Hypercyclic composition operators on spaces of real analytic functions

José Bonet and Paweł Domański

Abstract

We study the dynamical behaviour of composition operators C_φ defined on spaces $\mathscr{A}(\Omega)$ of real analytic functions on an open subset Ω of $\mathbb{R}^d$. We characterize when such operators are topologically transitive, i.e. when for every pair of non-empty open sets there is an orbit intersecting both of them. Moreover, under mild assumptions on the composition operator, we investigate when it is sequentially hypercyclic, i.e., when it has a sequentially dense orbit. If φ is a self map on a simply connected complex neighbourhood U of $\mathbb{R}$, $U \neq \mathbb{C}$, then topological transitivity, hypercyclicity and sequential hypercyclicity of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ are equivalent.

1 Introduction and Notation

The aim of this article is to study hypercyclic and topologically transitive composition operators $C_\varphi(f) := f \circ \varphi$ on spaces $\mathscr{A}(\Omega)$ of real analytic functions defined on an open subset Ω of $\mathbb{R}^d$, $\varphi : \Omega \to \Omega$ a real analytic self map. Recall that an operator $T : X \to X$, X a locally convex space, is called *topologically transitive* whenever for each pair of non-empty open sets U, V in X there is $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. A vector $x \in X$ is called *hypercyclic* (or *sequentially hypercyclic*) if the x-orbit $\{T^n x : n \in \mathbb{N}\}$ of T is dense (or sequentially dense, respectively) in X. Clearly, every sequentially hypercyclic operator on a locally convex space X is hypercylic, and hypercyclic operators are topologically transitive. In Theorem 2.3 we characterize when composition operators $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ are topologically transitive. This happens exactly when φ is injective, φ' is never singular on Ω and φ runs away on Ω. In Proposition 3.5 we analyze when the operator is sequentially hypercyclic. If φ is a self map on a simply connected complex neighbourhood U of $\mathbb{R}$, $U \neq \mathbb{C}$, then topological transitivity, hypercyclicity and sequential hypercyclicity of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ are equivalent, and these equivalent conditions are characterized in this case in terms of φ in Theorem 3.6. Several illustrating examples are presented in Section 4.

There is a huge literature about the dynamical behavior of various linear continuous operators on Banach, Fréchet and more general locally convex spaces; see the survey paper by Grosse-Erdmann [20] and the recent books by Bayart and Matheron [2] and by Grosse-Erdmann and Peris [22]. Composition operators on different function spaces have been also extensively

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investigated; see, for instance, [30], [3], [5] and [21]. For general theory of composition operators on Banach spaces of holomorphic functions we refer the reader to [13], [29]. A systematic investigation of composition operators on spaces of real analytic functions has been undertaken by Langenbruch and the second author; see for example [17] and the recent paper [16] with Goliński. These papers concentrate on aspects different from the dynamical behaviour of the operator. An investigation of those composition operators on $\mathscr{A}(\Omega)$ such that the orbit of every element is bounded was undertaken by the authors in [7]. In that paper we also characterized mean ergodic composition operators on $\mathscr{A}(\Omega)$. Operators with the orbits of all the elements bounded are called power bounded.

A description of the natural topology on $\mathscr{A}(\Omega)$, that goes back to Martineau, is given, for instance, in [18]. The space $\mathscr{A}(\Omega)$ has very good properties: it is nuclear, separable, complete, barrelled and even ultrabornological, satisfies the closed graph theorem, but surprisingly it has no Schauder basis by [18].

To be precise, the space $\mathscr{A}(\Omega)$ is equipped with the unique locally convex topology such that for any $U \subset \mathbb{C}^d$ open, $\mathbb{R}^d \cap U = \Omega$, the restriction map $R : H(U) \longrightarrow \mathscr{A}(\Omega)$ is continuous and for any compact set $K \subset \Omega$ the restriction map $r : \mathscr{A}(\Omega) \longrightarrow H(K)$ is continuous. We endow the space $H(U)$ of holomorphic functions on U with the compact-open topology and the space $H(K)$ of germs of holomorphic functions on K with its natural locally convex inductive limit topology:

$$H(K) = \text{ind}_{n\in\mathbb{N}}\, H^\infty(U_n),$$

where $(U_n)_{n\in\mathbb{N}}$ is a basis of $\mathbb{C}^d$-neighbourhoods of K. Martineau proved that there is exactly one topology on $\mathscr{A}(\Omega)$ satisfying the condition above. Endowed with this topology one has the following description as a countable projective limit:

$$\mathscr{A}(\Omega) = \text{proj}_{N\in\mathbb{N}}\ H(K_N).$$

Here $(K_N)_N$ is a fundamental sequence of compact subsets of Ω. For our purpose, it is important to recall that a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{A}(\Omega)$ tends to f if and only if there is a complex neighbourhood W of Ω such that each f_n and f extend to W and $f_n \to f$ uniformly on compact subsets of W. Analogously one defines the topology on $\mathscr{A}(\Omega)$ when Ω is a real analytic manifold [33]. A long survey on the space of real analytic functions with very precise description of its topology is contained [15].

We say that a map $\varphi : \Omega \to \Omega$ *runs away* on Ω if for every compact set $K \Subset \Omega$ there is $n \in \mathbb{N}$ such that $\varphi^n(K) \cap K = \emptyset$. This term was invented by Bernal and Montes [3]. Clearly, if φ runs away on Ω it has no fixed point in Ω.

Proposition 1.1 *A continuous map $\varphi : (a,b) \to (a,b)$ runs away on the open interval $(a,b) \subset \mathbb{R}$ if and only if it has no fixed point.*

Proof: If $\varphi : (a,b) \to (a,b)$ has a fixed point, it does not run away on (a,b). Conversely, if $\varphi : (a,b) \to (a,b)$ has no fixed points, then either $\varphi(x) > x$ for every $x \in (a,b)$ or $\varphi(x) < x$ for every $x \in (a,b)$, by the mean value theorem. Accordingly, the sequence $(\varphi^n(x))_{n\in\mathbb{N}}$ is monotonic for every $x \in (a,b)$ so either convergent (necessarily to a fixed point; a contradiction) or tends to a or to b. Assume for example that all these sequences converge to b. By Dini's theorem, for every compact subset K of (a,b), φ^n tends to b uniformly on K. Thus φ runs away on (a,b). □

