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Mean Ergodic semigroups of operators

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Abstract We present criteria for determining mean ergodicity of C_0–semigroups of linear operators in a sequentially complete, locally convex Hausdorff space X. A characterization of reflexivity of certain spaces X with a basis via mean ergodicity of equicontinuous C_0–semigroups acting in X is also presented. Special results become available in Grothendieck spaces with the Dunford–Pettis property.

Keywords C_0-semigroup · (reflexive) locally convex space · (uniform) mean ergodic

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1 Introduction

For a continuous linear operator T in a Banach space X, its Cesàro means are defined by

$$T_{[n]} := \frac{1}{n}\sum_{m=0}^{n-1} T^m, \quad n \in \mathbb{N}. \tag{1}$$

If $\{T_{[n]}\}_{n=1}^{\infty}$ is convergent in the strong operator topology, then T is called *mean ergodic.* The interest in such operators has its origins in statistical mechanics and probability theory. In such settings, one also considers continuous processes ϕ_t, with t specifying time, which in many situations satisfy $\phi_t(\phi_s(u)) = \phi_{t+s}(u)$ for all points u in a phase space and all times s, t. The abstract setting then consists of a *semigroup* of continuous linear operators $(T(t))_{t\geq 0}$ acting in X (i.e., $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$) and one investigates the long term behaviour of $(T(t))_{t\geq 0}$ via its Cesàro averages $C(r) := r^{-1}\int_0^r T(t)dt$ for $r > 0$. If this limit exists in the strong operator topology, then $(T(t))_{t\geq 0}$ is called *mean ergodic.* Fix $t_0 > 0$. Given $n \in \mathbb{N}$ it turns out that $C(nt_0) = T(t_0)_{[n]}C(t_0)$ and one sees the connection between the Cesàro averages $\{C(nt_0)\}_{n=1}^{\infty}$ of the semigroup $(T(t))_{t\geq 0}$ and the discrete Cesàro means $\{T(t_0)_{[n]}\}_{n=1}^{\infty}$ of the individual operator $T(t_0)$. So, there is a need to simultaneously investigate individual mean ergodic operators and mean ergodic systems of operators (i.e., semigroups). This has happened ever since the origins of the subject (i.e., the 1930's) and both theories are adequately addressed in the monographs, [8, Ch.VIII], [14, Ch.XVIII], [21], for example.

Much of modern analysis occurs in locally convex Hausdorff spaces (briefly, lcHs) which are *non–normable.* So, there is an interest in extending the Banach space results for mean ergodicity to this setting. For individual operators (especially Banach space results related to the theory of bases, [13]) this has been carried out in recent years, [1], [3], [4], [6], [7], [27], [28]. For certain aspects of the theory of mean ergodic *semigroups* of operators in the non–normable setting we refer to [9], [21, Ch.2], [31, Ch.III, §7], and the references therein. The aim of this paper is to develop this topic further, also in some rather different directions.

The natural framework is a sequentially complete lcHs X and a C_0–semigroup of continuous linear operators $(T(t))_{t\geq 0}$ acting in X. An adequate theory is available if $(T(t))_{t\geq 0}$ is equicontinuous, [33, Ch.IX]. However, this is too restrictive as there exist examples of non–equicontinuous C_0–semigroups; see §2. For X a Fréchet space, the theory of certain types of non–equicontinuous C_0–semigroups (called locally equicontinuous) is developed in [25]. But, even the Fréchet space setting is somewhat restrictive; this condition on X is relaxed in [18]. To define the Cesàro averages $\{C(r)\}_{r\geq 0}$ and establish some of their properties requires aspects of vector–valued Riemann integration; the relevant information is given in an Appendix.

In Section 3 results concerning mean ergodicity of certain classes of C_0–semigroups of operators $(T(t))_{t\geq 0}$ are established. We present sufficient conditions for various classes of C_0–semigroups to be mean ergodic (eg. Proposition

3, Corollary 2 and Theorem 4). A basic result needed in this regard is the fundamental mean ergodic theorem of W.F. Eberlein, [9]. In Theorem 5 we characterize the reflexivity of a complete, barrelled lcHs X with a Schauder basis via the condition that *every* equicontinuous C_0–semigroup of operators in X should be mean ergodic; for Banach spaces with a basis, see [26].

Section 4 concerns C_0–semigroups $(T(t))_{t\geq 0}$ in Grothendieck lcHs' with the Dunford–Pettis property (briefly, GDP) and their mean ergodicity. Certain classes of such C_0–semigroups are automatically uniformly continuous (cf. Theorem 7). Criteria are given which ensure that the dual semigroup $(T(s)')_{s\geq 0}$ is strongly continuous in X'_β (the strong dual of X). Mean ergodicity of $(T(t))_{t\geq 0}$ in Grothendieck spaces is equivalent to the subspace span $\{x' - T(s)'x' : x' \in X', s \geq 0\}$ of X' having the same closure for the weak–star topology as in X'_β; see Theorem 9. Consequently, if $(T(t))_{t\geq 0}$ is a mean ergodic, strongly continuous C_0–semigroup in a Grothendieck space and $\lim_{t\to\infty} T(t)/t = 0$ (for the topology of uniform convergence on bounded sets of X), then also $(T(s)')_{s\geq 0}$ is mean ergodic in X'_β. For GDP–Banach spaces, some of these results are due to H.P.Lotz, [22], [23].

2 Preliminaries.

Let X be a locally convex Hausdorff space (briefly, lcHs) and Γ_X a system of continuous seminorms determining the topology of X. The strong operator topology τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators from X to itself (from X to another lcHs Y we write $\mathcal{L}(X,Y)$) is determined by the seminorms $q_x(S) := q(Sx)$, for $S \in \mathcal{L}(X)$, with $x \in X$ and $q \in \Gamma_X$, in which case we write $\mathcal{L}_s(X)$. Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X. The topology τ_b of uniform convergence on bounded sets is defined in $\mathcal{L}(X)$ via the seminorms $q_B(S) := \sup_{x\in B} q(Sx)$, for $S \in \mathcal{L}(X)$, with $B \in \mathcal{B}(X)$, $q \in \Gamma_X$; in this case we write $\mathcal{L}_b(X)$. The identity operator on X is denoted by I.

By X_σ we denote X with its weak topology $\sigma(X, X')$, where X' is the topological dual space of X. The strong topology in X (resp. X') is denoted by $\beta(X, X')$ (resp. $\beta(X', X)$) and we write X_β (resp. X'_β); see [19, §21.2] for the definition. Then $\beta(X', X)$ is generated by the seminorms $p^B(x') := \sup_{x\in B} |\langle x, x'\rangle|$, for $x' \in X'$, with $B \in \mathcal{B}(X)$. The strong dual $(X'_\beta)'_\beta$ of X'_β is denoted by X''. By X'_{σ^*} we denote X' with its weak–star topology $\sigma(X', X)$. Given $T \in \mathcal{L}(X)$, its *dual operator* $T' \colon X' \to X'$ is defined by $\langle x, T'x'\rangle = \langle Tx, x'\rangle$ for $x \in X$, $x' \in X'$. Then $T' \in \mathcal{L}(X'_\sigma)$ and $T' \in \mathcal{L}(X'_\beta)$, [20, p.134].

Definition 1 Let X be a lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a 1–parameter family of operators. The map $t \mapsto T(t)$, for $t \in [0, \infty)$, is denoted by $T \colon [0, \infty) \to \mathcal{L}(X)$.

We say that $(T(t))_{t\geq 0}$ is a *semigroup* if it satisfies

(i) $T(s)T(t) = T(s + t)$ for $s, t \geq 0$, with $T(0) = I$.

A semigroup $(T(t))_{t\geq 0}$ is *locally equicontinuous* if, for fixed $K > 0$, the set $\{T(t) : 0 \leq t \leq K\}$ is equicontinuous, i.e., given $p \in \Gamma_X$ there exist $q \in \Gamma_X$

and $M > 0$ with

$$p(T(t)x) \leq Mq(x)\,, \quad x \in X,\, t \in [0,K]. \tag{2}$$

A semigroup $(T(t))_{t\geq 0}$ is said to be a C_0*–semigroup* if it satisfies

(ii) $\lim_{t\to 0^+} T(t) = I$ in $\mathcal{L}_s(X)$.

If the C_0–semigroup $(T(t))_{t\geq 0}$ satisfies the additional condition that

(iii) $\lim_{t\to t_0} T(t) = T(t_0)$ in $\mathcal{L}_s(X)$, for each $t_0 > 0$,

then it is called a *strongly continuous* C_0–semigroup.

A semigroup $(T(t))_{t\geq 0}$ is said to be *exponentially equicontinuous* if there exists $a \geq 0$ such that $(e^{-at}T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is equicontinuous, i.e.,

$$\forall p \in \Gamma_X\; \exists q \in \Gamma_X, M_p > 0 \text{ with } p(T(t)x) \leq M_p e^{at} q(x)\, \forall t \geq 0, x \in X. \tag{3}$$

If $a = 0$, then we simply say *equicontinuous*. Finally, a semigroup $(T(t))_{t\geq 0}$ is called *uniformly continuous* if $T\colon [0,\infty) \to \mathcal{L}_b(X)$ is continuous, i.e.,

(iv) $\lim_{t\to t_0} T(t) = T(t_0)$ in $\mathcal{L}_b(X)$, for each $t_0 > 0$ (with $t \to 0^+$ if $t_0 = 0$).

Remark 1 (i) If X is barrelled, then every strongly continuous C_0–semigroup is locally equicontinuous, [18, Proposition 1.1].

(ii) Every exponentially equicontinuous semigroup is locally equicontinuous. A strongly continuous C_0–semigroup in a Banach space is always exponentially equicontinuous, [8, p.619]. For Fréchet spaces this need not be so. Indeed, in the sequence space $\omega = \mathbb{C}^{\mathbb{N}}$ (topology of coordinate convergence), $T(t)x := (e^{nt}x_n)_{n=1}^{\infty}$, for $t \geq 0$ and $x = (x_n)_{n=1}^{\infty} \in \omega$, is a strongly continuous C_0–semigroup which is not exponentially equicontinuous. Since ω is barrelled, by (i) we see $(T(t))_{t\geq 0}$ is locally equicontinuous. As ω is a Montel space, $(T(t))_{t\geq 0}$ is also uniformly continuous. For an example in a Fréchet space of smooth functions, see [25, pp.162–163].

(iii) A locally equicontinuous C_0–semigroup $(T(t))_{t\geq 0}$ is strongly continuous. Indeed, via Definition 1(i) we have $T(t_0+h) - T(t_0) = T(t_0)(T(h) - I)$ for $h \geq 0$ and fixed $t_0 > 0$. It follows from continuity of $T(t_0)$ and Definition 1(ii) that $T\colon [0,\infty) \to \mathcal{L}_s(X)$ is right continuous at points of $[0,\infty)$. Fix $t_0 > 0$. Given $p \in \Gamma_X$, let $K := t_0$ and choose $q \in \Gamma_X$ and $M > 0$ so that (2) holds. Then

$$p(T(t_0 + h)x - T(t_0)x) = p(T(t_0 + h)(x - T(-h)x)) \leq Mq(x - T(-h)x),$$

for $x \in X$ and $h \in [-t_0/2, 0]$. In combination with Definition 1(ii) this implies left continuity of $T\colon [0,\infty) \to \mathcal{L}_s(X)$ at t_0.

