Orbits of composition operators on spaces of real analytic functions

José Bonet

Instituto Universitario de Matemática Pura y Aplicada IUMPA

Universitat Politècnica de València

Functional Analysis: Applications to Complex Analysis and Partial Differential Equations
Bedlewo, Poland

May 2012

Aim of the lecture

Study the dynamics of composition operators

on spaces of analytic or real analytic functions.

We report on joint work with Pawel Domanski (Univ. Poznań, Poland)

Definitions

X **is a Hausdorff locally convex space (lcs).**

$\mathcal{L}(X)$ **is the space of all continuous linear operators on** X**.**

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Fréchet space, or more generally if the uniform boundedness principle is valid for operators defined on X, then T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded.

Definitions

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} T^m x, \quad x \in X, \tag{1}$$

exist in X.

A power bounded operator T is mean ergodic precisely when

$$X = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Im}(I - T)}, \tag{2}$$

where I is the identity operator, $\mathrm{Im(I - T)}$ denotes the range of $(I - T)$ and the bar denotes the "closure in X".

Definitions

Hypercyclic operators

An operator $T \in \mathcal{L}(X)$ is said to be *(sequentially) hypercyclic* if there is a vector $x \in X$ whose orbit $\{T^m(x)\}_{m=1}^{\infty}$ is (sequentially) dense in X.

Transitive operators

An operator $T \in \mathcal{L}(X)$ is said to be *topologically transitive* if for every pair of non-empty open subsets U, V in X there is n such that $T^n(U) \cap V \neq \emptyset$.

Proposition

If $T \in \mathcal{L}(X)$ is an operator on a separable complete metrizable lcs X, T is hypercyclic if and only if it is topologically transitive. This is a consequence of Baire category theorem.

Composition operators on spaces of analytic functions

$$C_\varphi(f) := f \circ \varphi$$

- $\varphi : U \to U$ holomorphic, $U \subset \mathbb{C}^d$ open set

$$C_\varphi : H(U) \to H(U)$$

- $\varphi : \Omega \to \Omega$ real analytic, $\Omega \subset \mathbb{R}^d$ open set

$$C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$$

Iterates:

$$(C_\varphi)^n := \underbrace{C_\varphi \circ C_\varphi \circ \cdots \circ C_\varphi}_{n \text{ times}} = C_{\varphi^n}$$

Universal functions for composition operators on $H(U)$ have been investigated by **Bernal, Bonilla, Godefroy, Grosse-Erdmann, León, Luh, Montes, Mortini, Shapiro, Zajac and others.**

Power bounded composition operators on spaces of analytic functions

Theorem 1

Let $U \subseteq \mathbb{C}^d$ be a connected domain of holomorphy. The following assertions are equivalent:

(a) $C_\varphi : H(U) \to H(U)$ is power bounded;

(b) $C_\varphi : H(U) \to H(U)$ is uniformly mean ergodic;

(c) $C_\varphi : H(U) \to H(U)$ is mean ergodic;

(d) The map φ has **stable orbits** on U:
$\forall\ K \Subset U\ \exists\ L \Subset U$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$;

(e) There is a fundamental family of (connected) compact sets (L_j) in U such that $\varphi(L_j) \subseteq L_j$ for every $j \in \mathbb{N}$.

Power bounded composition operators on spaces of analytic functions

Theorem 2

Let U be a connected open subset of $\mathbb{C}$ and let $\varphi : U \to U$ be a holomorphic self-map. If C_φ is power bounded, then either φ is an automorphism of U or all orbits of φ tends to a constant $u \in U$ such that u is a fixed point of φ.

If additionally $U = \mathbb{D}$, then in the automorphism case φ has also a fixed point u and

$$\varphi = \varphi_u^{-1} \circ r_\theta \circ \varphi_u, \qquad \theta \in [0, 2),$$

where

$$\varphi_u(z) := \frac{u - z}{1 - \bar{u}z}, \quad r_\theta(z) = e^{i\theta\pi} z$$

Power bounded composition operators on spaces of analytic functions

Theorem 2 continued

Moreover,

(i) If φ is not an automorphism then the projection P associated to C_φ is given: $P(f)(z) = f(u)$.

(ii) If φ is an automorphism and $\theta = \frac{p}{q}$ is rational then $P(f)(z) = \frac{1}{q}\sum_{j=0}^{q-1} f(\varphi_u^{-1}(r_{\frac{jp}{q}}(\varphi_u(z))))$.

(iii) If φ is an automorphism and θ is irrational then $P(f)(z) = \frac{1}{2}\int_0^2 f(\varphi_u^{-1}(r_\theta(\varphi_u(z))))d\theta$.