We will use the Kobayashi semi-distance $k_V(\cdot,\cdot)$ on a complex manifold $V \subseteq \mathbb{C}^d$. Here the beautiful book [26] is a standard reference. Every holomorphic map $\varphi : V \to V$ is always non-expansive with respect to k_V. The manifold V is called *hyperbolic* if k_V is a distance (and then it induces the standard topology of V). Every domain biholomorphic to a bounded set is automatically hyperbolic [26, Cor. 4.1.10, Prop. 3.2.2]. In particular, every open subset of $\mathbb{C}$ with complement consisting of at least two points is hyperbolic. The manifold V is called *Kobayashi complete* if (V, k_V) is a complete metric space, or equivalently, if every ball in this space is relatively compact [26, Prop. 1.1.9]. Every bounded open set $V \subset \mathbb{C}^d$ such that each of its boundary point admits a weak peak function is Kobayashi complete [26, Cor. 4.1.11]. Both hyperbolicity and Kobayashi completeness are biholomorphic invariants. In fact every open subset of $\mathbb{C}$ with complement of at least two points is automatically hyperbolic complete. Kobayashi distance on $\mathbb{D}$ is just the Poincaré metric ([19]), denoted ρ throughout the paper.

For each set A and each function $f : A \to \mathbb{C}$ we denote $\|f\|_A := \sup_{x\in A} |f(x)|$. By $B(x, r)$ and $B_{k_V}(x, r)$ we denote, respectively, euclidean and Kobayashi closed balls of center x and radius r. The notation $K \Subset U$ means that K is a compact subset of the open set U. By $\bar{\mathbb{R}}$ we denote the extended real line $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. For non-explained notions from functional analysis we refer to [27]. For complex analysis of several variables see [24] and for real analytic manifolds see [23]. For dynamics of holomorphic maps see [1], [28].

2 Topologically transitive composition operators

Lemma 2.1 *Let $\varphi : \Omega \to \Omega$, $\Omega \subseteq \mathbb{R}^d$ open, be a real analytic injective map with derivative invertible at every point of Ω. There is a complex neighbourhood U of Ω such that φ is defined and holomorphic on U, it is injective there and its derivative is never singular on U. In particular, φ is biholomorphic from U onto $\varphi(U)$.*

Proof: Clearly there is a complex neighbourhood U_1 of Ω such that $\det \varphi'$ does not vanish on U_1. We define

$$F : U_1 \times U_1 \to U_1 - U_1, \qquad F(z, w) := \varphi(z) - \varphi(w).$$

The zero set $V(F)$ of F consists of the points of the diagonal of $U_1 \times U_1$ and maybe some additional points; the set of these additional points will be denoted by V_1. Clearly $V_1 \cap (\Omega \times \Omega) = \emptyset$. Since φ' is never singular on U_1 the singular set of the analytic set $V(F)$ is empty, i.e. $V(F)$ is a manifold. On the other hand, if $x \in \overline{V_1} \cap (\Omega \times \Omega)$, then $x \in V(F)$ and must belong to the diagonal of $U_1 \times U_1$ and thus x belongs to the singular part of $V(F)$; a contradiction. We have proved that V_1 is disjoint with some neighbourhood of $\Omega \times \Omega$. Thus for every compact $K \subseteq \Omega$ there is a complex neighbourhood V_K of K such that φ is injective on V_K.

Let (K_k) be a compact exhaustion of $\Omega = \bigcup_{k\in\mathbb{N}} K_k$, additionally $K_0 := \emptyset$. We construct inductively a family of complex neighbourhoods $(U_n^k)_{k=0,\dots,n}$ of K_n decreasing in k for each n and such that

1. φ is injective on U_n^k for every n and $k \leq n$;

2. φ is injective on $U_n^n \cup U_m^n$ for every n and $n \leq m$.

We define $U_0^0 = \emptyset$ and U_n^0 to be a complex neighbourhood of K_n such that φ is injective on U_n^0.

Assume that we have found $U_n^0, U_n^1, \dots, U_n^{k-1}$ for every $n \in \mathbb{N}$. First, we define U_k^k. Let us take an arbitrary complex neighbourhood V of K_k which is relatively compact in U_k^{k-1}. Define

$$W_V := \{z \in \overline{V} : \ \exists\, w \in \Omega \setminus U_k^{k-1} \ \ \varphi(z) = \varphi(w)\}.$$

If for every V the set W_V is non-empty, then we find a point $x \in K_k$ and two sequences of elements (z_j), (w_j) such that

$$\lim_{j\to\infty} z_j = x, \qquad (w_j) \subseteq \Omega \setminus U_k^{k-1}, \qquad \varphi(z_j) = \varphi(w_j) \quad \text{for every } j \in \mathbb{N}.$$

Since φ' is invertible in x, by the inverse mapping theorem, there is a real neighbourhood A of $\varphi(x)$ and a real neighbourhood B of x, $B \subseteq U_k^{k-1}$, such that φ maps B bijectively onto A. Clearly, there is $j \in \mathbb{N}$ such that $\varphi(w_j) = \varphi(z_j) \in A$ and φ takes value $\varphi(w_j)$ at some point in B. This contradicts injectivity of φ on Ω. Accordingly, we can find a relatively compact complex neighbourhood U_k^k of K_k with $\overline{U_k^k} \subset U_k^{k-1}$ such that

$$\varphi(\overline{U_k^k}) \cap \varphi(\Omega \setminus U_k^{k-1}) = \emptyset.$$

Hence, there is a complex neighbourhood W of $\Omega \setminus U_k^{k-1}$ such that $\varphi(U_k^k) \cap \varphi(W) = \emptyset$. As φ is injective on U_k^{k-1} the sets

$$U_n^k := U_n^{k-1} \cap (U_k^{k-1} \cup W) \qquad \text{for } n > k$$

satisfy the required conditions. This completes the inductive definition of U_n^k for $n, k \leq n$.

Now, φ is injective on the complex neighbourhood $\bigcup_{n=0}^{\infty} U_n^n$ of Ω. □

Lemma 2.2 *Every compact set $K \subset \mathbb{R}^d$ has a basis of complex neighbourhoods consisting of polynomial polyhedra, i.e., sets of the form:*

$$\left\{ z \in \mathbb{C}^d : |P_j(z)| < 1 \quad \textit{for } j = 1, \ldots, n \right\}$$

for finitely many polynomials $P_1, \ldots, P_n$.

Proof: It follows from [25, Lemma 2.7.4]. □

Now, we are ready to formulate a characterization of transitivity of composition operators.