(iv) Let $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be an equicontinuous semigroup. For $p \in \Gamma_X$ define $\tilde{p}(x) := \sup_{t\geq 0} p(T(t)x)$, for $x \in X$. Then $\tilde{p}$ is well-defined, is a seminorm and

$$p(x) \leq \tilde{p}(x) \leq M_p q(x) \leq M_p \tilde{q}(x), \quad x \in X, \tag{4}$$

with $M_p > 0$, $q \in \Gamma_X$ chosen via (3) (for $a = 0$). So, $\tilde{\Gamma}_X := \{\tilde{p} :\ p \in \Gamma_X\}$ also generates the given lc–topology of X. Moreover, for $\tilde{p} \in \tilde{\Gamma}_X$, we have

$$\tilde{p}(T(t)x) = \sup_{s\geq 0} p(T(t)T(s)x) = \sup_{s\geq 0} p(T(t+s)x) \leq \tilde{p}(x), \quad x \in X,\, t \geq 0. \tag{5}$$

A similar result to Proposition 1.6 of [17] (in Banach spaces see [14, Corollary, p.306, Theorem 10.6.5]) holds for semigroups of operators in lcHs'.

Proposition 1 *Let X be a quasicomplete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous semigroup. Then $(T(t))_{t\geq 0}$ is a C_0–semigroup if and only if $t \mapsto \langle T(t)x, x'\rangle$ is continuous on $[0,\infty)$ for each $x \in X$, $x' \in X'$.*

Proof If $(T(t))_{t\geq 0}$ is a C_0–semigroup, then $T\colon [0,\infty) \to \mathcal{L}_s(X)$ is continuous (cf. Remark 1(iii)) and so $\langle T(\cdot)x, x'\rangle$ is continuous on $[0,\infty)$ for each $x \in X$, $x' \in X'$.

Assume then that $\langle T(\cdot)x, x'\rangle$ is continuous for each $x \in X$, $x' \in X'$. Fix $x \in X$. For $r > 0$, the function $\varphi_{r,x}(s) := T(s)x$, $s \in [0, r]$, is $\sigma(X, X')$–continuous. So, $\varphi_{r,x}([0, r])$ is compact in X_σ. Since X is also quasicomplete for its Mackey topology, [19, 18.4.4, p.210], Krein's theorem implies the closed convex hull $H_{r,x}$ of $\varphi_{r,x}([0, r])$ is compact in X_σ, [19, 24.5,4$'$, p.325]. It follows from [29, Theorem 3.27] applied in X_σ that the weak integral $I(r, x) := \frac{1}{r}\int_0^r \varphi_{r,x}(s)ds \in H_{r,x}$ exists in the sense that $\langle I(r, x), x'\rangle = \frac{1}{r}\int_0^r \langle T(s)x, x'\rangle ds$, for $x' \in X'$. Since $s \mapsto \langle T(s)x, x'\rangle$ is continuous on $[0, r]$, the theory of scalar integration yields

$$\lim_{r\to 0^+} \frac{1}{r}\int_0^r \langle T(s)x, x'\rangle ds = \langle T(0)x, x'\rangle = \langle x, x'\rangle, \quad x' \in X'.$$

That is, $\sigma(X, X')$-$\lim_{r\to 0^+} I(r, x) = x$, for each $x \in X$. In particular, the subspace $H := \text{span}\,\{I(r, x) : x \in X, r > 0\}$ is dense in X (being dense in X_σ).

To complete the proof that $(T(t))_{t\geq 0}$ is a C_0–semigroup, it suffices to show $\lim_{t\to 0^+} T(t)y = y$, for $y \in H$, as H is dense with $\{T(t) : t \in [0, 1]\} \subseteq \mathcal{L}(X)$ equicontinuous and so the existence of $\lim_{t\to 0^+} T(t)x = x$ for an arbitrary $x \in X$ follows from [20, 39.4.1, p.138]. To this effect, fix $r > 0$, $x \in X$ and set $y := I(r, x)$. For $s \in (0, 1]$ we have $y - T(s)y = I(r, x) - T(s)I(r, x)$. Applying $x' \in X'$ and arguing as in the proof of Lemma 1(v) below, with the weak integral $I(r, x)$ replacing the Riemann integral $\frac{1}{r}\int_0^r T(s)x\, ds$, yields $y - T(s)y = \frac{1}{r}(I - T(r))sI(s, x)$. But, $sI(s, x) \in sH_{s,x} \subseteq sH_{1,x}$ with $H_{1,x}$ compact in X_σ and hence, bounded. As $T(r) \in \mathcal{L}(X_\sigma)$, the set $T(r)H_{1,x}$ is also compact in X_σ. Moreover, $y - T(s)y \in \frac{s}{r}(H_{1,x} + T(r)H_{1,x})$. Since $\frac{1}{r}(H_{1,x} + T(r)H_{1,x}) \in \mathcal{B}(X)$ and is independent of s, we conclude that $\lim_{s\to 0^+}(y - T(s)y) = 0$.

If X is a *sequentially complete* lcHs and $(T(t))_{t\geq 0}$ is a locally equicontinuous C_0–semigroup on X, then the linear operator A defined by

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t},$$

for $x \in D(A) := \{x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t}$ exists in $X\}$, is closed with $\overline{D(A)} = X$, [18, Propositions 1.3 & 1.4]. The operator $(A, D(A))$ is called the *infinitesimal generator* of $(T(t))_{t\geq 0}$. Moreover, A and $(T(t))_{t\geq 0}$ commute, [18, Proposition 1.2(1)], i.e., for each $t \geq 0$ we have $\{T(t)x : x \in D(A)\} \subseteq D(A)$

and $AT(t)x = T(t)Ax$ for all $x \in D(A)$. Also known, [18, Proposition 1.2(2)], is that

$$T(t)x - x = \int_0^t T(s)Ax\,ds = \int_0^t AT(s)x\,ds, \quad x \in D(A). \tag{6}$$

For each $x \in X$ and $t > 0$, the integrals occuring in (6) are Riemann integrals of an X–valued, continuous function on $[0,t]$; see Section 5. The closedness of A ensures that $\operatorname{Ker} A := \{x \in D(A) : Ax = 0\}$ is a closed subspace of X.

3 Mean ergodic theorems for C_0-semigroups in lcHs

Let X be a lcHs and $T \subseteq \mathcal{L}(X)$ be a semigroup under composition. Following Eberlein, [9, Definitiom 2.1], [21, Chapter 2, §2.1, p.75], we recall that T admits a *T–ergodic net* $\{A_\lambda : \lambda \in \Lambda\}$ if Λ is a directed set and

(E1) $A_\lambda \in \mathcal{L}(X)$ for all $\lambda \in \Lambda$,
(E2) $A_\lambda x \in \overline{\operatorname{co}\{Sx : \ S \in T\}}$, for all $x \in X$, $\lambda \in \Lambda$,
(E3) $\{A_\lambda : \lambda \in \Lambda\}$ is equicontinuous,
(E4l) for every $x \in X$ and $S \in T$

$$\lim_\lambda (SA_\lambda x - A_\lambda x) = 0,$$

(E4r) for every $x \in X$ and $S \in T$

$$\lim_\lambda (A_\lambda Sx - A_\lambda x) = 0.$$

Define a subspace of X, [21, p.77], necessarily closed in X, by

$$\operatorname{Fix}(T) := \{x \in X : Sx = x, \ \forall S \in T\} = \cap_{S\in T} \operatorname{Ker}(I - S).$$

We recall Eberlein's Theorem, [9, Theorem 3.1], [21, Chapter 2, Theorem 1.5, p.76];

Theorem 2 *If $T = (T(t))_{t\ge 0} \subseteq \mathcal{L}(X)$ is a 1–parameter semigroup on a lcHs X admitting a T–ergodic net $\{A_\lambda : \lambda \in \Lambda\}$ then, for every $x, y \in X$, the following assertions are equivalent.*

(i) $y \in \operatorname{Fix}(T(\cdot))$ *and* $y \in \overline{\operatorname{co}\{T(t)x : \ 0 \le t < \infty\}}$.
(ii) $y = \lim_\lambda A_\lambda x$.
(iii) $y = \sigma(X, X')$-$\lim_\lambda A_\lambda x$.
(iv) y *is a* $\sigma(X, X')$ *cluster point of* $\{A_\lambda x : \lambda \in \Lambda\}$.

Remark 2 The implications (ii)⇒(iii)⇒(iv) of Theorem 2 (as proved in [9], [21]) are clear. For the proof of (iv)⇒(i) one only needs (E1), (E2) and (E4l). Furthermore, the equicontinuity of $\{A_\lambda : \lambda \in \Lambda\}$ is unnecessary for (iv)⇒(i). Finally, the proof of (i)⇒(ii) does not require (E4l).

In the rest of this Section we suppose that $T = (T(t))_{t\geq 0}$ is a locally equicontinuous C_0–semigroup on a sequentially complete lcHs X. Hence, T is always strongly continuous (cf. Remark 1(iii)). The linear operators $C(0) := I$ and $C(r)$, for $r > 0$, given by

$$x \mapsto C(r)x := \frac{1}{r}\int_0^r T(t)x\,dt, \quad x \in X, \tag{7}$$

are called the *Cesàro means* of $(T(t))_{t\geq 0}$. For $x \in X$ and $r > 0$, the integral in (7) is the Riemann integral of the continuous X–valued function $t \mapsto T(t)x$, for $t \in [0, r]$, relative to the lc–topology of X; see Section 5.

The Cesàro means $\{C(r)\}_{r\geq 0}$ belong to $\mathcal{L}(X)$. Indeed, by local equicontinuity, for fixed $r > 0$ and $p \in \Gamma_X$ there exist $q \in \Gamma_X$ and $M > 0$ (depending on p, r) with

$$p(T(t)x) \leq Mq(x), \quad x \in X,\, 0 \leq t \leq r, \tag{8}$$

and hence, by Proposition 11(vii), we have that

$$p(C(r)x) \leq \frac{1}{r}\int_0^r p(T(t)x)dt \leq Mq(x), \quad x \in X. \tag{9}$$

This implies $C(r) \in \mathcal{L}(X)$ for $r > 0$. The definition of the Riemann integral $\int_0^r T(t)x\,dt$ as a limit of Riemann sums, together with continuity of $T(t)$, $C(r) \in \mathcal{L}(X)$, implies $T(t)C(r) = C(r)T(t)$ for $r,\ t \geq 0$. In case X is barrelled, the Cesàro means are well defined in $\mathcal{L}(X)$ whenever the C_0–semigroup $(T(t))_{t\geq 0}$ is strongly continuous (via Remark 1(i)). Moreover, if $(T(t))_{t\geq 0}$ is exponentially equicontinuous, then (3) and (9) imply (in the notation of (3)) that $p(e^{-ar}C(r)x) \leq M_p q(x)$, for $x \in X$, $r > 0$. Whenever $(T(t))_{t\geq 0}$ is equicontinuous, the seminorm q in (8) and (9) can be chosen independent of $r \geq 0$ and so $\{C(r)\}_{r\geq 0}$ is also equicontinuous.

According to the following result the Cesàro means $\{C(r)\}_{r\geq 0}$, which commute with $(T(t))_{t\geq 0}$, satisfy some of the properties of a T–ergodic net.

Lemma 1 *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}$ be a locally equicontinuous C_0–semigroup on X. Then the following assertions are valid.*

(i) $C(r)x \in \overline{\mathrm{co}\{T(t)x :\ 0 \leq t \leq r\}}$*, for all* $x \in X$, $r \geq 0$.
(ii) $x - C(r)x \in \overline{\mathrm{span}\,\{u - T(t)u : u \in X, t \geq 0\}}$*, for all* $x \in X$, $r \geq 0$.
(iii) $\lim_{r\to 0^+} C(r) = I = C(0)$ *in* $\mathcal{L}_s(X)$.
(iv) *The X–valued function $r \mapsto rC(r)x$ is continuous in $[0, \infty)$ for every $x \in X$.*
(v) *The following identities hold:*

$$(I-T(t))C(r) = C(r)(I-T(t)) = \frac{1}{r}(I-T(r))\int_0^t T(s)ds, \quad r,\, t > 0. \tag{10}$$

(vi) *The function $r \mapsto C(r)$ is continuous from $[0, \infty)$ into $\mathcal{L}_s(X)$.*

Proof (i) Follows from Proposition 11(iv).

(ii) This follows from $x - C(r)x = \frac{1}{r}\int_0^r (x - T(s)x)\,ds$, for $x \in X$ and $r > 0$, and Proposition 11(iv).