- $\Omega \subseteq \mathbb{R}^d$ open connected set.
- The space of real analytic functions $\mathscr{A}(\Omega)$ is equipped with the unique locally convex topology such that for each $U \subseteq \mathbb{C}^d$ open, $\mathbb{R}^d \cap U = \Omega$, the restriction map $R : H(U) \longrightarrow \mathscr{A}(\Omega)$ is continuous and for each compact set $K \subseteq \Omega$ the restriction map $r : \mathscr{A}(\Omega) \longrightarrow H(K)$ is continuous. In fact,

$$\mathscr{A}(\Omega) = \mathrm{proj}_{N\in\mathbb{N}}\ H(K_N) = \mathrm{proj}_{N\in\mathbb{N}}\ \mathrm{ind}_{n\in\mathbb{N}}\ H^\infty(U_{N,n}).$$

- $(f_n)_{n\in\mathbb{N}}$ tends to f in $\mathscr{A}(\Omega)$ if and only if there is a complex neighbourhood W of Ω such that each f_n and f extend to W and $f_n \to f$ uniformly on compact subsets of W.
- $\mathscr{A}(\Omega)$ is complete, separable, barrelled and Montel. **Domański, Vogt, 2000,** proved that the space $\mathscr{A}(\Omega)$ has no Schauder basis.

Power bounded composition operators on spaces of real analytic functions

Theorem 3

Let $\varphi : \Omega \to \Omega$, $\Omega \subseteq \mathbb{R}^d$ open connected, be a real analytic map. Then the following assertions are equivalent:

(a) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is power bounded;

(b) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is uniformly mean ergodic;

(c) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is mean ergodic;

(d) $\forall\, K \Subset \Omega\ \exists\, L \Subset \Omega\ \ \forall\, U$ complex neighbourhood of L $\exists\, V$ complex neighbourhood of K:

$$\forall\, n \in \mathbb{N} \quad \varphi^n \text{ is defined on } V \text{ and } \varphi^n(V) \subseteq U;$$

(e) For every complex neighbourhood U of Ω there is a complex (open!) neighbourhood $V \subseteq U$ of Ω such that φ extends as a holomorphic function to V and $\varphi(V) \subseteq V$.

Theorem 4

Let $a, b \in \overline{\mathbb{R}}$ and let $\varphi :]a, b[\to]a, b[$ be real analytic. The following are equivalent:

(a) $C_\varphi : \mathscr{A}(]a, b[) \longrightarrow \mathscr{A}(]a, b[)$ is power bounded;

(b) there exists a complex neighbourhood U of $]a, b[$ such that $\varphi(U) \subseteq U$, $\mathbb{C} \setminus U$ contains at least two points, and φ has a (real) fixed point u, or equivalently, there is a fundamental family of such neighbourhoods of $]a, b[$;

(c) φ is one of the following forms:

1. $\varphi = \mathrm{id}$;
2. $\varphi^2 = \mathrm{id}$;
3. As $n \to \infty$ the sequence φ^n tends to a constant function $\equiv u \in]a, b[$ in $\mathscr{A}(]a, b[)$.

Power bounded composition operators on spaces of real analytic functions

Theorem 4 continued

If u is the fixed point of φ then the above cases in (c) correspond to:

1. $\varphi'(u) = 1$;
2. $\varphi'(u) = -1$;
3. $|\varphi'(u)| < 1$.

Moreover, C_φ is uniformly mean ergodic and the projection $P := \lim_{N\in\mathbb{N}} \frac{1}{N}\sum_{n=1}^{N} C_{\varphi^n}$ is of the following form:

1. $P = \mathrm{id}$;
2. $P(f) = \frac{f+f\circ\varphi}{2}$, $\mathrm{ImP} = \{\mathrm{f} : \mathrm{f} = \mathrm{f}\circ\varphi\}$, $\ker P = \{f : f = -f\circ\varphi\}$;
3. $P(f) = f(u)$, $\mathrm{ImP} =$ the set of constant functions, $\ker P = \{f : f(u) = 0\}$.

Example:

Even if $\varphi :]-1,1[\to]-1,1[$ maps $]-1,1[$ in one compact set, it does not follow that $C_\varphi : \mathscr{A}(]-1,1[) \to \mathscr{A}(]-1,1[)$ is power bounded. Define

$$\varphi :]-1,1[\to]-1,1[, \varphi(x) = \frac{x}{1+x^2}.$$

We have $\varphi(]-1,1[) =]-1/2, 1/2[$. By Theorem 4, C_φ is not power bounded since $\varphi'(0) = 1$ and $\varphi^n \to 0$.

By Theorem 3 there is no complex neighbourhood V such that $\varphi(V) \subseteq V$. Observe that φ has two singularities i and $-i$.