Theorem 2.3 *Let $\varphi : \Omega \to \Omega$ be an analytic map on an open subset Ω of $\mathbb{R}^d$. The composition operator $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is topologically transitive if and only if φ is injective, φ' is never singular on Ω and φ runs away on Ω.*

Proof: *Necessity.* If C_φ is topologically transitive, then for each pair of functions f, $g \in \mathscr{A}(\Omega)$, each $\varepsilon > 0$ and each compact set $K \Subset \Omega$ there is $h \in \mathscr{A}(\Omega)$ and $n \in \mathbb{N}$ such that

$$\|f - h\|_K \leq \varepsilon, \qquad \|C_{\varphi^n}(h) - g\|_K \leq \varepsilon.$$

This is so since $\|\cdot\|_K$ is a continuous seminorm on $\mathscr{A}(\Omega)$. Now, we consider

$$f(w) = 0, \quad g(w) = 4\varepsilon \qquad \text{for every } w \in \Omega.$$

Then $\|h\|_K \leq \varepsilon$ and if $\varphi^n(K) \cap K \neq \emptyset$ for every $n \in \mathbb{N}$, then for every $n \in \mathbb{N}$ at some point $x \in K$ we have $|C_{\varphi^n}(h)(x)| \leq \varepsilon$. This contradicts the definition of g. So φ runs away.

Analogously, if $\varphi(z) = \varphi(w)$ for some $z, w \in \Omega$ then $C_{\varphi^n}(h)$ cannot approximate g such that $g(z) \neq g(w)$. Thus we have proved that φ is injective.

Assume that $\varphi'(w) \in L(\mathbb{C}^d)$ is singular for some $w \in \Omega$. This means that there is a vector v such that $(\varphi^n)'(w)v = 0$ for every $n \in \mathbb{N}$. Since differentiation is a continuous map on $\mathscr{A}(\Omega)$

transitivity implies that for every $f, g \in \mathscr{A}(\Omega)$ and every $\varepsilon > 0$, $K \Subset \Omega$ there are $h \in \mathscr{A}(\Omega)$ and $n \in \mathbb{N}$ such that for every $x \in K$

$$|f'(x)v - h'(x)v| \leq \varepsilon, \qquad |(C_{\varphi^n}(h))'(x)v - g'(x)v| \leq \varepsilon.$$

Let us take $g(x) = \langle x, v\rangle$ for every $x \in \Omega$. Then $g'(w)v = \langle v, v\rangle$. Clearly, $(C_{\varphi^n}(h))'(w)v = 0$ for any $n > 0$; a contradiction for $\varepsilon < \langle v, v\rangle$. We have proved that φ' must be invertible everywhere on Ω.

Sufficiency. Take a compact set $K \subset \Omega$ and arbitrary $f, g \in \mathscr{A}(\Omega)$. There is $n \in \mathbb{N}$ such that

$$\varphi^n(K) \cap K = \emptyset,$$

i.e., there exists $\varepsilon > 0$ such that

$$[K + B(0, \varepsilon)] \cap [\varphi^n(K) + B(0, \varepsilon)] = \emptyset.$$

By Lemma 2.1 there is a complex neighbourhood U of Ω such that $\varphi|_U$ is biholomorphic.

Let us take a closed complex neighbourhood $\tilde{W}$ of K contained in $K + B(0, \varepsilon)$ such that

$$\tilde{W}, \varphi(\tilde{W}), \varphi^2(\tilde{W}), \dots, \varphi^n(\tilde{W}) \Subset U, \quad \varphi^n(\tilde{W}) \subseteq \varphi^n(K) + B(0, \varepsilon)$$

and such that f, g are defined on $\tilde{W}$. Then $g \circ \varphi^{-n}$ is defined on $\varphi^n(\tilde{W})$ and it is holomorphic there. Let $W_1 \subset \tilde{W} \cup \varphi^n(\tilde{W})$ be a polynomial polyhedron satisfying $K \subset W_1$ and $\varphi^n(K) \subset W_1$. Fix $\delta > 0$. We can apply [25, Theorem 2.7.7] to get a polynomial h such that

$$\|h - f\|_{\tilde{W} \cap W_1} \leq \delta, \qquad \|h - g \circ \varphi^{-n}\|_{\varphi^n(\tilde{W}) \cap W_1} \leq \delta.$$

Hence for any complex neighbourhood $W \subset \tilde{W} \cap W_1$ of K such that $\varphi^n(W) \subset W_1$ we have

$$\|h \circ \varphi^n - g\|_W \leq \delta.$$

We have proved that for every $f, g \in \mathscr{A}(\Omega)$ and every compact $K \Subset \Omega$ there is a complex neighbourhood W of K such that for every $\delta > 0$ we have

$$C_{\varphi^n}(B_W(f, \delta)) \cap B_W(g, \delta) \neq \emptyset,$$

where $B_W(f, \delta)$ denotes the closed ball of center f and radius δ with respect of the norm $\|\cdot\|_W$. Since for every neighbourhood V_f of f and V_g of g in $\mathscr{A}(\Omega)$ there are a compact set $K \Subset \Omega$ such that for every complex neighbourhood W of K there is $\delta > 0$ such that $B_W(f, \delta) \subset V_f$ and $B_W(g, \delta) \subset V_g$, the assertion we have just shown implies that C_φ is topologically transitive. □

Remark. Using the proof of Theorem 2.3 one can easily show that the composition operator is never transitive on $H(K)$ for any compact set K. Indeed, $\varphi : K \to K$ never "runs away on K". However, there are hypercyclic operators on $H(K)$ by [8]; see also [31, Th. 1.10].

3 Hypercyclic composition operators induced by self maps on a complex neighbourhood

We start with a necessary condition for $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ to be hypercyclic that follows from Theorem 2.3, since hypercyclic operators are topologically transitive.

Corollary 3.1 *If $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is hypercyclic then φ is injective, runs away on Ω and φ' is never singular on Ω.*

Clearly, if U is a complex neighbourhood of an open subset $\Omega \subset \mathbb{R}^d$, then $H(U)$ is dense in $\mathscr{A}(\Omega)$. Thus if $\varphi : U \to U$ is holomorphic, $\varphi(\Omega) \subset \Omega$, then hypercyclicity of $C_\varphi : H(U) \to H(U)$ implies sequential hypercyclicity of $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$.

The proof of the following sufficient condition is based on the same idea of proofs given in [3], [4], [5].