(iii) Fix $x \in X$. Let $p \in \Gamma_X$ and $\varepsilon > 0$. As $(T(t))_{t\geq 0}$ is a C_0–semigroup, there is $R > 0$ such that $p(x - T(s)x) < \varepsilon$, for $s \in [0, R]$. Since $x - C(r)x = \frac{1}{r}\int_0^r (x - T(s)x)\,ds$ for $r > 0$, it then follows from Proposition 11(vii) that

$$p(x - C(r)x) \leq \frac{1}{r}\int_0^r p(x - T(s)x)\,ds < \varepsilon, \quad r \in (0, R].$$

Since p and ε are arbitrary we deduce that $\lim_{r\to 0^+} C(r)x = x$.

(iv) Fix $x \in X$ and $K > 0$. Suppose that $s, r \in (0, K)$, say $0 < s < r$. Proposition 11(ii) implies that $rC(r)x - sC(s)x = \int_s^r T(t)x\,dt$. Given $p \in \Gamma_X$ it then follows that $p(rC(r)x - sC(s)x) = p\left(\int_s^r T(s)T(t-s)x\,dt\right)$. Apply Proposition 11(vi) and then Proposition 11(iii) yields $p(rC(r)x - sC(s)x) = p\left(T(s)\int_s^r T(t-s)x\,dt\right) = p\left(T(s)\int_0^{r-s} T(u)x\,du\right)$. By local equicontinuity of $(T(t))_{t\geq 0}$ there exist $q \in \Gamma_X$ and $M > 0$ such that (2) holds. So, by the previous identity, $p(rC(r)x - sC(s)x) \leq Mq\left(\int_0^{r-s} T(u)x\,du\right)$. Then Proposition 11(vii) gives

$$p(rC(r)x - sC(s)x) \leq (r-s)M \sup\{q(T(u)x) : u \in [0, K]\} \tag{11}$$

with $\sup\{q(T(u)x) : u \in [0, K]\} < \infty$ by compactness of $[0, K]$ and continuity of $u \mapsto T(u)x$. It is clear from (11) that $r \mapsto rC(r)x$ is continuous in $(0, \infty)$. Continuity of $r \mapsto rC(r)x$ at $r = 0$ follows from part (iii).

(v) From the definition, for $x \in X$ fixed, we have

$$(I - T(t))C(r)x = \frac{1}{r}\left(\int_0^r T(u)x\,du - T(t)\int_0^r T(v)x\,dv\right).$$

So, to verify (10) it suffices to show that $\int_0^r T(u)x\,du - T(t)\int_0^r T(v)x\,dv = (I - T(r))\int_0^t T(s)x\,ds$ or, by rearranging, that $\int_0^t T(s)x\,ds - \int_0^r T(u)x\,du = T(r)\int_0^t T(s)x\,ds\ T(t)\int_0^r T(v)x\,dv$. In view of Proposition 11(ii), with $r \leq t$ say, this is equivalent to showing the identity $\int_r^t T(s)x\,ds = T(r)\int_0^t T(s)x\,ds - T(t)\int_0^r T(v)x\,dv$. By applying Proposition 11(vi) to the right side of this identity and using the semigroup property, we only need to verify

$$\int_r^t T(s)x\,ds = \int_0^t T(r+s)x\,ds - \int_0^r T(t+v)x\,dv. \tag{12}$$

But, by a change of variables, the right side of (12) is precisely $\int_r^{r+t} T(w)x\,dw - \int_t^{r+t} T(w)x\,dw$ (cf. Proposition 11(iii)) which, in view of Proposition 11(ii), is precisely the left side of (12), as required.

(vi) This is a consequence of parts (iii) and (iv).

Remark 3 As noted prior to Lemma 1, the maps $\{C(r)\}_{r\geq 0}$ satisfy (E1). Moreover, Lemma 1 ensures that (E2) is also satisfied by $\{C(r)\}_{r\geq 0}$ and implies the equivalence of

(i) $\{C(r)\}_{r\geq 0}$ satifies (E4l),
(ii) $\{C(r)\}_{r\geq 0}$ satisfies (E4r),
(iii) $\tau_s\text{-}\lim_{r\to\infty} \frac{1}{r}T(r)\int_0^t T(s)\,ds = 0$ for all $t > 0$.

A locally equicontinuous C_0–semigroup $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is *mean ergodic* if the net $\{C(r)\}_{r\geq 0}$ converges to some operator in $\mathcal{L}_s(X)$ as $r \to \infty$. If $\{C(r)\}_{r\geq 0}$ converges in $\mathcal{L}_b(X)$ as $r \to \infty$, then $(T(t))_{t\geq 0}$ is said to be *uniformly mean ergodic*.

Also, recall that a subset $A \subseteq X$ is *relatively sequentially compact* (resp. *relatively sequentially* $\sigma(X, X')$*–compact*) if every sequence in A has a convergent subsequence in X (resp. in X_σ).

By invoking Theorem 2, we now collect various results concerning mean ergodicity of semigroups.

Remark 4 (i) Assume that the locally equicontinuous C_0–semigroup $T = (T(t))_{t\geq 0}$ satisfies the requirement

$$\lim_{t\to\infty} \frac{1}{t}T(t) = 0 \text{ in } \mathcal{L}_s(X). \tag{13}$$

Then condition (iii) of Remark 3 is satisfied. Accordingly, if for $x \in X$ there exists $\lim_{r\to\infty} C(r) =: Px$, then Theorem 2 and Remarks 2 and 3 imply that $Px \in \mathrm{Fix}(T(\cdot))$ and $Px \in \overline{\mathrm{co}\{T(t)x : \ 0 \leq t < \infty\}}$.

(ii) It follows from part (i) that if T is mean ergodic and satisfies (13) (resp., condition (iii) of Remark 3), then $\mathrm{Im}\, P \subseteq \mathrm{Fix}(T(\cdot))$, where

$$P := \tau_s\text{-}\lim_{r\to\infty} C(r). \tag{14}$$

On the other hand, $C(r)x = x$ for every $x \in \mathrm{Fix}(T(\cdot))$ and hence, $x = Px$. Thus

$$\mathrm{Im}\, P = \mathrm{Fix}(T(\cdot)). \tag{15}$$

Moreover, by [21, Lemma 1.8, p.78] we have

$$\mathrm{Ker}\, P = \overline{\mathrm{span}\,\{x - T(t)x : \ t \geq 0, \ x \in X\}}. \tag{16}$$

Note that the completeness requirement of X in [21, Lemma 1.8, p.78] is not actually needed in the proof and that the equicontinuity requirement of $\{C(r)\}_{r\geq 0}$ is needed only to show the closedness of $\mathrm{Ker}\, P$. This is non–trivial in the more general situation considered in [21] because P there is neither necessarily continuous nor defined on the whole space X. Of course, if T is assumed to be mean ergodic, then the closedness of $\mathrm{Ker}\, P$ follows directly from this hypothesis.

Next, let $y_1, y_2 \in \overline{\mathrm{co}\{T(t)x : \ 0 \leq t < \infty\}} \cap \mathrm{Fix}(T(\cdot))$ for some fixed $x \in X$. It follows that

$$y_1 - y_2 = (y_1 - x) - (y_2 - x) \in \overline{\mathrm{span}\,\{z - T(t)z : \ t \geq 0, \ z \in X\}}.$$

Thus, (16) implies that $P(y_1-y_2)=0$. On the other hand, $y_1,\, y_2\in \mathrm{Fix}(T(\cdot))$ implies that $C(r)(y_1-y_2)=y_1-y_2$ for every $r\geq 0$. Hence, $0=P(y_1-y_2)=y_1-y_2$. This, together with (15), shows that for every $x\in X$ the set $\overline{\mathrm{co}\{T(t)x:\ 0\leq t<\infty\}}\cap \mathrm{Fix}(T(\cdot))$ contains exactly one element, i.e., Px. Therefore, by [9, Lemma 4.1] we obtain that P is a *projection* as well as $PS=SP=P$ for every $S\in\mathcal{L}(X)$ with the property that $Sy\in\overline{\mathrm{co}\{T(t)y:\ 0\leq t<\infty\}}$ for all $y\in X$.

(iii) If T is a mean ergodic, locally equicontinuous C_0–semigroup, then $\{C(r)x\}_{r\geq 0}$ is relatively sequentially compact for each $x\in X$ via the continuity of the map $r\mapsto C(r)x$ in $[0,\infty)$ (cf. Lemma 1(vi)) and the mean ergodicity of T.

(iv) If T is a mean ergodic, locally equicontinuous C_0–semigroup, then necessarily condition (iii) in Remark 3 is satisfied. Indeed, by changing variables and applying the semigroup law we have

$$\begin{aligned} C(r+t) &= \frac{1}{r+t}\int_0^r T(s)ds+\frac{1}{r+t}\int_r^{r+t}T(s)ds\\ &= \frac{r}{r+t}C(r)+\frac{1}{r+t}\int_0^t T(r+s)ds\\ &= \frac{r}{r+t}\left[C(r)+\frac{T(r)}{r}tC(t)\right],\quad r,\,t>0. \end{aligned}\tag{17}$$

By the mean ergodicity of T, it follows from (17) that, for any $t>0$ fixed, $\frac{T(r)}{r}C(t)=\frac{1}{t}\frac{T(r)}{r}tC(t)\to 0$ in $\mathcal{L}_s(X)$ as $r\to\infty$.

Remark 5 (i) As noted prior to Lemma 1, equicontinuity of $(T(t))_{t\geq 0}$ implies equicontinuity of $\{C(r)\}_{r\geq 0}$. Also, (5) implies that (13) is satisfied.

(ii) If X is barrelled and $(T(t))_{t\geq 0}$ is a strongly continuous C_0–semigroup in X, then mean ergodicity of $(T(t))_{t\geq 0}$ implies that $\{C(r)\}_{r\geq 0}$ is equicontinuous. That is, $\{C(r)\}_{r\geq 0}$ satisfies (E3).

(iii) Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}$ be a locally equicontinuous C_0–semigroup on X. Then

$$\mathrm{Ker}\,A=\mathrm{Fix}(T(\cdot))\quad\text{and}\quad \overline{\mathrm{Im}\,A}=\overline{\mathrm{span}\,\{x-T(t)x:\ t\geq 0,\, x\in X\}},\tag{18}$$

where $(A,D(A))$ is the infinitesimal generator of $(T(t))_{t\geq 0}$. The first identity in (18) follows from the definition of the infinitesimal generator and (6).

Let $x\in D(A)$ and set $y:=Ax$, i.e., $y=\lim_{t\to 0^+}(T(t)x-x)/t$ in X. Then, $y\in\overline{\mathrm{span}\,\{x-T(t)x:\ t\geq 0,\, x\in X\}}$. Conversely, let $z=x-T(t)x$ for some $x\in X$, $t>0$. As $D(A)$ is dense in X, there is a net $(x_\alpha)_{\alpha\in J}\subseteq D(A)$ with $x_\alpha\to x$ in X. By continuity, the net $(I-T(t))x_\alpha=x_\alpha-T(t)x_\alpha\to z$ in X. Now $x_\alpha\in D(A)$ and so $AT(s)x_\alpha\in\mathrm{Im}\,A$ for $s\in[0,t]$. By properties of the Riemann integral, $\int_0^t AT(s)x_\alpha ds\in\overline{\mathrm{Im}\,A}$. Then (6) implies $x_\alpha-T(t)x_\alpha=-\int_0^t AT(s)x_\alpha ds\in\overline{\mathrm{Im}\,A}$, for $\alpha\in J$. Passing to the limit we obtain $z\in\overline{\mathrm{Im}\,A}$.

(iv) The identity $\mathrm{Ker}\,A=\mathrm{Fix}(T(\cdot))$ in (18) is valid even if X is *not* sequentially complete and $(T(t))_{t\geq 0}$ is only a strongly continuous C_0–semigroup (*not* necessarily locally equicontinuous). Indeed, for $x \in X$ and $t > 0$, the formulae (6) remain valid under these weaker hypotheses; see Proposition 1.2 and its proof in [18].

