Holomorphic dependence of diagonal operators between sequence spaces

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Complex and harmonic analysis 2011
Málaga
July, 2011

Joint work with C. Gómez, D. Jornet and E. Wolf

What do we want to study?

AIM

Investigate **properties of diagonal operators defined on Köthe echelon spaces in case the diagonal depends holomorphically on a parameter** $z \in \mathbb{D}$.

Köthe matrix

$A = (a_n(i))_{i,n\in\mathbb{N}}$ a matrix of non-negative numbers is a *Köthe matrix* if for each $i \in \mathbb{N}$ and $n \in \mathbb{N}$

$$0 < a_n(i) \leq a_{n+1}(i)$$

Köthe echelon spaces

For $1 \leq p < \infty$,

$$\lambda_p(A) = \{x \in \mathbb{C}^{\mathbb{N}} \ : \ q_n(x) := \Big(\sum_{i=1}^{\infty} (a_n(i)|x_i|)^p\Big)^{1/p} < \infty \text{ for all } n \in \mathbb{N}\}.$$

Examples

Clearly

$$\lambda_p(A) = \bigcap_{n\in\mathbb{N}} \ell_p(a_n).$$

- $H(\mathbb{D}) \simeq \lambda_1(A)$ with $a_n(i) = (n/(n+1))^i$.
- $H(\mathbb{C}) \simeq \lambda_1(A)$ with $a_n(i) = n^i$.
- $\mathcal{S} \simeq C^\infty([0,1]) \simeq \mathcal{D}([0,1]) \simeq \lambda_1(A)$ with $a_n(i) = i^n$.

Köthe echelon spaces are Fréchet spaces, i.e., metrizable complete locally convex spaces.

Diagonal operators

$\lambda_p \equiv \lambda_p(A)$.

$f_i : \mathbb{D} \to \mathbb{C}$ holomorphic, $i \in \mathbb{N}$, $(f_i)_{i\in\mathbb{N}}$ bounded for the co-topology.

Diagonal operators in Köthe echelon spaces

We consider the following operator-valued function

$$\begin{array}{rcl} \psi : \mathbb{D} & \to & L_b(\lambda_p, \lambda_p) \\ z & \rightsquigarrow & \psi(z)(x) = (f_i(z)x_i)_{i\in\mathbb{N}} \end{array}$$

Notation

X, Y Fréchet spaces.

$L_b(X, Y)$ linear continuous operators between X and Y endowed with the topology of uniform convergence on bounded subsets of X. If X and Y are Banach spaces, this is the topology of the operator norm.

$H(\mathbb{D}, X)$ space of vector-valued analytic functions.

$\psi : \mathbb{D} \to L_b(X, Y)$ an analytic operator-valued function

Proposition

Let $f_i : \mathbb{D} \to \mathbb{C}$, $i \in \mathbb{N}$, be holomorphic functions such that $(f_i)_{i\in\mathbb{N}}$ is bounded for the co-topology. Then

(a) $\psi \in H(\mathbb{D}, L_b(\lambda_p, \lambda_p))$.

(b) If $(f_i)_{i\in\mathbb{N}}$ tends to 0 in the co-topology, then $\psi(z)$ is **Montel** for all $z \in \mathbb{D}$, i.e. each $\psi(z)$ maps bounded sets into relatively compact sets.

In case X and Y are Banach spaces, Montel operators are exactly compact operators.

Proof

For (a) we use

Theorem (Grosse-Erdmann)

E complete lcs, $\psi : \Omega \to E$ locally bounded, and Ω domain in $\mathbb{C}$. If $H \subset E'$ is $\sigma(E', E)$-dense in E' and $u \circ \psi$ holomorphic for $u \in H$, then $\psi \in H(\Omega, E)$.

And we check

- $\psi : \mathbb{D} \to L_b(\lambda_p, \lambda_p)$ is locally bounded
- $G = \mathrm{span}\{u \otimes y \ : \ u \in \lambda_p', y \in \lambda_p\}$ is weak*-dense in $L_b(\lambda_p, \lambda_p)'$
- Finally, $(u \otimes y) \circ \psi(z) = \sum_i u_i y_i f_i(z)$ holomorphic in $H(\mathbb{D})$.

Operator weighted composition operators

X, Y Fréchet spaces.

$\varphi : \mathbb{D} \to \mathbb{D}$, $\psi : \mathbb{D} \to L_b(X, Y)$ analytic.