Proposition 3.2 *Let $\varphi : U \to U$ be a holomorphic map, where U is an open subset of $\mathbb{C}^d$. If the map C_φ is hypercylic on $H(U)$, then φ is injective on U, its derivative is never singular on U and φ runs away on U.*

Proof: If $\varphi(z) = \varphi(w)$ then every function in the image of C_{φ^n} has the same values in z and in w, thus C_φ cannot be hypercyclic. If $\varphi'(z)v = 0$, then inductively $(\varphi^n)'(z)v = 0$ for every $n \in \mathbb{N}$ and every function in im C_{φ^n} has a derivative at z vanishing on the vector v. Again C_φ cannot be hypercylic. If $\varphi^n(K) \cap K \neq \emptyset$ for every $n \in \mathbb{N}$ then for any function $f \in H(U)$ with $f(K) \subseteq B(0,r)$ the function $C_{\varphi^n}(f)$ takes some values in $B(0,r)$ and so it cannot approximate any function with moduli of all values bigger than r. □

Unfortunately, for arbitrary open $\Omega \subset \mathbb{R}^d$ and real analytic $\varphi : \Omega \to \Omega$ such that C_φ is hypercyclic we do not know if there is a complex neighbourhood U of Ω such that $\varphi(U) \subset U$. Even if such a neighbourhood exists we do not know if φ is injective on U and if the derivative is invertible at every $z \in U$. Let us note that if V is a hyperbolic neighbourhood of $(a,b) \subseteq \mathbb{R}$ such that φ is holomorphic on V and $\varphi(V) \subset V$, with $\varphi : (a,b) \to (a,b)$ real analytic, then there are many neighbourhoods W with $\varphi(W) \subset W$: one can take, for instance, $\{z \in \Omega : k_V(z,(a,b)) < \varepsilon\}$ for every $\varepsilon > 0$ where k_V is the Kobayashi metric for V. But even in this situation it is not clear whether we can choose this neighbourhood in such a way that φ is injective there. In case $U \subset \mathbb{C}$ simply connected the condition in Proposition 3.2 is also sufficient for hypercyclicity of $C_\varphi : H(U) \to H(U)$ [21, Th. 3.21]. For similar results in many variable case see [34].

Although we cannot characterize hypercyclic composition operators $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$, we can do it under some additional assumptions. We start with some auxiliary results:

Lemma 3.3 *Let U be an open complete hyperbolic complex neighbourhood of $\Omega \subset \mathbb{R}^d$ in $\mathbb{C}^d$ such that $U \cap \mathbb{R}^d = \Omega$. If $\varphi : U \to U$, $\varphi(\Omega) \subset \Omega$, and φ runs away on Ω, then it runs away on U.*

Proof: Fix $x \in U$. Since U is hyperbolic complete, the Kobayashi disc $B_{k_U}(x,r)$ is relatively compact in U for every r and for every compact subset of U there is r such that the compact set is contained in $B_{k_U}(x,r)$. Let $K \Subset U$, $K \subset B_{k_U}(x,r)$. Thus

$$\varphi^n(K) \subset B_{k_U}(\varphi^n(x), r).$$

On the other hand, since φ runs away on $U \cap \mathbb{R}^d$ there exists $n \in \mathbb{N}$ such that

$$\varphi^n(B_{k_U}(x,3r) \cap \mathbb{R}^d) \cap B_{k_U}(x,3r) = \emptyset,$$

in particular, $\varphi^n(x) \notin B_{k_U}(x,3r)$. Therefore,

$$\varphi^n(K) \cap K \subset B_{k_U}(\varphi^n(x), r) \cap B_{k_U}(x,r) = \emptyset.$$

□

Proposition 3.4 *Let W be an arbitrary complex neighbourhood of $\mathbb{R}$ and let $\varphi : W \to W$, $\varphi(\mathbb{R}) \subset \mathbb{R}$, be holomorphic.*

(i) If $W \neq \mathbb{C}$, then there exists a complete hyperbolic complex neighbourhood $U \subset W$ of $\mathbb{R}$ such that $\varphi(U) \subset U$.

(ii) If in addition U is finitely connected, then there exists a simply connected complex neighbourhood $V \subset U$ of $\mathbb{R}$, symmetric with respect to the real axis such that $\varphi(V) \subset V$. Moreover, there is a biholomorphism $\psi : V \to \mathbb{D}$ such that $\psi(\mathbb{R}) = (-1, 1)$.

Proof: (i) Since $\varphi(\mathbb{R}) \subset \mathbb{R}$ we get $\varphi(\bar{z}) = \overline{\varphi(z)}$ for all $z, \bar{z} \in W$. Assume that $W = \mathbb{C} \setminus \{w\}$, $w \notin \mathbb{R}$. Select $z \in W$ with $\varphi(z) = \bar{w}$. If $\bar{z} \in W$, then $\varphi(\bar{z}) = \overline{\varphi(z)} = w$, which contradicts $\varphi(W) \subset W$. Therefore $\bar{z} \notin W$, hence $\bar{z} = w$. Setting $U := W \setminus \{\bar{w}\}$, we have $\varphi(U) \subset U$. We have proved that there exists a complex neighbourhood U of $\mathbb{R}$, $\varphi(U) \subset U$ such that $\mathbb{C} \setminus U$ contains at least 2 points. By [26, Cor. 3.7.3], U is complete hyperbolic.

(ii) Suppose now that U is finitely connected. If we take $\varepsilon > 0$ small enough, then

$$U_\varepsilon := \{z \in U : k_U(z, \mathbb{R}) < \varepsilon\}$$

is contained in a simply connected neighbourhood of $\mathbb{R}$ contained in U and it is clearly not equal $\mathbb{C}$. Set $V_1 := \{z \in U_\varepsilon | \bar{z} \in U_\varepsilon\}$, since $\varphi(\bar{z}) = \overline{\varphi(z)}$ we have $\varphi(V_1) \subset V_1$.

Let us define V to be V_1 together with its holes, i.e., all compact closed-open subsets of the complement. Since $V_1 \subset U_\varepsilon$ and U_ε is included in a simply connected neighbourhood of $\mathbb{R}$ contained in U, we have $V \subset U$. We claim that that $\varphi(V) \subset V$. Indeed, if φ is constant then it must be real and the result holds. If φ is not constant, then φ is open. Let L be any subset of $\mathbb{C} \setminus V_1$ closed-open in it and compact. We show that $\varphi(L)$ is contained in V. To see this, first observe that $\varphi(L) \cap (\mathbb{C} \setminus V_1) = \varphi(V_1 \cup L) \cap (\mathbb{C} \setminus V_1)$. Thus $\varphi(L) \cap (\mathbb{C} \setminus V_1)$ is both closed and open in $(\mathbb{C} \setminus V_1)$ so it is a connected component of the latter set, and in fact a bounded one. Such a component is just the hole of V_1 and so it is contained in V.

