)

Statement of the problem

Linear partial differential operator (P.D.O.) with constant coefficients

$$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

$f_\lambda \in \mathscr{D}'(\Omega)$ depends on $\lambda \in U$ holomorphically, smoothly, real analytically, continuously,...

Does there exist $u_\lambda \in \mathscr{D}'(\Omega)$ depending on $\lambda \in U$ in the same way such that

$$P(D)u_\lambda = f_\lambda, \qquad \forall \lambda \in U?$$

Notation

- $P(z) := \sum_{|\alpha|\leq m} a_\alpha z^\alpha, a_\alpha \in \mathbb{C}$
- $P_m(z) := \sum_{|\alpha|=m} a_\alpha z^\alpha$

$$P(D) := \sum_{|\alpha|\leq m} a_\alpha D^\alpha$$

where $\partial_j := \dfrac{\partial}{\partial x_j}$, $\quad D_j := \dfrac{1}{i}\dfrac{\partial}{\partial x_j}$, $\quad D^\alpha := D_1^{\alpha_1} \dots D_N^{\alpha_N}$

- $\mathcal{E}(\Omega)$, $\mathcal{D}'(\Omega)$, $C(\Omega)$, $\mathcal{A}(\Omega)$

Surjectivity of $P(D)$

Malgrange, Ehrenpreis, Hörmander ($\leq$ 1962)

Theorem 1. $P(D)$ is surjective on $\mathcal{E}(\Omega)$ if and only if Ω is P-convex:

$$\forall K \subset \Omega \quad \exists L \subset \Omega \quad \forall \varphi \in \mathcal{D}(\Omega) \quad \mathrm{supp} P(D)\varphi \subset K \Rightarrow \mathrm{supp}\varphi \subset L$$

- Every convex open set is P-convex.
- If P is elliptic, every open set is P-convex.

Surjectivity of $P(D)$

Malgrange, Ehrenpreis, Hörmander ($\leq$ 1962)

Theorem 2. $P(D)$ is surjective on $\mathcal{D}'(\Omega)$ if and only if Ω is P convex and P-convex for singular supports:

$$\forall K \subset \Omega \ \ \exists L \subset \Omega \ \forall \mu \in \mathcal{E}'(\Omega) \quad \text{sing supp} P(D)\mu \subset K \Rightarrow \text{sing supp}\mu \subset L$$

The singular support is the complement in Ω of the largest open set on which μ is $\mathcal{C}^\infty$.

Historical Remarks

- **Trèves, 1962**: smooth and holomorphic dependence
- **F.E. Browder, 1962**: real analytic dependence
- **F. Mantlik, 1990-92**: parameter dependence of fundamental solutions
- **Grothendieck, late 1950's, Vogt, 1985**: case of $P(D)$ on $\mathcal{E}(\Omega)$
- **Domanski, Bonet**: early results for continuous and real analytic dependence in 1996.

Clarifying the dependence

$G \subset \mathbb{R}^N$, open.
$G \to \mathcal{D}'(\Omega),\ \lambda \to f_\lambda$ is $\mathcal{C}^\infty$ if for each $\varphi \in \mathcal{D}(\Omega)$ the map on G
$\lambda \to \langle f_\lambda, \varphi \rangle$ is $\mathcal{C}^\infty$

Clarifying the dependence

$G \subset \mathbb{R}^N$, open.
$G \to \mathcal{D}'(\Omega)$, $\lambda \to f_\lambda$ is $\mathcal{C}^\infty$ if for each $\varphi \in \mathcal{D}(\Omega)$ the map on G
$\lambda \to \langle f_\lambda, \varphi \rangle$ is $\mathcal{C}^\infty$

$G \subset \mathbb{C}$, open.
$G \to \mathcal{D}'(\Omega)$, $\lambda \to f_\lambda$ is **holomorphic** if for each $\varphi \in \mathcal{D}(\Omega)$ the map on G
$\lambda \to \langle f_\lambda, \varphi \rangle$ is holomorphic.

Clarifying the real analytic dependence

$G \subset \mathbb{R}^N$, $f : G \to E$ is **real analytic** if $u \circ f : G \to \mathbb{C}$ is real analytic for each $u \in E'$.

- This concept does **not** coincide with the fact that f is locally given by a vector valued Taylor series convergent in E.