(v) Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup on X. If (13) is satisfied and $\{C(r)\}_{r\geq 0}$ is equicontinuous, then $\{C(r)\}_{r\geq 0}$ is a T–ergodic net. Arguing as in the proof of [21, p.78, Lemma 1.8], where the completeness requirement on X is not actually needed, yields the analogous fact from there that

$$\overline{\operatorname{span}\{x - T(t)x : t \geq 0, x \in X\}} = \{x \in X : \lim_{r\to\infty} C(r)x = 0\}.$$

(vi) Recall that $y \in Y$ is a cluster point of the net $\{u_\alpha\}$ in a topological space Y if, for each neighbourhood $\mathcal{U}$ of y and each β, there is $\alpha \geq \beta$ such that $u_\alpha \in \mathcal{U}$. A subset $A \subseteq Y$ is called *relatively countably compact* if every sequence in A has a cluster point in Y. So, if the set $\{C(r)x\}_{r\geq 0}$ is relatively countably $\sigma(X, X')$–compact for some $x \in X$, then $\{C(r)x\}_{r\geq 0}$ has a cluster point in X_σ. Indeed, if $r_n \uparrow \infty$, then the sequence $\{C(r_n)x\}_{n=1}^\infty$ has a cluster point y in X_σ. Let $\mathcal{U}$ be a neighbourhoood of y in X_σ and $R \in [0, \infty)$. Select $N \in \mathbb{N}$ satisfying $r_N > R$. As y is a cluster point of $\{C(r_n)x\}_{n=1}^\infty$ in X_σ there is $n > N$ such that $C(r_n)x \in \mathcal{U}$. Then $r_n > R$ and we can conclude that y is also a cluster point of $\{C(r)x\}_{r\geq 0}$ in X_σ.

For Banach spaces X the formulae (18) are known, [11, p.278], [21, p.83].
Collecting various facts from above yields the following result.

Proposition 3 *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup on X. Suppose that the following three conditions are satisfied.*

(i) *Condition* (13) *is satisfied, i.e.,* $\frac{T(t)}{t} \to 0$ *in* $\mathcal{L}_s(X)$ *as* $t \to \infty$.
(ii) $\{C(r)\}_{r\geq 0}$ *is equicontinuous.*
(iii) *The net*

$$\{C(r)x\}_{r\geq 0} \textit{ is relatively countably } \sigma(X, X')\textit{–compact}, \quad \forall x \in X. \tag{19}$$

Then $(T(t))_{t\geq 0}$ is mean ergodic and the limit $P = \tau_s$-$\lim_{r\to\infty} C(r)$ is a projection.

Proof Conditions (i), (ii) imply that $\{C(r)x\}_{r\geq 0}$ is a T–ergodic net; cf. Remarks 3 and Remark 5(v). So, the result follows from Theorem 2 and Remark 4(ii) (via Remark 5(vi)).

Remark 6 Regarding Proposition 3, let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ a locally equicontinuous C_0–semigroup.

(i) In a lcHs X all relatively $\sigma(X, X')$–compact sets and all relatively sequentially $\sigma(X, X')$–compact sets are necessarily relatively countably $\sigma(X, X')$–compact. These are the only implications between these three notions which hold in general. All three notions coincide whenever X_σ is *angelic*, [12, p.31]. A large list of spaces X for which X_σ is angelic occurs in [12, §3.10]. In particular, X_σ is angelic for all Fréchet (hence, all Banach) spaces X.

So, if $(T(t))_{t\geq 0}$ is mean ergodic, then Remark 4(iii) implies that (19) holds.

(ii) If X is barrelled, then (iii) implies (ii). Indeed, relatively countably $\sigma(X, X')$–compact sets are bounded, [19, §24, p.310]. So, (iii) ensures $\{C(r)\}_{r\geq 0}$ is bounded in $\mathcal{L}_s(X)$ and hence, equicontinuous (X is barrelled), [20, p.137]. That is, (ii) holds.

(iii) Let X be barrelled. Then conditions (i) and (iii) of Proposition 3 alone imply that $(T(t))_{t\geq 0}$ is mean ergodic. Indeed, (ii) is then automatic (by (ii)).

(iv) Suppose (i), (ii) of Proposition 3 hold. Then $\{C(r)\}_{r\geq 0}$ is a T–ergodic net, cf. Remarks 3 and 5(v). As observed in Remark 5(v) the completeness of X is not needed in Lemma 1.8 of [21, p.78]. Then an examination of the proof of part (b) of the Koliha, Nagel, Sato Theorem, [21, Theorem 1.9, p.79], reveals that also there X is not required to be complete and we can conclude

Fact. $(T(t))_{t\geq 0}$ *is mean ergodic if and only if* $\mathrm{Fix}\,(T(\cdot))$ *separates* $\mathrm{Fix}\,(T(\cdot)')$.

A locally equicontinuous C_0–semigroup $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is called *weakly mean ergodic* if $\lim_{r\to\infty} C(r) = P$ exists in $\mathcal{L}_s(X_\sigma)$

Corollary 1 *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be an equicontinuous C_0–semigroup. The following assertions are equivalent.*

(i) $(T(t))_{t\geq 0}$ *is mean ergodic.*
(ii) $(T(t))_{t\geq 0}$ *is weakly mean ergodic.*
(iii) *Condition* (19) *is satisfied.*

Proof (i)⇒(ii) is clear.

(i)⇒(iii). This was pointed out in Remerk 6(i).

(iii)⇒(i). This follows from Proposition 3 and Remarks 4(ii), 5(i).

(ii)⇒(iii). Lemma 1(vi) implies that $r \mapsto C(r)$ is continuous from $[0,\infty)$ into $\mathcal{L}_s(X_\sigma)$. Fix $x \in X$ and let $\{C(r_n)x\}_{n=1}^\infty$ be any sequence in $\{C(r)x\}_{r\geq 0}$. If there exists $M > 0$ with $r_n \leq M$ for all $n \in \mathbb{N}$, then the continuity of $C(\cdot)x\colon [0, M] \to X_\sigma$ implies that $\{C(r)x\}_{r\in[0,M]}$ is $\sigma(X, X')$–compact and so $\{C(r_n)x\}_{n=1}^\infty$ has a cluster point in X_σ. Otherwise there exists a sequence $r_{n(k)} \uparrow \infty$ in which case $\lim_{k\to\infty} C(r_{n(k)})x = y$ exists in X_σ. Hence, y is a cluster point of $\{C(r_n)x\}_{n=1}^\infty$ in X_σ; argue along the lines of Remark 5(vi). So, (19) holds for $\{C(r)x\}_{r\geq 0}$.

Recall that a lcHs X is *semi–reflexive* (resp. *semi–Montel*) if and only if every bounded subset of X is relatively $\sigma(X, X')$–compact (resp. relatively compact), [24, Proposition 23.18].

Corollary 2 *Let X be a lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ an equicontinuous C_0–semigroup.*

(i) *If X is semi–reflexive, then $(T(t))_{t\geq 0}$ is mean ergodic.*
(ii) *If X is semi–Montel, then $(T(t))_{t\geq 0}$ is uniformly mean ergodic.*

Proof (i) The space X is necessarily quasicomplete, [15, p.229], and by Remark 5(i) the net $\{C(r)\}_{r\geq 0}$ is equicontinuous. So, for each $x \in X$, the set $\{C(r)x\}_{r\geq 0} \in \mathcal{B}(X)$, i.e., $\{C(r)x\}_{r\geq 0}$ is relatively $\sigma(X,X')$–compact. Then Remark 6(i) shows that (19) holds and so the mean ergodicity of $(T(t))_{t\geq 0}$ follows from Corollary 1.

(ii) As X is quasicomplete and semi–reflexive, [15, Propositions 11.5.1 & 11.5.2], $(T(t))_{t\geq 0}$ is mean ergodic by part (i), i.e., $P := \lim_{r\to\infty} C(r)$ exists in $\mathcal{L}_s(X)$. Since $\{C(r)\}_{r\geq 0}$ is equicontinuous (see Remark 5(i)), also $H := \{P\} \cup \{C(r)\}_{r\geq 0}$ is equicontinuous. So, τ_s and the topology τ_c in $\mathcal{L}(X)$ of uniform convergence on precompact subsets of X coincide on H, [20, (2) p.139]. But, bounded sets in X are relatively compact (so, $\tau_c = \tau_b$). Hence, $C(r) \to P$ in $\mathcal{L}_b(X)$ as $r \to \infty$, i.e., $(T(t))_{t\geq 0}$ is uniformly mean ergodic.

An individual operator $T \in \mathcal{L}(X)$ is *mean ergodic* (resp. *uniformly mean ergodic*) if its Cesàro means $\{T_{[n]}\}_{n=1}^{\infty}$ (cf. (1)) converge in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$).

Theorem 4 *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup such that $\{C(r)\}_{r\geq 0}$ is equicontinuous.*

(i) *If $T(t_0)$ is mean ergodic for some $t_0 > 0$, then $(T(t))_{t\geq 0}$ is mean ergodic.*
(ii) *If $T(t_0)$ is uniformly mean ergodic for some $t_0 > 0$, then $(T(t))_{t\geq 0}$ is uniformly mean ergodic.*

Proof (i) For any fixed $r > 0$, set $n = \left[\frac{r}{t_0}\right]$ so that $r = t_0(n+\alpha)$ with $0 \leq \alpha < 1$. Arguing as in Remark 4(iv) we obtain

$$\begin{aligned} C(r) &= t_0\frac{n}{r}\left[\frac{1}{n}\sum_{m=0}^{n-1}(T(t_0))^m C(t_0) + \frac{T(t_0 n)}{t_0 n} t_0\alpha C(t_0\alpha)\right] \\ &= t_0\frac{n}{r}\left[T(t_0)_{[n]}C(t_0) + \frac{(T(t_0))^n}{t_0 n} t_0\alpha C(t_0\alpha)\right]. \end{aligned} \tag{20}$$

Fix $x \in X$. By continuity of $\alpha \mapsto t_0\alpha C(t_0\alpha)x$ the set $\{t_0\alpha C(t_0\alpha)x : 0 \leq \alpha \leq 1\}$ is compact in X. As $T(t_0)$ is mean ergodic, $\lim_{n\to\infty} T(t_0)_{[n]}$ exists in $\mathcal{L}_s(X)$. So, from

$$\frac{(T(t_0))^n}{n} = T(t_0)_{[n]} - \frac{(n-1)}{n}T(t_0)_{[n-1]}, \quad n \in \mathbb{N} \tag{21}$$

(routine to verify), it follows τ_s-$\lim_{n\to\infty} \frac{(T(t_0))^n}{n} = 0$. Accordingly,

$$\frac{(T(t_0))^n}{t_0 n}(t_0\alpha C(t_0\alpha)x) \to 0, \text{ as } n \to \infty, \tag{22}$$

uniformly for $\alpha \in [0,1]$. As $\frac{t_0 n}{r} = 1 - \frac{t_0\alpha}{r} \to 1$ as $n \to \infty$ (i.e., as $r \to \infty$), by (20) and (22) we can conclude that $C(r)x$ converges to some Px in X as $r \to \infty$. Since x is arbitrary and $\{C(r)\}_{r\geq 0}$ is equicontinuous, we obtain that $P \in \mathcal{L}(X)$.