Since V is simply connected not equal to $\mathbb{C}$ there is a unique biholomorphism $\psi : V \to \mathbb{D}$ such that $\psi(0) = 0$ and $\psi'(0) > 0$. Since V is symmetric we can define $\psi_1 : V \to \mathbb{D}$, $\psi_1(z) = \overline{\psi(\bar{z})}$. It is biholomorphic and $\psi_1(0) = 0$, $\psi_1'(0) > 0$. Thus by uniqueness $\psi = \psi_1$. This implies that $\psi|_\mathbb{R} \subset \mathbb{R}$. □

Proposition 3.5 *Let $\varphi : \Omega \to \Omega$ be analytic, $\Omega \subset \mathbb{R}$ open. Assume that there is a finitely connected complex neighbourhood $U \neq \mathbb{C}$ of Ω such that φ extends holomorphically to U and $\varphi(U) \subset U$. If φ runs away on Ω and there is a complex neighbourhood $V \subset U$ of Ω such that φ^n is injective on V for every $n \in \mathbb{N}$, then $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is sequentially hypercyclic.*

Proof: Since $U \neq \mathbb{C}$, we can apply Proposition 3.4 and may assume that $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic and $\varphi((-1, 1)) \subset (-1, 1)$. Moreover, there is a complex neighbourhood $V \subset \mathbb{D}$ of $(-1, 1)$ such that φ^n are injective on V for every $n \in \mathbb{N}$. By Lemma 3.3, φ runs away on $\mathbb{D}$.

Without loss of generality we may assume that

$$V = \bigcup_{j \in \mathbb{N}} (K_j + \varepsilon_j \mathbb{D}),$$

where $(K_j)_{j \in \mathbb{N}}$ is a fundamental system of compact sets in $(-1, 1)$ and $(\varepsilon_j)_{j \in \mathbb{N}}$ is a sequence of positive numbers and $(K_j + \varepsilon_j \mathbb{D})$ is relatively compact in $\mathbb{D}$. In particular, V is simply connected.

Let $(p_k)_{k \in \mathbb{N}}$ be a dense sequence of polynomials in $H(V)$ equipped with the compact open topology. We define inductively a sequence of polynomials $(f_k)_{k \in \mathbb{N}}$ and an increasing sequence of positive integers $(n_j)_{j\in\mathbb{N}}$. Take $f_0 \equiv 0$. Assume that $f_0, f_1, \ldots, f_{k-1}$ are defined. Then

$$\left(\bigcup_{j<k} \varphi^{n_j}\left(\bigcup_{l\leq j}(K_l + \varepsilon_l \mathbb{D})\right)\right) \cup \left(\bigcup_{j\leq k}(K_j + \varepsilon_j \mathbb{D})\right) =: L_k$$

is an open relatively compact set in $\mathbb{D}$. We take M_k to be L_k with filled holes. It is still open and relatively compact in $\mathbb{D}$ and the sequence of M_k is increasing. Since φ runs away on $\mathbb{D}$ there is $n_k \in \mathbb{N}$ bigger than n_{k-1} such that

$$\varphi^{n_k}\left(\overline{\bigcup_{j\leq k}(K_j + \varepsilon_j \mathbb{D})}\right) \cap \overline{M_k} = \emptyset.$$

As $\bigcup_{j\leq k}(K_j + \varepsilon_j \mathbb{D})$ is simply connected and φ^{n_k} is injective then $\varphi^{n_k}\left(\bigcup_{j\leq k}(K_j + \varepsilon_j \mathbb{D})\right)$ is simply connected and the complement of

$$\varphi^{n_k}\left(\bigcup_{j\leq k}(K_j + \varepsilon_j \mathbb{D})\right) \cup M_k$$

is connected. By Runge theorem, there is a polynomial f_k such that

$$|f_k(z) - p_k \circ \varphi^{-n_k}(z)| \leq 2^{-k} \quad \text{for } z \in \varphi^{n_k}\left(\bigcup_{j\leq k}(K_j + \varepsilon_j \mathbb{D})\right)$$

while

$$|f_k(z) - f_{k-1}(z)| \leq 2^{-k} \quad \text{for } z \in M_k.$$

The sequence $(f_j)_{j\in\mathbb{N}}$ is uniformly convergent on every M_k thus its limit f is holomorphic on the union of $(M_k)_{k\in\mathbb{N}}$. Let $z \in \bigcup_{j\leq k}(K_j + \varepsilon_j \mathbb{D})$. Then

$$|f\circ\varphi^{n_k}(z) - p_k(z)| \leq |(f - f_k)\circ\varphi^{n_k}(z)| + |(f_k - p_k\circ\varphi^{-n_k})\circ\varphi^{n_k}(z)| \leq \sum_{m=k+1}^{\infty} 2^{-m} + 2^{-k} = 2^{-k+1}.$$

Thus the sequence $(f\circ\varphi^{n_k})_{k\in\mathbb{N}}$ is a dense sequence in $H(V)$ and thus it is sequentially dense in $\mathscr{A}((-1,1))$. □

Remark. We explain the difficulties to prove the converse of Proposition 3.5. Suppose that $\varphi : \Omega \to \Omega$ is real analytic on an open interval Ω of $\mathbb{R}$ such that there is complex neighbourhood U of Ω such that φ extends to U and $\varphi(U) \subset U$. If $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is sequentially hypercyclic, then φ runs away on Ω by Theorem 2.3. Now, let $f \in \mathscr{A}(\Omega)$ be a sequentially hypercyclic vector in $\mathscr{A}(\Omega)$ with respect to C_φ. There is an increasing sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ such that $f\circ\varphi^{k_n}$ (as a function defined on Ω) tends to g as n tends to ∞, with $g(z) = z$ in $\mathscr{A}(\Omega)$. Accordingly, there is a connected complex neighbourhood $V \subset U$ of Ω such that each $f\circ\varphi^{k_n}$ can be extended to a holomorphic function f_{k_n} on V and the extended sequence converges to $g(z) = z$ uniformly on the compact subsets of V. However, it is not clear a priori whether these extensions f_{k_n} actually coincide with the composition $f\circ\varphi^{k_n}$, since it is not even clear if the composition is defined. If the compositions were defined on the set V, e.g. in case f was an entire function, then we could conclude that all the iterates φ^n would be injective on V.