Transitive composition operators on spaces of real analytic functions

Theorem 5

Let $\varphi : \Omega \to \Omega$, $\Omega \subseteq \mathbb{R}^d$ open connected, be a real analytic map. Then the following assertions are equivalent:

(a) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is topologically transitive;

(b) φ is injective, φ' is never singular on Ω and φ runs away on Ω, i.e. for every compact set $K \Subset \Omega$ there is $n \in \mathbb{N}$ such that $\varphi^n(K) \cap K = \emptyset$.

In particular, if $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is hypercyclic then φ is injective, runs away on Ω and φ' is never singular on Ω.

The concept of a function running away on an open set was introduced by Bernal and Montes in 1995

Hypercyclic composition operators on spaces of real analytic functions

Theorem 6

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, $\varphi((-1,1)) \subset (-1,1)$. The following assertions are equivalent:

(a) $C_\varphi : \mathscr{A}(-1,1) \to \mathscr{A}(-1,1)$ is sequentially hypercyclic;

(b) $C_\varphi : \mathscr{A}(-1,1) \to \mathscr{A}(-1,1)$ is hypercyclic;

(c) $C_\varphi : \mathscr{A}(-1,1) \to \mathscr{A}(-1,1)$ is topologically transitive;

(d) φ runs away on $(-1,1)$ and φ' does not vanish on $(-1,1)$;

(e) φ has no fixed point on $(-1,1)$ and φ' does not vanish on $(-1,1)$;

(f) there is a one connected complex neighbourhood W of $(-1,1)$ in $\mathbb{D}$ such that $\varphi(W) \subset W$, φ is injective on W and φ runs away on W;

(g) there is a one connected complex neighbourhood W of $(-1,1)$ in $\mathbb{D}$ such that $\varphi(W) \subset W$ and $C_\varphi : H(W) \to H(W)$ is hypercyclic.

Hypercyclic composition operators on spaces of real analytic functions

Remark

If $\varphi : \mathbb{R} \to \mathbb{R}$ extends to a self map on some complex finitely connected neighbourhood $U \neq \mathbb{C}$, then we can take without loss of generality $U = \mathbb{D}$ the unit disc and $\mathbb{R}$ corresponds to $(-1, 1)$. That is why the theorem above is interesting and more general than it might look.

Hypercyclic composition operators on spaces of real analytic functions

Comments on the Proof of Theorem 6

1. (f) $\Leftrightarrow$ (g) by Bernal, Montes, Grosse-Erdmann, Mortini Theorem on composition operators on $H(W)$.
2. (g)$\Rightarrow$(a) follows from the hypercyclic comparison principle, (a)$\Rightarrow$(b)$\Rightarrow$(c) are trivial, and (c)$\Rightarrow$(d) follows from Theorem 5.
3. (d) $\Leftrightarrow$ (e) because φ runs away on an interval I if and only if φ has no fixed point on I.
4. The assumption that φ extends to a self map of $\mathbb{D}$ is needed only for (d)$\Rightarrow$(f). The proof uses the classification of selfmaps on the unit disc and Denjoy-Wolff theorem.

1. There are examples of self maps $\varphi : \mathbb{D} \to \mathbb{D}$ such that $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is not hypercyclic (since φ is not injective on $\mathbb{D}$) but $C_\varphi : \mathscr{A}((-1,1)) \to \mathscr{A}((-1,1))$ is hypercyclic.

Define

$$\varphi(z) := \frac{(z+1)^3}{32} + \frac{3}{4}.$$

Clearly $\varphi(\mathbb{D}) \subset \mathbb{D}$ and φ is not injective on $\mathbb{D}$ since $\varphi(\alpha e^{\frac{\pi}{3}i} - 1) = \varphi(\alpha e^{-\frac{\pi}{3}i} - 1)$ for suitably small $\alpha > 0$.

However, φ is a self map on

$$U := \mathbb{D} \cap \left\{ z = x + iy : |y| < \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} \right\},$$

and $C_\varphi : H(U) \to H(U)$ is hypercyclic.

Examples

2. The following family of self maps $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi((-1,1)) \subset (-1,1)$ has been investigated by Cowen and Ko, 2010.

Take $a_0, a_1 \in \mathbb{R}$ and define

$$\varphi(z) := a_0 + \frac{a_1 z}{1 - a_0 z}, \ \ z \in \mathbb{D}.$$

Then φ is a linear fractional map with real coefficients and φ maps the unit disc into itself if and only if

$$|a_0| < 1 \quad \text{and} \quad -1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2.$$

This is equivalent to the fact that φ maps the interval $(-1,1)$ into itself.

It is enough to consider $0 \leq a_0 < 1$ and distinguish three cases.

Case 1. $0 \leq a_0 < 1$ and $a_1 = -1 + a_0^2$.

In this case $\varphi(z) = (a_0 - z)/(1 - a_0 z)$ is an automorphism of $\mathbb{D}$ such that $\varphi(1) = -1$ and $\varphi(-1) = 1$.