The operator-weighted composition operator

$$\begin{array}{rcccl} W_{\psi,\varphi} : & H(\mathbb{D}, X) & \longrightarrow & H(\mathbb{D}, Y) & \\ & f & \rightsquigarrow & W_{\psi,\varphi} f : & \mathbb{D} \longrightarrow Y \\ & & & & z \rightsquigarrow \psi(z)[(f \circ \varphi)(z)] \end{array}$$

Continuity

The operator $W_{\psi,\varphi} : H(\mathbb{D}, X) \longrightarrow H(\mathbb{D}, Y)$ is well-defined and continuous.

An auxiliary operator

The auxiliary operator T_ψ

Let $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic. We consider the operator

$$\begin{aligned} T_\psi : X &\to H(\mathbb{D}, Y), \\ x &\rightsquigarrow T_\psi(x) : \mathbb{D} \to Y, \\ & \qquad\qquad z \rightsquigarrow T_\psi(x)(z) = \psi(z)[x] \end{aligned}$$

- T_ψ is well defined and linear
- T_ψ inherits the properties of $W_{\psi,\varphi}$

The operators in the case of Köthe echelon spaces

- $\psi : \mathbb{D} \to L_b(\lambda_p, \lambda_p), \ \psi(z)(x) = (f_i(z)x_i)_{i\in\mathbb{N}},$
- $W_{\psi,\varphi} : H(\mathbb{D}, \lambda_p) \longrightarrow H(\mathbb{D}, \lambda_p), \varphi(z) = id(z) = z,$

$$g(z) = (g_i(z))_i \to W_{\psi,id}g(z) = (f_i(z)g_i(z))_i.$$

- $T_\psi : \lambda_p \to H(\mathbb{D}, \lambda_p),$

$$T_\psi((x_i)_i)(z) = \psi(z)((x_i)_i) = (f_i(z)x_i)_i.$$

Montel operators

Theorem

Let X and Y be Fréchet spaces. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$, $\psi \neq 0$, be analytic mappings. Then the following assertions are equivalent:

1. The operator
$$W_{\psi,\varphi} : \quad H(\mathbb{D}, X) \quad \longrightarrow \quad H(\mathbb{D}, Y)$$
is Montel
2. $T_\psi : X \to H(\mathbb{D}, Y)$ is Montel
3. $\psi(z) : X \longrightarrow Y$ is Montel for each $z \in \mathbb{D}$.

Proof

Idea of the proof

(3) $\Rightarrow$ (2) : Let $\psi(z) : X \to Y$ be Montel for all $z \in \mathbb{D}$.
Is $T_\psi : X \to H(\mathbb{D}, Y)$ Montel?

- The function ψ is holomorphic with values in $L_b(X, Y)$, and then $\psi(z) = \sum_{m=0}^{\infty} A_m z^m, A_m \in L_b(X, Y)$.
- $A_0 = \psi(0)$ is Montel and A_m are Montel, for $m \geq 1$ by Cauchy Integral Formula.
- For $\psi_n(z) = \sum_{m=0}^{n} A_m z^m$, the operator T_{ψ_n} is Montel.
- Finally, T_{ψ_n} tends to T_ψ in $L_b(X, H(\mathbb{D}, Y))$.

(2) $\Rightarrow$ (1) requires results on tensor products due to Ruess.

The weighted spaces case

Weights

Let v be a strictly positive continuous **weight** on the open unit disk $\mathbb{D}$ in the complex plane which is radial (that is, $v(z) = v(|z|)$ for every $z \in \mathbb{D}$), strictly decreasing with respect to $|z|$ and $\lim_{r\to 1} v(r) = 0$.

Examples

- The **standard weights** are $v(z) = (1-|z|)^\alpha$, $\alpha > 0$.
- $v(r) = \exp(-\frac{1}{(1-r)^\alpha})$, $\alpha > 0$.
- $v(r) = (1 - \log(1-r))^{-\alpha}$, $\alpha > 0$.

The weighted spaces case

Weighted spaces of holomorphic functions

$$H_v^\infty(\mathbb{D},X) := \{f \in H(\mathbb{D},X) : \quad \sup_{z\in\mathbb{D}} v(z)p(f(z)) < \infty \quad \forall p \in cs(X)\}$$

$$H_v^0(\mathbb{D},X) := \{f \in H_v^\infty(\mathbb{D},X) : \quad \lim_{|z|\to 1} v(z)p(f(z)) = 0 \quad \forall p \in cs(X)\}$$

endowed with the natural topology.

If we do not assume that $\lim_{r\to 1} v(r) = 0$, then $H_v^\infty(\mathbb{D},X) = H^\infty(\mathbb{D},X)$.

The weighted spaces case

Theorem

Let X, Y be Fréchet spaces. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic maps.

$W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \longrightarrow H_w^\infty(\mathbb{D}, Y)$ is continuous if, and only if, the set

$$\left\{ \frac{w(z)}{\tilde{v}(\varphi(z))}\psi(z);\ z \in \mathbb{D} \right\}$$

is equicontinuous in $L_b(X, Y)$.