Proposition 3.4 implies that if $\varphi : \mathbb{R} \to \mathbb{R}$ extends to a self map on some complex finitely connected neighbourhood $U \neq \mathbb{C}$, then we can take without loss of generality $U = \mathbb{D}$ the unit disc and $\mathbb{R}$ corresponds to $(-1,1)$. That is why the following theorem is interesting and more general than it might look.

Theorem 3.6 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, $\varphi((-1,1)) \subset (-1,1)$. Then the following assertions are equivalent:*

(a) $C_\varphi : \mathscr{A}(-1,1) \to \mathscr{A}(-1,1)$ is sequentially hypercyclic;

(b) $C_\varphi : \mathscr{A}(-1,1) \to \mathscr{A}(-1,1)$ is hypercyclic;

(c) $C_\varphi : \mathscr{A}(-1,1) \to \mathscr{A}(-1,1)$ is topologically transitive;

(d) φ runs away on $(-1,1)$ and φ' does not vanish on $(-1,1)$;

(e) there is a simply connected complex neighbourhood W of $(-1,1)$ in $\mathbb{D}$ such that $\varphi(W) \subset W$, φ is injective on W and φ runs away on W;

(f) there is a simply connected complex neighbourhood W of $(-1,1)$ in $\mathbb{D}$ such that $\varphi(W) \subset W$ and $C_\varphi : H(W) \to H(W)$ is hypercyclic.

Remark. The implications (e)⇒(f)⇒(a)⇒(b)⇒(c)⇒(d) hold for every analytic $\varphi : (-1,1) \to (-1,1)$ without any assumption. The assumption that φ extends to a self map of $\mathbb{D}$ is needed only for (d)⇒(e). Example 4.2 shows that (d) does not imply (e) in general.

Every selfmap φ of $\mathbb{D}$ has either a fixed point in $\mathbb{D}$ or it has the so-called Denjoy-Wolff point $\tau \in \partial\mathbb{D}$, i.e., φ^n tends uniformly on compact subsets of $\mathbb{D}$ to τ [29, p. 78] or [30]. According to [29, p. 78, The Grand Iteration Theorem (b)] (see also [10]), all selfmaps φ of the unit disc can be classified as

1. *elliptic*: those with a fixed point inside the disc;
2. *hyperbolic*: those which have Denjoy-Wolff point $\tau \in \partial\mathbb{D}$ but the angular derivative at this point $\varphi'(\tau)$ is strictly smaller than 1;
3. *parabolic:* those which have Denjoy-Wolff point $\tau \in \partial\mathbb{D}$ but its angular derivative there $\varphi'(\tau) = 1$.

Recall that a disc internally tangent to the boundary of the unit disc at some point w is called a *horodisc at w*.

We need some auxiliary results.

Lemma 3.7 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, $\varphi((-1,1)) \subset (-1,1)$ and its real orbits tend to 1. Then for every compact set $K \subset \mathbb{D}$ and every horodisc h at 1 there is $N \in \mathbb{N}$ such that for every $n \geq N$ we have $\varphi^n(K) \subset h$.*

Proof: Let us define $w_0 = 0$, $w_{n+1} = \varphi(w_n)$. Then w_n tends to 1 radially. It is easily seen that for every $r < 1$ there is $N \in \mathbb{N}$ such that for every $n \geq N$ the hyperbolic ball $B_{k_\mathbb{D}}(w_n, r) \subset h$. Now, if $K \Subset \mathbb{D}$ then $K \subset B_{k_\mathbb{D}}(0, r)$ for some $r < 1$. Clearly, $\varphi^n(K) \subset B_{k_\mathbb{D}}(w_n, r) \subset h$. □

We thank M. Contreras for the following remark: The statement of Lemma 3.7 does not hold if we drop the assumption that $\varphi((-1,1)) \subset (-1,1)$. The map $T(z) := (1+z)/(1-z)$ transforms the unit disc in the right half plane in a bijective form taking 1 to ∞. Define $\varphi(z) := T^{-1}(T(z)+i)$

for $z \in \mathbb{D}$. The map φ is a selfmap of the unit disc whose Denjoy-Wolff point is 1, as all the orbits tend to 1. To see that the thesis of the lemma fails, we work in the right halfplane. The horodiscs are mapped by T into semiplanes of the form $\{w \in \mathbb{C} : \mathrm{Re}\,(z) > \alpha\}$, $\alpha > 0$. The function φ in the semiplane acts as $w \to w + i$, that keeps the horodiscs. However, the iterations of a fixed compact set K do not enter a horodisc unless they are already contained in it.

Lemma 3.8 *[29, Th. 5.3] Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, $\varphi((-1,1)) \subset (-1,1)$, and φ runs away on $(-1,1)$. Then $\varphi(h) \subset h$ for any horodisc h at the Denjoy-Wolff point of φ.*

Proof of Theorem 3.6: (a)⇒(b)⇒(c): Obvious.

(c)⇒(d): Theorem 2.3.

(d)⇒(e): By Lemma 3.3, φ runs away on $\mathbb{D}$. Thus, by the Denjoy-Wolff Theorem and the classification of self maps of $\mathbb{D}$ mentioned above (see [29, p. 78], [30] or [10, p.10, Th. 4.1]) it is of either hyperbolic or parabolic type and φ^n converges on compact sets uniformly to its Denjoy-Wolff point τ on the boundary which must be real in our case. Without loss of generality we assume that $\tau = 1$. Clearly then $\varphi(x) > x$ for every $x \in (-1,1)$. By [11, Th. 2.2] and [9, Lemma 5 and remarks below] (or [10, Th. 6.2, 6.3]) there is a simply connected set $V \subset \mathbb{D}$ such that $\varphi(V) \subset V$, φ is injective on V and for every compact $K \Subset \mathbb{D}$ there is $N \in \mathbb{N}$ such that $\varphi^n(K) \subset V$ holds for every $n \geq N$. By Lemmas 3.7 and 3.8 there are plenty of such sets since for any horodisc h at 1 the set $h \cap V$ also satisfies the above requirements.