(ii) Proceed as in (i) to deduce via (21) that τ_b-$\lim_{n\to\infty} \frac{(T(t_0))^n}{n} = 0$. As

$$t_0 \frac{n}{r} \frac{(T(t_0))^n}{t_0 n} t_0 \alpha C(t_0 \alpha) = \frac{(T(t_0))^n}{n} \left(\frac{t_0 \alpha n}{r} C(t_0 \alpha) \right)$$

with $\frac{t_0 \alpha n}{r} \in [0,1]$ and $t_0 \alpha \in [0, t_0]$ for all $\alpha \in [0,1]$ and $r \geq 0$ we see that

$$\left(t_0 \frac{n}{r} \frac{(T(t_0))^n}{t_0 n} t_0 \alpha C(t_0 \alpha) \right)(B) \subseteq \frac{(T(t_0))^n}{n} \left(\cup_{\substack{\beta \in [0,1] \\ s \in [0,t_0]}} \beta C(s)(B) \right) =: \frac{(T(t_0))^n}{n}(B^*)$$

for each $B \in \mathcal{B}(X)$. But, equicontinuity of $\{C(r)\}_{r \geq 0}$ implies that also $B^* \in \mathcal{B}(X)$ and hence, $t_0 \frac{n}{r} \frac{(T(t_0))^n}{t_0 n} t_0 \alpha C(t_0 \alpha) \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ uniformly for $\alpha \in [0,1]$. Furthermore, since $\frac{t_0 n}{r} \in [0,1]$ for all $r \geq 0$ we see, for each $n \in \mathbb{N}$, that

$$\left(t_0 \frac{n}{r} T(t_0)_{[n]} C(t_0) \right)(B) \subseteq T(t_0)_{[n]} C(t_0) \left(\cup_{\gamma \in [0,1]} \gamma B \right) =: T(t_0)_{[n]}(D)$$

with $D \in \mathcal{B}(X)$. So, also $t_0 \frac{n}{r} T(t_0)_{[n]} C(t_0)$ converges in $\mathcal{L}_b(X)$ as $n \to \infty$. Then (20) implies $\lim_{r\to\infty} C(r)$ exists in $\mathcal{L}_b(X)$, i.e., $(T(t))_{t \geq 0}$ is uniformly mean ergodic.

Remark 7 Theorem 4 ensures an equicontinuous C_0–semigroup $(T(t))_{t\geq 0}$ in a sequentially complete lcHs X is mean ergodic if there is $t_0 > 0$ such that $T(t_0)$ is mean ergodic. For results of this kind in Banach spaces we refer to [30], [32], and the references therein. The converse is false, both in Banach spaces, [11, p.345], [21, p.83], [32, Example (d), p.75], and also in (non–trivial) non–normable Fréchet spaces. Indeed, let $Y := \{f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f(s)ds = 0\}$. Then Y is closed in $L^1(\mathbb{R})$ and hence, $(Y, \|\cdot\|_1)$ is a Banach space. The left translation semigroup $(T(t))_{t\geq 0}$ on Y is an equicontinuous C_0–semigroup which is mean ergodic, but $\{T(t_0)_{[n]}\}_{n=1}^\infty$ fails to converge for every $t_0 > 0$, [11, Ch.V, 4.13.3]. The space $X := Y^{\mathbb{N}}$ endowed with the product topology is a non–normable Fréchet space and $S(t)f := (T(t)f_n)_n$, for $f = (f_n)_n \in X$, $t \geq 0$, defines an equicontinuous C_0–semigroup in X which is mean ergodic but, for no $t_0 > 0$ is $S(t_0)$ mean ergodic.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a lcHs X is called a *basis* if, for every $x \in X$, there is a unique sequence $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ such that the series $\sum_{n=1}^\infty \alpha_n x_n$ converges to $x \in X$. By setting $f_n(x) := \alpha_n$ we obtain a linear form $f_n \colon X \to \mathbb{C}$. If $(f_n)_{n\in\mathbb{N}} \subseteq X'$, then $(x_n)_{n\in\mathbb{N}}$ is called a *Schauder basis* for X.

The next result extends its Banach space counterpart, [26, Theorem 3.4].

Theorem 5 *Let X be a complete, barrelled lcHs with a Schauder basis. Then the following assertions are equivalent.*

(i) *X is reflexive.*
(ii) *Every equicontinuous C_0–semigroup on X is mean ergodic.*
(iii) *Every equicontinuous, uniformly continuous C_0–semigroup on X is mean ergodic.*

Proof (i)⇒(ii). As a reflexive space is semi–reflexive, Corollary 2(i) implies (ii).

(ii)⇒(iii). Obvious.

(iii)⇒(i). Suppose (i) fails, i.e., X is *not* reflexive. By [1, Theorem 1.2] the space X admits a non–shrinking basis. So, there is a *power bounded* operator $A \in \mathcal{L}(X)$ (i.e., $\{A^n\}_{n=1}^\infty$ is equicontinuous) which is *not* mean ergodic, [3, Theorem 3.4].

As A is power bounded, given $p \in \Gamma_X$ there exist $M_p > 0$ and $q \in \Gamma_X$ such that $p(A^n x) \le M_p q(x)$ for all $x \in X$, $n \in \mathbb{N}$. So, the seminorms $\{\overline{p} : p \in \Gamma_X\}$ given by $\overline{p}(x) := \sup_{n\ge 0} p(A^n x)$ also generate the lc–topology of X because $p(x) \le \overline{p}(x) \le M_p q(x) \le M_p \overline{q}(x)$, for $x \in X$, $p \in \Gamma_X$. Moreover,

$$\overline{p}(A^n x) = \sup_{m\ge 0} p(A^m A^n x) = \sup_{h \ge n} p(A^h x) \le \overline{p}(x), \quad x \in X, p \in \Gamma_X. \tag{23}$$

So, given $t \ge 0$, the operator $T(t) := \sum_{k=0}^\infty \frac{t^k A^k}{k!}$ exists in $\mathcal{L}(X)$ and satisfies

$$\overline{p}(T(t)x) \le e^t \overline{p}(x), \quad x \in X, p \in \Gamma_X, \tag{24}$$

[33, p.245]. Moreover, $(T(t))_{t\ge 0}$ is a uniformly continuous C_0–semigroup with (A, X) as infinitesimal generator. Indeed, uniform continuity follows via (23), which implies for $p \in \Gamma_X$ and $B \in \mathcal{B}(X)$ that $\sup_{x\in B} \overline{p}((T(t) - I)x) \le K(e^t - 1)$, for $t \ge 0$, where $K := \sup_{x\in B} \overline{p}(x)$. Then (24) shows that $S(t) := e^{-t}T(t)$, for $t \ge 0$, is an equicontinuous, uniformly continuous C_0–semigroup on X with $(A - I, X)$ as infinitesimal generator. Now, Remark 5(iii) implies (see (18)) that $\mathrm{Ker}(A - I) = \mathrm{Fix}\,(S(\cdot))$. Observe the lcHs X'_{σ^*} is quasi-complete, [19, §27]. Moreover, $r \mapsto S(r)'$, for $r \ge 0$, is a C_0–semigroup in $\mathcal{L}(X'_{\sigma^*})$ with infinitesimal generator (A', X'), [18, §2 & Proposition 2.1]. Actually, $(S(r)')_{r\ge 0}$ is an equicontinuous, uniformly continuous semigroup in $\mathcal{L}(X'_\beta)$ because $S(r)' \in \mathcal{L}(X'_\beta)$, with $S(r)' = e^{-r} \sum_{k=1}^\infty \frac{r^k (A')^k}{k!}$ for $r \ge 0$, and $A' \in \mathcal{L}(X'_\beta)$ is power bounded. As X'_β is quasicomplete, [15, Proposition 11.2.4, p.222], Remark 5(iii) yields $\mathrm{Ker}(A' - I) = \mathrm{Fix}\,(S(\cdot)')$. By hypothesis, the equicontinuous, uniformly continuous semigroup $(S(t))_{t\ge 0} \subseteq \mathcal{L}(X)$ is mean ergodic. Equicontinuity of $(S(t))_{t\ge 0}$ implies the hypotheses of Remark 6(iv) are satisfied. So, $\mathrm{Fix}\,(S(\cdot))$ separates $\mathrm{Fix}\,(S(\cdot)')$, i.e., $\mathrm{Ker}(A-I)$ separates $\mathrm{Ker}(A' - I)$. An examination of the proof of Sine's Theorem for an individual operator in a Banach space, as given in [21, p.74], shows it is based purely on a duality argument and so carries over to lcHs', [27, Theorem 13], thereby allowing us to conclude that $X = \mathrm{Ker}(A - I) \oplus \overline{\mathrm{Im}(A - I)}$. As A is power bounded, we have $\overline{\mathrm{Im}(A - I)} = \{x \in X : \lim_{n\to\infty} A_{[n]}x = 0\}$, [33, p.213]. Given $x \in X$, write $x = u + v$ with $u \in \mathrm{Ker}(A - I)$ and $v \in \overline{\mathrm{Im}(A - I)}$. Then $A_{[n]}x = u + A_{[n]}v$, for $n \in \mathbb{N}$, and so $\lim_{n\to\infty} A_{[n]}x = u$ exists in X. As X is barrelled, the limit operator $x \mapsto \lim_{n\to\infty} A_{[n]}x$ belongs to $\mathcal{L}(X)$, i.e., A is mean ergodic. This contradicts the choice of A and so X must be reflexive.

4 Semigroups and mean ergodicity in GDP–spaces

In Banach GDP–spaces every strongly continuous C_0–semigroup is uniformly continuous, [22, Theorem 6], [23, Theorem 3]. This also holds for exponentially equicontinuous C_0-semigroups acting in Fréchet GDP–spaces which, additionally, are also quojections, [4, Theorem 3.5]. In this section we extend this fact to quasicomplete, barrelled, GDP–spaces and to C_0–semigroups which are only locally equicontinuous. The methods introduced also allow us to characterize mean ergodicity of C_0-semigroups $(T(t))_{t\geq 0}$ satisfying (13) and acting in complete, barrelled, Grothendieck spaces in terms of the annihilator $(\mathrm{Fix}\,(T(\cdot)))^{\perp}$. If, additionally, $(T(t))_{t\geq 0}$ is uniformly mean ergodic, then $(T(s)')_{s\geq 0}$ is mean ergodic in X'_β.

A lcHs X is a *Grothendieck space* if sequences in X' which are convergent for $\sigma(X',X)$ also converge for $\sigma(X',X'')$. Reflexive lcHs' are Grothendieck spaces. A lcHs X has the *Dunford–Pettis property* (briefly, DP) if every element of $\mathcal{L}(X,Y)$, for Y any quasicomplete lcHs, which transforms elements of $\mathcal{B}(X)$ into relatively $\sigma(Y,Y')$–compact subsets of Y, also transforms $\sigma(X,X')$–compact subsets of X into relatively compact subsets of Y, [10, pp.633-634]. It suffices if Y runs through all Banach spaces, [5, p.79]. A reflexive lcHs has the DP property if and only if it is Montel, [10, p.634]. A Grothendieck lcHs X with the DP property is called a GDP-space. Every Montel lcHs is a GDP-space, [5, Remark 2.2], [2, Corollary 3.8]. For examples of Banach GDP–spaces, see [22], [23].

Proposition 6 *Let X be a quasicomplete, barrelled lcHs and $(T(t))_{t\geq 0}\subseteq\mathcal{L}(X)$ be a semigroup.*

(i) *If $(T(t))_{t\geq 0}$ is a strongly continuous C_0–semigroup and X is a Grothendieck space, then $(T(s)')_{s\geq 0}$ is a locally equicontinuous C_0–semigroup in X'_β.*
(ii) *If $(T(s)'')_{s\geq 0}$ is a strongly continuous C_0–semigroup on X''_β, then $(T(s)')_{s\geq 0}$ is a locally equicontinuous C_0–semigroup on X'_β.*

Proof (i) It is known that $(T(s)')_{s\geq 0}\subseteq\mathcal{L}(X'_\beta)$, [20, p.134], and routine to check that $(T(s)')_{s\geq 0}$ is a semigroup. To verify that $(T(s)')_{s\geq 0}$ is locally equicontinuous fix $K>0$ and $B\in\mathcal{B}(X)$. By Remark 1(i), $(T(t))_{t\geq 0}$ is locally equicontinuous and so $C:=\{T(t)x : t\in[0,K], x\in B\}\in\mathcal{B}(X)$. Accordingly, for each $x'\in X'$,

$$\sup_{s\in[0,K]} p^B(T(s)'x') = \sup_{s\in[0,K]}\sup_{x\in B}|\langle x, T(s)'x'\rangle| \leq \sup_{y\in C}|\langle y, x'\rangle| = p^C(x'),$$

which establishes the locally equicontinuity of $(T(s)')_{s\geq 0}\subseteq\mathcal{L}(X'_\beta)$.