Moreover $1 - (1 - a_0)^{1/2} \in [0, 1)$ is a fixed point.

Thus φ is elliptic and the operator $C_\varphi : \mathscr{A}((-1, 1)) \to \mathscr{A}((-1, 1))$ is power bounded and mean ergodic by Theorem 3. In particular it is not topologically transitive.

Examples

Case 2. $0 \leq a_0 < 1$ and $-1 + a_0^2 < a_1 < (1 - a_0)^2$.

In this case $-1 < \varphi(-1) < 1$, $-1 < \varphi(1) < 1$ and φ maps the closed unit disc $\overline{\mathbb{D}}$ into the open unit disc.

Therefore C_φ is power bounded on $H(\mathbb{D})$ and on $\mathscr{A}((-1,1))$ by Theorems 1 and 3, hence it is not topologically transitive.

Case 3. $0 \leq a_0 < 1$ and $a_1 = (1 - a_0)^2$.

If $a_0 = 0$, then $\varphi(z) = z, z \in \mathbb{D}$, and C_φ coincides with the identity that is not topologically transitive on $\mathscr{A}((-1, 1))$.

In case $0 < a_0 < 1$, we get $\varphi'(x) > 0$ for each $x \in (-1, 1)$, φ is not an automorphism of the disc, $\varphi(1) = 1, -1 < \varphi(-1) < 1$ and there are no fixed points in $(-1, 1)$.

We can apply our Theorem 6 to conclude that
$C_\varphi : \mathscr{A}((-1, 1)) \to \mathscr{A}((-1, 1))$ is sequentially hypercyclic.

Examples

3. Our last example shows that the implication (d) implies (f) in the Theorem 6 does not hold in general.

Just take

$$\varphi(z) := z^2, z \in (0,1).$$

$C_\varphi : \mathscr{A}((0,1)) \to \mathscr{A}((0,1))$ is topologically transitive by Theorem 5 on the characterization of transitive operators.

However, there is no complex neighbourhood U of $(0,1)$ such that $\varphi^n(z) = z^{2^n}$ is injective on U for every $n \in \mathbb{N}$. Note that $\varphi^n(\frac{1}{2}\exp(\pm 2\pi i/2^n)) = 1/2^{2^n}$ for each $n \in \mathbb{N}$.

Examples

Peris has shown very recently that $C_\varphi : \mathscr{A}((0,1)) \to \mathscr{A}((0,1))$ for $\varphi(z) := z^2, z \in (0,1)$, is even sequentially hypercyclic.

Accordingly *not every sequentially hypercyclic operator* $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ *satisfies that there is complex neighbourhood* U *of* Ω *such that* φ *extends holomorphically to* U, $\varphi(U) \subset U$, *and* $C_\varphi : H(U) \to H(U)$ *is hypercyclic*

Idea of Peris' argument

1. $\varphi(z) := z^2, z \in (0,1)$.
2. $(p_n(z))_n$ a dense sequence of polynomials. Set $f_n(z) := z(1-z)p_n(z), n \in \mathbb{N}$.
3. $S_n := C_{\gamma_n}$, $\gamma_n(z) := \exp(2^{-n} \log z)$, with $\log z$ the branch of the logarithm defined on $\mathbb{C} \setminus]-\infty, 0]$. Clearly $(C_\varphi^k \circ S_k)h = h$ for each polynomial h.
4. A sequentially hypercyclic vector for $C_\varphi : \mathscr{A}((0,1)) \to \mathscr{A}((0,1))$ is given explicitly by

$$f := \sum_{j=1}^{\infty} S_{n_j} f_{n_j}$$

for certain subsequence $(n_j)_j$ of $\mathbb{N}$.

Open problems

Open Problems

(1) Is the operator $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$, $\varphi(z) = \exp(z)$, (sequentially) hypercyclic? It is topologically transitive by Theorem 5.

(2) Is there a hypercyclic (composition) operator $T : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$, $\Omega \subset \mathbb{R}^d$, which is not sequentially hypercyclic? Is there a transitive (composition) operator $T : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$, $\Omega \subset \mathbb{R}^d$, which is not hypercyclic?

(3) Is there a mean ergodic operator on a Banach space (or a locally convex space) which is hypercyclic? Recall that no power bounded operator can be hypercyclic, but there are mean ergodic operators which are not power bounded.

References

1. **J. Bonet, P. Domański,** Mean ergodic composition operators on spaces of holomorphic functions, RACSAM 105 (2011), 389–396.
2. **J. Bonet, P. Domański,** Power bounded composition operators on spaces of analytic functions, Collectanea Math. 62 (2011), 69–83.
3. **J. Bonet, P. Domański,** Hypercyclic composition operators on spaces of real analytic functions, Math. Proc. Cambridge Phil. Soc. .