Take $\mathbb{R} \to \mathcal{D}(\mathbb{R})$,

$$f_\lambda(x) := \frac{1}{1+\lambda^2 x^2}, \ x \in \mathbb{R},$$

is real analytic but not strongly real analytic in the sense explained above.

Clarifying the real analytic dependence

The coincidence of the two concepts of real analytic vector valued maps $f : G \to E$, $G \subset \mathbb{R}^N$, was investigated by **Kriegl** and **Michor** in 1991. Their research was continued by **Domanski, Bonet** in 1996.

Theorem (Bonet, Domański, 1996)

Let E be a Fréchet space. Every real analytic map $f : G \to E$ is strongly real analytic if and only if the Fréchet space E satisfies condition (DN) od Vogt.
A nuclear Fréchet space satisfies condition (DN) if and only if it is isomorphic to a closed subspace of the Schwartz space S of rapidly decreasing functions on the real line.

Existence of continuous linear right inverse

- Suppose that there exists a solution operator for $P(D)$. This means that there is a continuous, linear operator

$$R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$$

such that $P(D)Ru = u$ for each $u \in \mathcal{D}'(\Omega)$.

- If we try to solve the parameter dependence problem $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ with $f_\lambda \in \mathscr{D}'(\Omega)$ depends on $\lambda \in U$, it is enough to take $u_\lambda := Rf_\lambda$.

- Schwartz asked as a problem to characterize the linear P.D.O. with constant coefficients which admit a solution operator.

Existence of continuous linear right inverse

- Hyperbolic operators : **YES**.
- Elliptic operators: **NO**, Grothendieck, late 1950's.
- Parabolic operators: **NO**, Cohhon, 1970.
- Hypoelliptic operators: **NO**, Vogt, 1983.
- **Complete solution**: Meise, Taylor, Vogt, 1990.
  Characterizations in terms of the existence of shifted fundamental solutions with big lacunas in the support, or, for convex domains, in terms of a Phragmén Lindelöf condition for plurisubharmonic functions on the zero variety of the polynomial $P(z)$ in $\mathbb{C}^N$.
- The Zeilon operator $P(D)$ for $P(z) = z_1^3 + z_2^3 + z_3^3$ admits a solution operator on $\mathcal{D}'(\mathbb{R}^3)$ which is not hyperbolic.

Main Results

Assume that $P(D)$ is surjective on $\mathcal{D}'(\Omega)$.

(1) (Bonet, Domański, 1996)

Ω arbitrary, $K \subset \mathbb{R}^d$ compact.

$$P(D) : C(K, \mathcal{D}'(\Omega)) \to C(K, \mathcal{D}'(\Omega))$$

is surjective.
It is always possible to solve with continuous dependence.
In fact, for all X Banach,

$$P(D) : \mathcal{D}'(\Omega, X) \to \mathcal{D}'(\Omega, X)$$

is surjective.

Main Results

(2) (Bonet, Domański, 2006,08)

If Ω is convex then the following operators are surjective:

(a) $P(D) : H(U, \mathscr{D}'(\Omega)) \longrightarrow H(U, \mathscr{D}'(\Omega))$ for every Stein manifold U.

(b) $P(D) : C^\infty(U, \mathscr{D}'(\Omega)) \longrightarrow C^\infty(U, \mathscr{D}'(\Omega))$ for every smooth manifold U.

(c) $P(D) : \mathscr{D}'(\Omega, E) \longrightarrow \mathscr{D}'(\Omega, E)$ for every FN space $E \in (\Omega)$ (for instance, $E \simeq H(U), C^\infty(U), \mathscr{S}, \Lambda_r(\alpha)$).

(d) $P(D) : \mathscr{D}'(\Omega, E') \longrightarrow \mathscr{D}'(\Omega, E')$ for every FN space $E \in (DN)$ (for instance, $E' \simeq \mathscr{S}', H(\{0\}), \Lambda'_\infty(\alpha), \mathscr{D}'(U)$).

For elliptic $P(D)$ the converse to (d) holds (Vogt, 1983).
Examples of spaces with (Ω): $H(U)$, $C^\infty(U)$, $\mathscr{S}$, $\Lambda_r(\alpha)$.

Main Results

(3) (Domański, 200?)

Let Ω be convex and let U be a real analytic manifold.

$$P(D) : \mathcal{A}(U, \mathcal{D}'(\Omega)) \to \mathcal{A}(U, \mathcal{D}'(\Omega)) \quad \text{is surjective}$$

$$\Updownarrow$$

$$P(D) : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$$

admits a continuous linear right inverse, **in one of the following cases**:

1. $P(D)$ is homogeneous,
2. $P(D)$ is of order two.
3. $PD)$ is hypoelliptic.