Applying Proposition 1 to $(T(s)')_{s\geq 0}$ in X'_β the proof is complete (X'_β is quasicomplete being the strong dual of a barrelled space, [15, Proposition 11.2.4]), provided we verify $s\mapsto\langle T(s)'x', x''\rangle$ is continuous on $[0,\infty)$, for $x'\in X'$, $x''\in X''$. Let $x'\in X'$. Fix $s^*>0$ and let $(s_n)_{n=1}^\infty\subseteq[0,\infty)$ converge to s^*. We need to check $T(s_n)'x'\to T(s^*)'x'$ in $(X',\sigma(X',X''))$ as $n\to\infty$. Since

X is a Grothendieck space, it suffices to prove $\langle x, T(s_n)'x'\rangle \to \langle x, T(s^*)'x'\rangle$, for $x \in X$. But, this is so as

$$|\langle x, T(s_n)'x'\rangle - \langle x, T(s^*)'x'\rangle| = |\langle (T(s_n) - T(s^*))x, x'\rangle|, \quad n \in \mathbb{N},$$

and the right–side converges to 0 as $n \to \infty$ because $(T(t))_{t\geq 0}$ is a strongly continuous C_0–semigroup. For $s^* = 0$ we take $(s_n)_{n=1}^{\infty} \subseteq [0, \infty)$ with $s_n \to 0^+$.

(ii) As X is barrelled, it is a *topological* subspace of X''_β, [15, Proposition 11.2.2]. Also, $T(s)''x = T(s)x$ for $s \geq 0$, $x \in X$, implies $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is a strongly continuous C_0–semigroup which, by Remark 1(i), is locally equicontinuous. Arguing as in (i) we conclude $(T(s)')_{s\geq 0} \subseteq \mathcal{L}(X'_\beta)$ is a locally equicontinuous semigroup.

The C_0–property again follows from Proposition 1 applied to $(T(s)')_{s\geq 0}$ in X'_β. Indeed, fix $s^* > 0$, $x' \in X'$ and let $s_n \to s^*$ in $[0, \infty)$. Then, for $x'' \in X''$, we have

$$|\langle T(s_n)'x', x''\rangle - \langle T(s^*)'x', x''\rangle| = |\langle x', (T(s_n)'' - T(s^*)'')x''\rangle|, \quad n \in \mathbb{N},$$

with the right–side converging to 0 as $n \to \infty$ because $(T(s)'')_{s\geq 0}$ is a strongly continuous C_0–semigroup in X''_β. So, $T(s_n)'x' \to T(s^*)'x'$ in $(X', \sigma(X', X''))$. For $s^* = 0$ we take $(s_n)_{n=1}^{\infty} \subseteq [0, \infty)$ with $s_n \to 0^+$.

Lemma 2 *Let X be a lcHs with $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ a locally equicontinuous semigroup. If τ_b-$\lim_{t\to 0^+}(T(t) - I)^2 = 0$, then $(T(t))_{t\geq 0}$ is uniformly continuous.*

Proof Fix $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$. By local equicontinuity, $\sup_{t\in[0,1]} q_B(T(t) - I) < \infty$ and so $s_{q,B} := \limsup_{t\to 0^+} q_B(T(t) - I) < \infty$. Then $2(T(t) - I) = T(2t) - I - (T(t) - I)^2$ implies, for $x \in B$ and $t \in [0, 1/2]$, that

$$2q(T(t)x - x) \leq q((T(2t) - I)x) + q((T(t) - I)^2 x) \leq q((T(2t) - I)x) + q_B((T(t) - I)^2).$$

Form the sup over $x \in B$ and let $t \to 0^+$ gives $2s_{q,B} \leq \limsup_{t\to 0^+} q_B(T(2t) - I) \leq s_{q,B}$. So, $s_{q,B} = 0$ which implies $\lim_{t\to 0^+} T(t) = I$ in $\mathcal{L}_b(X)$.

Fix $t_0 > 0$. Then for $h > 0$ and fixed $B \in \mathcal{B}(X)$, $q \in \Gamma_X$, we have

$$q_B(T(t_0 + h) - T(t_0)) = q_B(T(t_0)(T(h) - I)) = q_{T(t_0)B}(T(h) - I).$$

Since $T(t_0)B \in \mathcal{B}(X)$, we have $\lim_{t\to t_0^+} T(t) = T(t_0)$ in $\mathcal{L}_b(X)$. By local equicontinuity, $C := \{T(t)x : t \in [0, t_0], x \in B\} \in \mathcal{B}(X)$. So, for $h \in [-t_0/2, 0]$, we have

$$q_B(T(t_0 + h) - T(t_0)) = q_B(T(t_0 + h)(I - T(-h))) \leq q_C(I - T(-h)),$$

which implies also τ_b-$\lim_{t\to t_0^-} T(t) = T(t_0)$, i.e., $T\colon [0, \infty) \to \mathcal{L}_b(X)$ is continuous.

Theorem 7 *Let X be a barrelled, quasicomplete, GDP lcHs. Then every strongly continuous C_0–semigroup on X is uniformly continuous.*

Proof Let $(T(t))_{t\geq 0}$ be a strongly continuous C_0–semigroup in X. Via Remark 1(i), $(T(t))_{t\geq 0}$ is locally equicontinuous. So, by Lemma 2, it suffices to show

$$\lim_{s\to 0^+} q_B((T(t)-I)^2)=0, \quad B\in\mathcal{B}(X),\, q\in\Gamma_X. \tag{25}$$

If (25) does *not* hold, then there is $\varepsilon>0$, a sequence $s_n\downarrow 0$ in $[0,\infty)$, a bounded sequence $C:=(x_n)_{n=1}^\infty\subseteq X$, a seminorm $q\in\Gamma_X$, and a sequence $(x'_n)_{n=1}^\infty\subseteq X'$ satisfying $|\langle y,x'_n\rangle|\leq q(y)$ for $y\in X$, $n\in\mathbb{N}$ such that, for all $n\in\mathbb{N}$, we have

$$\varepsilon<|\langle (T(s_n)-I)^2x_n,x'_n\rangle|=|\langle (T(s_n)-I)x_n,(T(s_n)'-I)x'_n\rangle|. \tag{26}$$

We first show $(T(s_n)'-I)x'_n\to 0$ in $(X',\sigma(X',X''))$. As X is a Grothendieck space, it suffices to show $\langle x,(T(s_n)'-I)x'_n\rangle\to 0$ as $n\to\infty$, for $x\in X$. But,

$$|\langle x,(T(s_n)'-I)x'_n\rangle|=|\langle (T(s_n)-I)x,x'_n\rangle|\leq q((T(s_n)-I)x), \quad n\in\mathbb{N},$$

which converges to 0 as $n\to\infty$ because $(T(t))_{t\geq 0}$ is a C_0–semigroup.

Proposition 6(i) implies $(T(s)')_{s\geq 0}\subseteq\mathcal{L}(X'_\beta)$ is a locally equicontinuous, strongly continuous C_0–semigroup. As $C\in\mathcal{B}(X)$, we see from $|\langle (T(s_n)-I)x_n,x'\rangle|\leq p^C((T(s_n)'-I)x')$, for $x'\in X'$, $n\in\mathbb{N}$, that $(T(s_n)-I)x_n\to 0$ in X_σ. But, X has the DP–property and so Proposition 3.3(i) of [2] implies the right–side of (26) converges to 0, which is impossible. So, (25) holds.

Remark 8 For X a quojection GDP Fréchet space it was shown in [4, Theorem 3.5] that every exponentially equicontinuous C_0–semigroup in X (locally equicontinuous and strongly continuous by Remark 1(ii), (iii)) is uniformly continuous with infinitesimal generator in $\mathcal{L}(X)$. Theorem 7 shows uniform continuity of the semigroup holds under milder conditions; metrizability of X can be replaced by quasicompleteness and barrelledness and, even within the class of Fréchet spaces, X need not be a quojection. However, there do exist nuclear Fréchet spaces (hence, GDP) and equicontinuous, uniformly continuous semigroups in such spaces whose infinitesimal generator is *not* in $\mathcal{L}(X)$, [4, Example 3.1]; such a space cannot be a quojection. For Banach spaces, Theorem 7 is due to H.P. Lotz, [22], [23].

In the Banach space setting the next result is proved in [23, Theorem 4].

Theorem 8 *Let X be a barrelled, quasicomplete lcHs with the DP–property and $(T(t))_{t\geq 0}\subseteq\mathcal{L}(X)$ be a semigroup such that $(T(s)'')_{s\geq 0}$ is a strongly continuous C_0–semigroup on X''_β. Then $(T(t))_{t\geq 0}$ is uniformly continuous.*

Proof From the proof of Proposition 6(ii) we see that $(T(t))_{t\geq 0}$ is a strongly continuous C_0–semigroup which is locally equicontinuous. We proceed via contradiction as in the proof of Theorem 7; our notation is from there. Set $D:=(x'_n)_{n=1}^\infty$, which is equicontinuous in X'. We need to show that

$$(T(s_n)'-I)x'_n\to 0 \text{ in } (X',\sigma(X',X'')),\ \text{for } n\to\infty, \tag{27}$$

and also that

$$(T(s_n) - I)x_n \to 0 \text{ in } (X, \sigma(X, X')), \text{ for } n \to \infty. \tag{28}$$

Fix $x'' \in X''$. Then, for $n \in \mathbb{N}$, we have $|\langle (T(s_n)' - I)x'_n, x''\rangle| = |\langle x'_n, (T(s_n)'' - I)x''\rangle| \leq \sup_{u \in D} |\langle u, (T(s_n)'' - I)x''\rangle|$. As $D \in \mathcal{B}(X'_\beta)$ and $(T(s)'')_{s\geq 0}$ is a C_0–semigroup in $X''_\beta = (X'_\beta)'_\beta$, it follows $\lim_{n\to\infty} \sup_{u\in D} |\langle u, (T(s_n)'' - I)x''\rangle| = 0$, i.e., (27) holds.

To verify (28), observe that $(T(s)')_{s\geq 0} \subseteq \mathcal{L}(X'_\beta)$ is a locally equicontinuous, strongly continuous C_0–semigroup (cf. Proposition 6(ii)) and X has the DP–property. Hence, the argument in the proof of Theorem 7 can be repeated.

Turning our attention to mean ergodicity, we first require a preliminary result.

Lemma 3 *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup on X which satisfies* (13).

(i) *If $(T(t))_{t\geq 0}$ is mean ergodic and $P \in \mathcal{L}(X)$ is the projection* (14)*, then*

$$\operatorname{Im} P' = \operatorname{Fix}(T(\cdot)'). \tag{29}$$

(ii) *Suppose that X is complete and $\{C(r)\}_{r\geq 0} \subseteq \mathcal{L}(X)$ is equicontinuous. Then the following topological direct sum is a closed subspace of X:*

$$\operatorname{Fix}(T(\cdot)) \oplus \overline{\operatorname{span}\{x - T(t)x : t \geq 0, x \in X\}}. \tag{30}$$

Proof (i) Via Remark 4(ii), $P' \in \mathcal{L}(X'_\beta)$ satisfies $P' = P'T(s)' = T(s)'P'$, for $s \geq 0$. In particular, $T(s)'P'x' = P'x'$ for $s \geq 0$, $x' \in X'$, i.e., $P'x' \in \operatorname{Fix}(T(\cdot)')$ for $x' \in X'$. So, $\operatorname{Im} P' \subseteq \operatorname{Fix}(T(\cdot)')$.