We apply Lemma 2.1 to find a complex neighbourhood U of $(-1,1)$ in $\mathbb{D}$ such that $\varphi|_U$ is injective. Let us assume that $[x,1) \subset V$. We find a horodisc h at 1 such that

$$\rho(\varphi(x), h) > \frac{1}{2},$$

where ρ is the Poincaré metric on $\mathbb{D}$, [19] We find r, $0 < r < 1$, such that

$$\{z \in \mathbb{D} : \rho(z, [x, \varphi(x)]) < r\} \subset V.$$

Then we assume without loss of generality that

$$U \subset \left\{z \in \mathbb{D} : \rho(z, (-1, \varphi(x)]) < \frac{r}{4}\right\} \cup V.$$

Fix a fundamental sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ in the set $(-1, \varphi(x)]$. For every $n \in \mathbb{N}$ there is k_n such that

$$\varphi^{k_n}(K_n) \Subset V \cap h.$$

Take a complex neighbourhood W_n of K_n such that $\varphi^{k_n}(W_n) \subset V \cap h$ and

$$W_n \cup \varphi(W_n) \cup \cdots \cup \varphi^{k_n}(W_n) \subset U$$

and define

$$W = \tilde{W} \cup (V \cap h),$$

with

$$\tilde{W} = \bigcup_{n\in\mathbb{N}} W_n \cup \varphi(W_n) \cup \cdots \cup \varphi^{k_n}(W_n) \subset U.$$

Clearly, $\varphi(W) \subset W$, φ is injective both on $\tilde{W}$ and on $V \cap h$.

We want to show that φ is injective on W. Since it is injective on V it suffices to show that if $z \in \tilde{W} \setminus V$, $w \in V \cap h$ and $z \neq w$, then $\varphi(z) \neq \varphi(w)$.

If $z \in \tilde{W} \setminus V \subset U \setminus V$ thus

$$\rho(z, (-1, \varphi(x)]) < \frac{r}{4}.$$

If $\rho(z, [x, \varphi(x)]) < \frac{r}{4}$ then z would belong to V; a contradiction. Therefore

$$\rho(z, (-1, x]) < \frac{r}{4}.$$

Hence $\rho(\varphi(z), (-1, \varphi(x)]) < \frac{r}{4} < \frac{1}{4}$. Since $w \in V \cap h$, $\varphi(w) \in h$ (see Lemma 3.8) and since $\rho(\varphi(w), (-1, \varphi(x)]) > \frac{1}{2}$, we have

$$\rho(\varphi(w), \varphi(z)) > \frac{1}{2} - \frac{1}{4} > 0$$

and $\varphi(w) \neq \varphi(z)$.

We can make W simply connected by [13, Ex. 2.4.4]. The map φ runs away on W by Lemma 3.3.

(e)⇒(f): It is [21, Theorem 3.21].

(f)⇒(a): Obvious, since $H(W)$ is dense in $\mathscr{A}((-1, 1))$. □

We give an alternative proof of the implication (d)⇒(a) in Theorem 3.6: By Lemma 3.3, the map φ runs away on $\mathbb{D}$. Clearly it has no fixed point in $\mathbb{D}$ thus φ has to be either hyperbolic or parabolic.

Parabolic case. By [11, Th. 2.2], there is a holomorphic map $\sigma : \mathbb{D} \to \mathbb{C}$ such that

$$\sigma \circ \varphi(z) = \sigma(z) + 1$$

and from the construction in the cited paper it follows that $\sigma((-1, 1)) \subset \mathbb{R}$. Moreover, there is an open set $V \subset \mathbb{D}$ such that $\sigma|_V$ is injective and eventually every orbit of φ is in V.

First, we show that σ on $(-1, 1)$ is injective and its derivative never vanishes there. Indeed,

$$\sigma(\varphi^n(z)) = \sigma(z) + n \tag{1}$$

thus

$$\sigma'(\varphi^n(z)) \cdot (\varphi^n)'(z) = \sigma'(z).$$

Since φ' never vanishes on $(-1, 1)$ we get

$$\sigma'(z) = 0 \quad \text{implies} \quad \sigma'(\varphi^n(z)) = 0.$$

For sufficiently big $n \in \mathbb{N}$ we have $\varphi^n(z) \in V$ and $\sigma'(\varphi^n(z)) \neq 0$. So we have proved that $\sigma'(z)$ does not vanish for any $z \in (-1, 1)$ and consequently is strictly monotonic, i.e., injective on $(-1, 1)$. Now, by Lemma 2.1, there is a 1-connected complex neighbourhood U of $(-1, 1)$ in $\mathbb{D}$ such that σ is injective on U. By (1) also φ^n is injective on U.

Hyperbolic case. By [9, Part. 2.3] there is a holomorphic map $\sigma : \mathbb{D} \to \mathbb{C}$ such that

$$\sigma \circ \varphi(z) = \frac{1}{\lambda}\sigma(z), \qquad 0 < \lambda < 1.$$

In fact, the whole construction in [9] is done in the upper half plane instead of the unit disc. It is easily seen from the construction that the imaginary positive axis is mapped by σ into itself. Transferring into the unit disc we will get that $\sigma((-1, 1)) \subset (-1, 1)$. Moreover, by [9, Lemma 5 and remarks below], there is an open set $V \subset \mathbb{D}$ such that $\sigma|_V$ is injective and eventually every orbit of φ is in V.

We will show again that σ is injective on $(-1,1)$ and its derivative never vanishes there. Indeed,

$$\sigma(\varphi^n(z)) = \frac{1}{\lambda^n}\sigma(z) \tag{2}$$

thus

$$\sigma'(\varphi^n(z)) \cdot (\varphi^n)'(z) = \frac{1}{\lambda^n}\sigma'(z).$$

Since φ' never vanishes on $(-1,1)$ we get that if $\sigma'(z) = 0$ for $z \in (-1,1)$ then $\sigma'(\varphi^n(z)) = 0$. This cannot be the case since for sufficiently big $n \in \mathbb{N}$ we have $\varphi^n(z) \in V$ and σ' never vanishes on V. We have proved that σ' never vanishes on $(-1,1)$ and consequently it is strictly monotonic, i.e., injective on $(-1,1)$. Again by Lemma 2.1, there is a 1-connected complex neighbourhood U of $(-1,1)$ in $\mathbb{D}$ such that σ is injective on U. By (2), also φ^n is injective on U.

Summarizing, both in parabolic and hyperbolic cases we have constructed a 1-connected complex neighbourhood U of $(-1,1)$ in $\mathbb{D}$ such that φ^n are injective on U for every $n \in \mathbb{N}$. Use Proposition 3.5. □

We conclude this section with the following open question:

Problem 3.9 *Is there a hypercyclic (composition) operator $T : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$, $\Omega \subset \mathbb{R}^d$, which is not sequentially hypercyclic? Is there a transitive (composition) operator $T : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$, $\Omega \subset \mathbb{R}^d$, which is not hypercyclic?*

4 Examples

There are examples of self maps $\varphi : \mathbb{D} \to \mathbb{D}$ such that $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is *not* hypercyclic (since φ is not injective on $\mathbb{D}$) but $C_\varphi : \mathscr{A}((-1,1)) \to \mathscr{A}((-1,1))$ is hypercyclic.