A functional analytic approach

$$H(U, \mathscr{D}'(\Omega)) \leftrightarrow H(U)\hat{\otimes}_{\varepsilon}\mathscr{D}'(\Omega) = L(H(U)', \mathscr{D}'(\Omega))$$

$$H(U, \mathscr{D}'(\Omega)) \ni u \longleftrightarrow T \in L(H(U)', \mathscr{D}'(\Omega))$$

$$T(\delta_\lambda) := u(\lambda), \quad P(D)(T) := P(D) \circ T$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker P(D) & \longrightarrow & \mathscr{D}' & \xrightarrow{P(D)} & \mathscr{D}' & \longrightarrow & 0 \\ & & & & & & \uparrow T & & \\ & & & & & & H(U)' & & \end{array}$$

The map

$$P(D) \ : \ L(H(U)', \mathscr{D}'(\Omega)) \longrightarrow L(H(U)', \mathscr{D}'(\Omega))$$

is surjective if and only if every T lifts.

Theorem

Let E be a Fréchet Schwartz space and $R : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be surjective.

$$R \otimes \mathrm{id}_E \; : \; \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E \longrightarrow \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E \simeq \mathscr{D}'(\Omega, E)$$

$$R \otimes \mathrm{id}_{E'} \; : \; \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E' \longrightarrow \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E' \simeq \mathscr{D}'(\Omega, E')$$

Then

$$\mathrm{Ext}^1(E, \ker R) = 0 \quad \Leftrightarrow \quad R \otimes \mathrm{id}_{E'} \text{ is surjective,}$$

$$\mathrm{Ext}^1(E', \ker R) = 0 \quad \Rightarrow \quad R \otimes \mathrm{id}_E \text{ is surjective}$$

If additionally $E \in\!(\Omega)$ then

$$\mathrm{Ext}^1(E', \ker R) = 0 \quad \Leftrightarrow \quad R \otimes \mathrm{id}_E \text{ is surjective.}$$

A functional analytic approach

Theorem

If Ω convex then $\ker P(D) \subseteq \mathscr{D}'(\Omega)$ has (PA) and $(P\Omega)$, i.e., it has dual interpolation estimate for small θ:

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m, \theta < \theta_0\ \exists\ k, C\ \forall\ y \in X_N' :$$

$$\|y\|_{M,m}^* \ \leq\ C \left(\|y\|_{K,k}^{*\ (1-\theta)} \cdot \|y\|_{N,n}^{*\ \theta} \right).$$

Invariants like this, and the theory of the splitting of short exact sequences of (PLS)-spaces (countable projective limits of (DFS)-spaces), permit to conclude when $\mathrm{Ext}^1(E', \ker P(D)) = 0$

Open problems

1. Show that $\Omega \times \mathbb{R}$ is P-convex for singular support if Ω is P-convex for singular support for arbitrary Ω. In other words, does the surjectivity of $P(D) : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ imply that $P(D)$ is surjective on $\mathcal{D}'(\Omega \times \mathbb{R})$?

Open problems

1. Show that $\Omega \times \mathbb{R}$ is P-convex for singular support if Ω is P-convex for singular support for arbitrary Ω. In other words, does the surjectivity of $P(D) : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ imply that $P(D)$ is surjective on $\mathcal{D}'(\Omega \times \mathbb{R})$?

2. Does the surjectivity of $P(D) : \mathcal{A}(U, \mathcal{D}'(\Omega)) \to \mathcal{A}(U, \mathcal{D}'(\Omega))$ imply that $P(D)$ has a continuous linear right inverse on $\mathcal{D}'(\Omega)$?

References

1. **J. Bonet, P. Domański,** Real analytic curves in Fréchet spaces and their duals, Monatsh. Math. 126 (1998), 13–36.
2. **J. Bonet, P. Domański,** Parameter dependence of solutions of differential equations on spaces of distributions and the splitting of short exact sequences. J. Funct. Anal. 230 (2006), 329–381.
3. **J. Bonet, P. Domanski,** The splitting of short exact sequences of PLS-spaces and smooth dependence of solutions of linear partial differential equations, Advances in Math. 217 (2008) 561-585.
4. **P. Domański,** Real analytic parameter dependence of solutions of differential equations, Preprint, 2008.