By Remark 1(iii), $(T(t))_{t\geq 0}$ is strongly continuous. Fix $x' \in \operatorname{Fix}(T(\cdot)')$ so that $T(s)'x' = x'$ for all $s \geq 0$. Then, for $r > 0$, $x \in X$, we have $\langle x, C(r)'x'\rangle = \langle \frac{1}{r}\int_0^r T(s)x\,ds, x'\rangle = \frac{1}{r}\int_0^r \langle T(s)x, x'\rangle\,ds = \frac{1}{r}\int_0^r \langle x, T(s)'x'\rangle\,ds = \langle x, x'\rangle$, which implies $C(r)'x' = x'$. For $x \in X$, it then follows from (14) that $\langle Px, x'\rangle = \lim_{r\to\infty}\langle C(r)x, x'\rangle = \lim_{r\to\infty}\langle x, C(r)'x'\rangle = \langle x, x'\rangle$ and hence, $\langle x, P'x'\rangle = \langle x, x'\rangle$, i.e., $P'x' = x'$. So, $x' \in \operatorname{Im} P'$.

(ii) This is due to Koliha, Nagel and Sato, [21, Theorem 1.9(a), p.79].

We recall for a subspace Y of a lcHs X that its *annihilator* $Y^\perp := \{x' \in X' : \langle y, x'\rangle = 0 \ \forall y \in Y\}$ coincides with its *polar* $Y^\circ := \{x' \in X' : |\langle y, x'\rangle| \leq 1 \ \forall y \in Y\}$. Observe that the weak topology of X'_β is $\sigma(X', X'')$.

Theorem 9 *Let X be a a Grothendieck lcHs which is complete and barrelled and $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a strongly continuous C_0–semigroup which satisfies* (13) *and such that $\{C(r)\}_{r\geq 0}$ is equicontinuous. Then the following assertions are equivalent.*

(i) *$(T(t))_{t\geq 0}$ is mean ergodic.*

(ii) *The subspace* $Z := \mathrm{span}\{x' - T'(s)x' : s \geq 0, x' \in X'\}$ *of* X' *satisfies*

$$\overline{Z}^{\sigma(X',X)} = \overline{Z}^{\sigma(X',X'')}, \tag{31}$$

that is, the closure of Z *in* X'_{σ^*} *coincides with its closure in* X'_β.

Proof Observe $(T(s)')_{s\geq 0} \subseteq \mathcal{L}(X'_\beta)$ is a locally equicontinuous C_0–semigroup (cf. Proposition 6(i)). So, by Remark 1(iii) it is strongly continuous. Also, as X is barrelled, X'_β is quasicomplete, [15, Proposition 11.2.4]. So, the Cesàro means $D(r)x' := \frac{1}{r}\int_0^r T(s)'x'\,ds$, for $x' \in X'$, exist in $\mathcal{L}(X'_\beta)$ for $r > 0$ (with $D(0) := I$). By properties of the Riemann integral it follows that $\langle C(r)x, x'\rangle = \langle x, D(r)x'\rangle$ for $x \in X$, $x' \in X'$, i.e., $D(r) = C(r)'$ for $r \geq 0$.

(i)⇒(ii). Since $(T(t))_{t\geq 0}$ is mean ergodic, the operator $P \in \mathcal{L}(X)$ given by $P := \tau_s\text{-}\lim_{r\to\infty} C(r)$ is a projection; see Remark 4(ii). According to (15) we have

$$\mathrm{Im}\, P = \mathrm{Fix}(T(\cdot)) = \cap_{t\geq 0} \mathrm{Ker}(I - T(t)) \tag{32}$$

and by (16) we have

$$\mathrm{Ker}\, P = \overline{\mathrm{span}\{x - T(t)x : t \geq 0, x \in X\}} = \overline{\cup_{t\geq 0}\mathrm{Im}\,(I - T(t))}. \tag{33}$$

Moreover, $P' \in \mathcal{L}(X'_\beta)$ is a projection and so

$$\mathrm{Im}\, P' = (\mathrm{Ker}\, P)^\perp, \quad \mathrm{Ker}\, P' = (\mathrm{Im}\, P)^\perp, \tag{34}$$

[20, p.2, (5)&(6)]. It then follows from (32) and (34) that

$$\begin{aligned} \mathrm{Ker}\, P' &= (\mathrm{Im}\, P)^\perp = [\cap_{t\geq 0} \mathrm{Ker}(I - T(t))]^\perp \\ &= \overline{\mathrm{span}\,(\cup_{s\geq 0}\mathrm{Im}\,(I - T(s)'))}^{\sigma(X',X)} \\ &= \overline{\mathrm{span}\{x' - T(s)'x' : s \geq 0,\, x' \in X'\}}^{\sigma(X',X)} = \overline{Z}^{\sigma(X',X)}, \end{aligned} \tag{35}$$

where we also used $[\cap_{t\geq 0} \mathrm{Ker}(I - T(t))]^\perp = \overline{\mathrm{span}\,(\cup_{s\geq 0}\mathrm{Im}\,(I - T(s)'))}^{\sigma(X',X)}$, [19, p.247, (11)], and $[\mathrm{Ker}(I - T(s))]^\perp = \overline{\mathrm{Im}\,(I - T(s)')}^{\sigma(X',X)}$, [20, p.2, (6)]. It follows that $\overline{Z}^{\sigma(X',X'')} \subseteq \overline{Z}^{\sigma(X',X)} \subseteq \mathrm{Ker}\, P'$.

Conversely, fix $x' \in \mathrm{Ker}\, P'$. Then, for $x \in X$, we have $\langle x, D(r)x'\rangle = \langle C(r)x, x'\rangle \to \langle Px, x'\rangle = \langle x, P'x'\rangle = 0$ as $r \to \infty$, i.e., $\lim_{r\to\infty} D(r)x' = 0$ in X'_{σ^*}. As X is a Grothendieck space, it follows $D(r_n)x' \to 0$ in $(X', \sigma(X', X''))$, whenever $r_n \to \infty$ in $(0, \infty)$. Moreover, the Riemann integral

$$x' - D(r_n)x' = \frac{1}{r_n}\int_0^{r_n} (x' - T(s)'x')ds, \tag{36}$$

exists via continuity of the X'_β–valued function $s \mapsto (I - T(s)')x'$. Applying Lemma 1(ii) to $(T(s)')_{s\geq 0} \subseteq \mathcal{L}(X'_\beta)$ we have $(x' - D(r_n)x') \in \overline{Z}^{X'_\beta} = \overline{Z}^{\sigma(X',X'')}$. Letting $n \to \infty$ in (36) we deduce that $x' \in \overline{Z}^{\sigma(X',X'')}$. This shows $\mathrm{Ker}\, P' \subseteq \overline{Z}^{\sigma(X',X'')}$ and hence, establishes equality in (31).

(ii)⇒(i). Suppose (31) holds. As $\mathrm{Fix}(T(\cdot)) = \cap_{t\geq 0}\mathrm{Ker}(I - T(t))$, the last three equalities in (35) (which do not require $(T(t))_{t\geq 0}$ to be mean ergodic) imply that $[\mathrm{Fix}(T(\cdot))]^{\perp} = \overline{Z}^{\sigma(X',X'')}$. Also, it follows from [15, Proposition 8.2.1(f)] that

$$[\overline{\mathrm{span}\cup_{t\geq 0}\mathrm{Im}\,(I - T(t))}]^{\perp} = \cap_{s\geq 0}\mathrm{Ker}(I - T(s)') = \mathrm{Fix}(T(\cdot)'). \quad (37)$$

Combining this with $(Y\oplus W)^{\perp} = Y^{\perp}\cap W^{\perp}$ for closed subspaces $Y, W \subseteq X$ and observing that $\mathrm{Fix}(T(\cdot)) \cap \overline{\mathrm{span}\{x - T(t)x : t \geq 0, x \in X\}} = \{0\}$ (cf. Lemma 3(ii)) we have (cf. also (37)) that

$$\begin{aligned}&[\mathrm{Fix}(T(\cdot)) \oplus \overline{\mathrm{span}\{x - T(t)x : t \geq 0, x \in X\}}]^{\perp}\\ &\quad = [\mathrm{Fix}(T(\cdot))]^{\perp} \cap [\overline{\mathrm{span}\{x - T(t)x : t \geq 0, x \in X\}}]^{\perp}\\ &\quad = \overline{Z}^{\sigma(X',X'')} \cap \mathrm{Fix}(T(\cdot)').\end{aligned}$$

But, equicontinuity of $\{C(r)\}_{r\geq 0}$ implies equicontinuity of $\{D(r) = C(r)'\}_{r\geq 0} \subseteq \mathcal{L}(X'_\beta)$ (by a similar argument for local equicontinuity as in the proof of Proposition 6(i)). Hence, by the proof of Lemma 3(ii) applied to $(T(s)')_{s\geq 0}$ in X'_β we see that $\overline{Z}^{\sigma(X',X'')} \cap \mathrm{Fix}(T(\cdot)') = \overline{Z}^{X'_\beta} \cap \mathrm{Fix}(T(\cdot)') = \{0\}$. It follows that the space $\mathrm{Fix}(T(\cdot)) \oplus \overline{\mathrm{span}\{x - T(t)x : t \geq 0, x \in X\}}$ is dense in X. Then Lemma 3(ii) yields

$$X = \mathrm{Fix}(T(\cdot)) \oplus \overline{\mathrm{span}\{x - T(t)x : t \geq 0, x \in X\}}.$$

To see $(T(t))_{t\geq 0}$ is mean ergodic, for $x \in X$ write $x = y + z$ with $y \in \mathrm{Fix}(T(\cdot))$ and $z \in \overline{\mathrm{span}\{x - T(t)x : t \geq 0, x \in X\}}$. By Remark 5(v) we have $\lim_{r\to\infty} C(r)z = 0$. Also, $C(r)y = \frac{1}{r}\int_0^r T(s)y\,ds = y$ for $r > 0$. So, $\lim_{r\to\infty} C(r)x = y$ exists in X.

We end this section with an application of the previous result.

Corollary 3 *Let X be a Grothendieck lcHs which is complete and barrelled and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a mean ergodic, strongly continuous C_0–semigroup which satisfies $\lim_{t\to\infty}\frac{T(t)}{t} = 0$ in $\mathcal{L}_b(X)$. Then $(T(s)')_{s\geq 0} \subseteq \mathcal{L}(X'_\beta)$ is mean ergodic.*

Proof By Remark 1(i), $(T(t))_{t\geq 0}$ is locally equicontinuous. The hypotheses on $(T(t))_{t\geq 0}$ imply (13). As $r \mapsto C(r)$ is continuous from $[0,\infty)$ into $\mathcal{L}_s(X)$ (see Lemma 1(vi)) and $P := \tau_s\text{-}\lim_{r\to\infty} C(r)$ exists, $\{C(r)\}_{r\geq 0}$ is bounded in $\mathcal{L}_s(X)$ and so, by barrelledness of X, is equicontinuous. By Remark 4(ii), $P \in \mathcal{L}(X)$ is a projection.

Lemma 3(i) gives $\mathrm{Im}\,P' = \mathrm{Fix}(T(\cdot)')$. In the proof of (i)⇒(ii) in Theorem 9 it was shown (with the notation from there) that $\mathrm{Ker}\,P' = \overline{Z}^{\sigma(X',X)}$; see (35). Then Theorem 9 itself gives $\mathrm{Ker}\,P' = \overline{Z}^{\sigma(X',X'')} = \overline{Z}^{X'_\beta}$. So, $X'_\beta = \mathrm{Im}\,P' \oplus \mathrm{Ker}\,P'$ yields $X'_\beta = \mathrm{Fix}(T(\cdot)') \oplus \overline{Z}^{X'_\beta}$. Given $x' \in X'$, write $x' = u' + v'$ with $u' \in \mathrm{Fix}(T(\cdot)')$ and $v' \in \overline{Z}^{X'_\beta}$. It was shown in the proof of

Lemma 3(i) that $C(r)'u' = u'$ for $r > 0$. Moreover, $(T(s)')_{s\geq 0} \subseteq \mathcal{L}(X'_\beta)$ is a locally equicontinuous C_0–semigroup (cf. Proposition 6(i))). Then the Cesàro means $D(r)x' := \frac{1}{r}\int_0^r T(s)'x'\,ds$, for $x' \in X'$, of $(T(s)')_{s\geq 0}$ exist (as Riemann integrals) in $\mathcal{L}_s(X'_\beta)$ with $D(r) = C(r)'$, $r > 0$; see the proof of Theorem 9. Then $\lim_{r\to\infty} D(r)u' = u'$ in X'_β.