Laitila and Tylli, 2009, have studied these operators for Banach spaces X and Y. Multiplication and composition operators are particular cases.

Theorem

Let X, Y be Fréchet spaces. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic maps. Then the following assertions are equivalent:

1. The weighted composition operator $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is Montel
2. (a) $T_\psi : X \to H_w^\infty(\mathbb{D}, Y)$ is Montel.

   (b) For every $B \in \mathcal{B}(X)$, $q \in cs(Y)$ and $\varepsilon > 0$ there is $r_0 \in (0, 1)$ such that if $|\varphi(z)| > r_0$ and $x \in B$, then we have the following inequality

$$\frac{w(z)}{\tilde{v}(\varphi(z))} q(\psi(z)[x]) \leq \varepsilon.$$

Examples.

Constant case

If the analytic operator-valued function $\psi : \mathbb{D} \to L_b(X, Y)$ is constant, that is, $\psi(z) = L \neq 0$ for all z, then

- The operator $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is continuous if and only if $C_\varphi : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ is continuous,
- The operator $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is Montel if and only if L is Montel and $C_\varphi : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ is compact.

Bonet, Domański, Lindström, Taskinen (1996), Contreras, Hernández-Díaz (2000), Bonet, Friz (2003)

The following always holds:

$$W_{\psi,\varphi} \text{ Montel } \Rightarrow T_{\psi} \text{ Montel } \Rightarrow \psi(z) \text{ Montel for all } z$$

Question

Is it true in the weighted case that

$$\psi(z) \textbf{ Montel for all } z \Rightarrow T_{\psi} \textbf{ Montel ?}$$

Laitila-Tylli: NO

The diagonal operator for $\lambda_p = \ell_1$

$$\psi : \mathbb{D} \to L(\ell_1, \ell_1), \ \psi(z)(x) = (z^i x_i)_{i \in \mathbb{N}},$$

satisfies that $\psi(z)$ is compact for all z, but $T_{\psi} : \ell_1 \longrightarrow H^{\infty}(\mathbb{D}, \ell_1)$ is not even weakly compact (here $v \equiv 1$).

Theorem

Let X, Y be Fréchet spaces.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic maps.

If $\psi \in H_w^0(\mathbb{D}, L_b(X, Y))$ and $\psi(z)$ is Montel for all z, then

$$T_\psi : X \longrightarrow H_w^\infty(\mathbb{D}, Y)$$

is a Montel operator.

Weighted case. Diagonal operators on Köthe echelon spaces

Proposition (Back to the diagonal operator)

Assume that $\|f_i\|_w := \sup_{z\in\mathbb{D}} w(z)|f_i(z)| \le 1$ for all $i \in \mathbb{N}$. For the operator $\psi : \mathbb{D} \to L_b(\lambda_p, \lambda_p)$, $\psi(z)(x) = (f_i(z)x_i)_{i\in\mathbb{N}}$, we have

- $T_\psi : \lambda_p \longrightarrow H_w^\infty(\mathbb{D}, \lambda_p)$, $T_\psi((x_i)_i) := (f_i(z)x_i)_i$, is well-defined and continuous.

Lemma

For every weight there is a sequence $(f_i)_{i\in\mathbb{N}}$ which tends to 0 for the co-topology, and such that $1 \ge \|f_i\|_w > \varepsilon$ for some $\varepsilon > 0$ and all $i \in \mathbb{N}$.

A Fréchet space X is called **Montel** if every bounded subset of X is relatively compact; i.e. if the identity map Id on X is a Montel operator.

The space $H(\Omega)$, Ω an open subset of $\mathbb{C}$, endowed with the compact open topology, is Montel

Theorem

Let w be a weight. If the sequence $(f_i)_{i\in\mathbb{N}}$ tends to 0 for the co-topology and there is $\varepsilon > 0$ such that $\varepsilon < \|f_i\|_w \leq 1$ for all $i \in \mathbb{N}$, then

(a) $\psi(z) : \lambda_p \rightarrow \lambda_p$ is Montel for all z.

(b) If λ_p is Montel, then T_ψ is a Montel operator.

(c) If λ_p is not Montel, then T_ψ is not a Montel operator.

References

1. **J. Bonet, M.C. Gómez-Collado, D.Jornet and E. Wolf,** Operator-weighted composition operators between vector-valued (weighted) spaces of analytic functions, Preprint 2011.

2. **J. Laitila and H.-O. Tylli,** Operator-weighted composition operators on vector-valued analytic function spaces, Illinois J. Math. 53 (2009), 1019-1032.