To show $(T(s)')_{s\geq 0}$ is mean ergodic, it remains to verify $\lim_{r\to\infty} D(r)v' = 0$ in X'_β, for $v' \in \overline{Z}^{X'_\beta}$. This follows from Remark 5(v) applied to $(T(s)')_{s\geq 0} \subseteq \mathcal{L}(X'_\beta)$, provided we check the hypotheses hold. As noted before, X'_β is quasi-complete and it was just observed above that $(T(s)')_{s\geq 0}$ is a locally equicontinuous C_0–semigroup on X'_β. As $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is mean ergodic and X is barrelled, Remark 5(ii) implies $\{C(r)\}_{r\geq 0} \subseteq \mathcal{L}(X)$ is equicontinuous. This implies equicontinuity of $\{D(r)\}_{r\geq 0} \subseteq \mathcal{L}(X'_\beta)$; cf. proof of (ii)⇒(i) in Theorem 9. It remains to show that $\frac{T(r)'}{r} \to 0$ in $\mathcal{L}_s(X'_\beta)$ as $r \to \infty$.

To this effect, fix $x' \in X'$. Then there exist $q \in \Gamma_X$ and $C > 0$ such that $|\langle x, x'\rangle| \leq Cq(x)$ for each $x \in X$. Now, for $B \in \mathcal{B}(X)$, we have

$$p^B\left(\frac{T(r)'x'}{r}\right) = \sup_{x\in B}\left|\left\langle \frac{T(r)x}{r}, x'\right\rangle\right| \leq C\sup_{x\in B} q\left(\frac{T(r)}{r}\right) = Cq_B\left(\frac{T(r)}{r}\right), \quad r > 0,$$

with q_B the continuous seminorm in $\mathcal{L}_b(X)$ corresponding to $q \in \Gamma_X$, $B \in \mathcal{B}(X)$. As τ_b-$\lim_{r\to\infty} \frac{T(r)}{r} = 0$ (by hypothesis), it follows $p^B\left(\frac{T(r)'x'}{r}\right) \to 0$. But, this is for arbitrary $x' \in X'$, $B \in \mathcal{B}(X)$, i.e., $\frac{T(r)'}{r} \to 0$ in $\mathcal{L}_s(X'_\beta)$ as $r \to \infty$.

5 Appendix: vector-valued Riemann integrals.

A partition P of a compact interval $[a, b] \subseteq \mathbb{R}$ is any finite set of numbers $a = t_0 < t_1 < \ldots < t_n = b$, for some $n \in \mathbb{N}$. We write $P = \{t_k\}_{k=0}^n$ and set $|P| := \max\{(t_k - t_{k-1}) : 1 \leq k \leq n\}$. Define $[P] := \prod_{k=1}^n [t_{k-1}, t_k] \subseteq \mathbb{R}^n$. The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}([a, b])$. Given $P, Q \in \mathcal{P}([a, b])$ and elements $\xi \in [P]$, $\eta \in [Q]$ define $(P, \xi) \geq (Q, \eta)$ to mean $|P| \leq |Q|$. Then $\geq$ turns $\mathcal{D} := \cup_{P\in\mathcal{P}([a,b])}\{(P, \xi) : \xi \in [P]\}$ into a directed set.

Let X be a lcHs and $f\colon [a, b] \to X$ be a bounded function. Given $P = \{t_k\}_{k=0}^n \in \mathcal{P}([a, b])$ and $\xi = (\xi_1, \ldots, \xi_n) \in [P]$, the element

$$R(f, P, \xi) = \sum_{k=1}^n f(\xi_k)(t_k - t_{k-1}) \in X \tag{38}$$

is called the *Riemann sum* of f relative to P and ξ. Then $\{R(f, P, \xi) : (P, \xi) \in \mathcal{D}\} \subseteq X$ is a *net* in X. Recall that f is *Riemann integrable* if the net $\{R(f, P, \xi) : (P, \xi) \in \mathcal{D}\}$ converges to an element of X, denoted by $\int_a^b f(t)dt$. More precisely, for every $\varepsilon > 0$ and $p \in \Gamma_X$ there is $(\overline{P}, \overline{\xi}) \in \mathcal{D}$ such that $p(R(f, P, \xi) - \int_a^b f(t)dt) < \varepsilon$, for all $(P, \xi) \geq (\overline{P}, \overline{\xi})$. Equivalently, there is $\delta > 0$ (i.e., $\delta := |\overline{P}|$) such that $\sup_{\xi\in[P]} p(R(f, P, \xi) - \int_a^b f(t)dt) < \varepsilon$, for all $P \in \mathcal{P}([a, b])$ with $|P| < \delta$.

Theorem 10 *Let X be a sequentially complete lcHs and $f\colon [a,b] \to X$ be a continuous function. Then f is Riemann integrable.*

Proof It suffices to consider $[a,b] = [0,1]$. Fix $p \in \Gamma_X$ and $\varepsilon > 0$. The space X equipped with the pseudometric $(x,y) \mapsto p(x-y)$ is a uniform space. Since $[0,1]$ is a compact uniform space, f is uniformly continuous, [16, Ch.6, Theorem 31]. So, there is $\delta > 0$ such that $p(f(s)-f(t)) < \varepsilon$ whenever $s,t \in [0,1]$ satisfy $|s-t| < \delta$. Fix any $\overline{P} \in \mathcal{P}([0,1])$ with $|\overline{P}| < \delta$. Suppose that $P, Q \in \mathcal{P}([0,1])$ satisfy $P \geq \overline{P}$ and $Q \geq \overline{P}$, and consider the common refinement $P \cup Q$. Let $\xi \in [P]$ and $\xi' \in [P \cup Q]$.

Let $[u,v]$ be any subinterval of $[0,1]$ determined by two adjacent points of P and let r be that coordinate of ξ such that $r \in [u,v]$. Note that $|v-u| < \delta$ and $|r-t_k| < \delta$ for every $k = 0,\ldots,n$, where $\{t_k\}_{k=0}^n$ are those points of $P \cup Q$ which lie in $[u,v]$ and hence, form a partition of $[u,v]$. Then

$$f(r)(v-u) - \sum_{k=1}^{n} f(\xi'_k)(t_k - t_{k-1}) = \sum_{k=1}^{n} (f(r) - f(\xi'_k))(t_k - t_{k-1})$$

and hence,

$$p\left(f(r)(v-u) - \sum_{k=1}^{n} f(\xi'_k)(t_k - t_{k-1}) \right) \leq \varepsilon \sum_{k=1}^{n} (t_k - t_{k-1}) = \varepsilon(v-u).$$

It follows that $p(R(f,P,\xi) - R(f,P\cup Q,\xi')) < \varepsilon$, for all choices of $\xi \in [P]$ and $\xi' \in [P \cup Q]$. A similar argument yields $p(R(f,Q,\eta) - R(f,P\cup Q,\xi')) < \varepsilon$, for all choices of $\eta \in [Q]$ and $\xi' \in [P \cup Q]$. Combining the previous two inequalities yields $p(R(f,P,\xi) - R(f,Q,\eta) < 2\varepsilon$, for all $\xi \in [P]$, $\eta \in [Q]$, whenever $P,Q \in \mathcal{P}([0,1])$ satisfy $P \geq \overline{P}$ and $Q \geq \overline{P}$. This establishes that the Riemann sums form a *Cauchy net* in X.

The partitions $(P_n, \xi_{[n]})$ with $P_n := \{\frac{k}{n} : k = 0,\ldots,n\}$ and $\xi_{[n]} := (0, \frac{1}{n}, \frac{2}{n}, \ldots, 1) \in [P_n]$, for $n \in \mathbb{N}$, form a *countable cofinal subset* of $\mathcal{D}$. Since $(P_n, \xi_{[n]}) \leq (P_m, \xi_{[m]})$ whenever $n \leq m$, this ensures that the Riemann sums

$$R(f, P_n, \xi_{[n]}) = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right), \quad n \in \mathbb{N},$$

form a Cauchy sequence in X which then converges to some $x \in X$. To see that the net of all Riemann sums of f also converges to x, fix $\varepsilon > 0$ and $p \in \Gamma_X$. Then there is $\overline{P} \in \mathcal{P}([0,1])$ such that $p(R(f,P,\xi) - R(f,Q,\eta)) < \varepsilon$, for all $\xi \in [P]$, $\eta \in [Q]$, whenever $P,Q \in \mathcal{P}([0,1])$ satisfy $P,Q \geq \overline{P}$. Choose $N \in \mathbb{N}$ such that $(\overline{P}, \xi) \leq (P_n, \xi_{[n]})$ for all $n \geq N$ and $\xi \in [\overline{P}]$. Via the previous inequality we have

$$p(R(f,P,\xi) - R(f,P_n,\xi_{[n]})) < \varepsilon, \quad \forall n \geq N,\ \xi \in [P], \tag{39}$$

whenever $P \in \mathcal{P}([0,1])$ satisfies $P \geq \overline{P}$ (i.e. $|P| \leq |\overline{P}|$). Let $n \to \infty$ in (39) gives $p(R(f,P,\xi) - x) \leq \varepsilon$, for all $P \geq \overline{P}$, $\xi \in [P]$. Hence, the net of Riemann sums of f converges to x.

A function $f\colon J \to X$, with $J \subseteq \mathbb{R}$ an open interval, is differentiable at $t_0 \in J$ if there is $f'(t_0) \in X$ satisfying $\lim_{t\to t_0} \frac{f(t)-f(t_0)}{t-t_0} = f'(t_0)$ with the limit existing in X. We write $f \in C^1(J,X)$ if f is diffentiable at each point of J and $f'\colon J \to X$ is continuous. A function $f\colon [a,b] \to X$ is a C^1*–function* means there is an open interval $J \supseteq [a,b]$ and $g \in C^1(J,X)$ such that the restriction $g|_{[a,b]} = f$.

Since the proofs of the following properties follow via routine duality arguments, approximation via Riemann sums, and the scalar theory of Riemann integration, we omit the details.

Proposition 11 *Let X be a sequentially complete lcHs, the real numbers a, b satisfy $a < b$ and $f\colon [a,b] \to X$ be a continuous function.*

(i) $\langle \int_a^b f(t)dt, x' \rangle = \int_a^b \langle f(t), x' \rangle dt$ *for all* $x' \in X'$.
(ii) $\int_a^u f(t)dt + \int_u^b f(t)dt = \int_a^b f(t)dt$ *for all* $u \in \mathbb{R}$ *with* $a < u < b$.
(iii) $\int_a^b (f\circ g)(t)g'(t)dt = \int_{g(a)}^{g(b)} f(t)dt$ *for all* $g \in C^1(\mathbb{R},\mathbb{R})$.
(iv) $\int_a^b f(t)dt$ *lies in the closed convex hull of the compact set* $(b-a)f([a,b])$.
(v) *If* $f\colon [a,b] \to X$ *is a* C^1*–function, then* $f(b) - f(a) = \int_a^b f'(t)dt$.
(vi) *If* $T \in \mathcal{L}(X)$*, then* $T(\int_a^b f(t)dt) = \int_a^b (T\circ f)(t)dt$.
(vii) $p(\int_a^b f(t)dt) \le \int_a^b p(f(t))dt$ *for all* $p \in \Gamma_X$.

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