Example 4.1 *A holomorphic non-injective map $\varphi : \mathbb{D} \to \mathbb{D}$ such that $C_\varphi : \mathscr{A}((-1,1)) \to \mathscr{A}((-1,1))$ is hypercyclic.*

Define

$$\varphi(z) := \frac{(z+1)^3}{32} + \frac{3}{4}.$$

Clearly $\varphi(\mathbb{D}) \subset \mathbb{D}$ and φ is not injective on $\mathbb{D}$ since $\varphi(\alpha e^{\frac{\pi}{3}i} - 1) = \varphi(\alpha e^{-\frac{\pi}{3}i} - 1)$ for suitably small $\alpha > 0$. We will prove that φ is a self map on

$$U := \mathbb{D} \cap \left\{ z = x + iy : |y| < \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} \right\}.$$

The map $z \mapsto z + 1$ maps U into

$$(\mathbb{D} + 1) \cap \left\{ z : |\arg z| < \frac{\pi}{6} \right\},$$

thus $z \mapsto \frac{(z+1)^3}{32}$ maps U injectively into

$$\left\{ z : \text{Re } z > 0, |z| < \frac{1}{4} \right\} =: V$$

Obviously $V + \frac{3}{4} \subset U$. We have proved that $\varphi(U) \subseteq U$.

Moreover, φ is strictly increasing on $(-1,1)$ and $\varphi(x) > x$ for all $x \in (-1,1)$. Thus for every $x \in (-1,1)$ its orbit $(\varphi^n(x))_{n\in\mathbb{N}}$ tends to 1 and φ runs away on $(-1,1)$. Hence φ is injective on U and runs away on U by Lemma 3.3. We can apply [21, Theorem 3.21] to conclude that $C_\varphi : H(U) \to H(U)$ is hypercyclic, so $C_\varphi : \mathscr{A}((-1,1)) \to \mathscr{A}((-1,1))$ is hypercyclic.

Our next example shows that the implication (d) implies (e) in Theorem 3.6 does not hold in general.

Example 4.2 *A topologically transitive composition operator $C_\varphi : \mathscr{A}((0,1)) \to \mathscr{A}((0,1))$ such that there is no complex neighbourhood U of $(0,1)$ such that φ^n is injective on U for every $n \in \mathbb{N}$. In particular, there is no open complex neighbourhood W of $(0,1)$ such that φ is a self map on W and C_φ is hypercyclic on $H(W)$. We are thankful to the referee for giving us a much easier and nicer example than our original one.*

Consider the map $\varphi(z) := z^2$ on $(0,1)$ (or equivalently $\varphi(z) := (z^2 + 2z - 1)/2$ on $(-1,1)$). The operator $C_\varphi : \mathscr{A}((-1,1)) \to \mathscr{A}((-1,1))$ is topologically transitive by Theorem 2.3 and Proposition 1.1. However, there is no complex neighbourhood U of $(0,1)$ such that $\varphi^n(z) = z^{2^n}$ is injective on U for every $n \in \mathbb{N}$. Note that $\varphi^n(\frac{1}{2}\exp(\pm 2\pi i/2^n)) = 1/2^{2^n}$ for each $n \in \mathbb{N}$.

Problem 4.3 *(1) Is the operator $C_\varphi : \mathscr{A}((0,1)) \to \mathscr{A}((0,1))$, $\varphi(z) = z^2$, (sequentially) hypercyclic?*

(2) Is the operator $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$, $\varphi(z) = \exp(z)$, (sequentially) hypercyclic? It is topologically transitive by Theorem 2.3.

(3) Does every sequentially hypercyclic operator $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ satisfy that there is complex neighbourhood U of Ω such that φ extends holomorphically to U, $\varphi(U) \subset U$ and $C_\varphi : H(U) \to H(U)$ is hypercyclic? Compare with Theorem 3.6 (a)⇒(f) and with the Remark after Proposition 3.5.

The following family of self maps $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi((-1,1)) \subset (-1,1)$ has been investigated by Cowen and Ko [12] in connection with Hermitian weighted composition operators on H^2.

Example 4.4 Take $a_0, a_1 \in \mathbb{R}$ and define

$$\varphi(z) := a_0 + \frac{a_1 z}{1 - a_0 z}, \ z \in \mathbb{D}.$$

Then φ is a linear fractional map with real coefficients. By [12, Lemma 2.2 and Corollary 2.3], φ maps the unit disc into itself if and only if

$$|a_0| < 1 \quad \text{and} \quad -1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2.$$

Moreover, by [14, Theorem 10], this is equivalent to the fact that φ maps the interval $(-1,1)$ into itself. We want to study the operator $C_\varphi : \mathscr{A}((-1,1)) \to \mathscr{A}((-1,1))$. To do this, it is enough to consider the case $0 \leq a_0 < 1$. Otherwise select, as in [12, proof of Corollary 2.3], θ such that $\tilde{\varphi}(z) := e^{i\theta}\varphi(e^{-i\theta}z) = |a_0| + a_1 z/(1 - |a_0|z)$. We also suppose that $a_1 \neq 0$, since otherwise $\varphi(z) = a_0$ for each $z \in \mathbb{D}$. Clearly $\varphi'(z) = a_1/(1 - a_0 z)^2$, hence $\varphi'(x) \neq 0, x \in (-1,1)$, and $a_1\varphi'(x) > 0$ for each $x \in (-1,1)$. By Proposition 1.1 and Theorem 3.6 $C_\varphi : \mathscr{A}((-1,1)) \to \mathscr{A}((-1,1))$ is (sequentially) hypercyclic if and only if φ has no fixed point on $(-1,1)$. This is easily seen to be equivalent to $0 < a_0 < 1$ and $a_1 = (1 - a_0)^2$. In the other cases, C_φ is not even topologically transitive. A finer analysis of the dynamics of C_φ requires the distinction of three cases.

Case 1. $0 \leq a_0 < 1$ and $a_1 = -1 + a_0^2$.

In this case $\varphi(z) = (a_0 - z)/(1 - a_0 z)$ is an automorphism of $\mathbb{D}$ such that $\varphi(1) = -1$ and $\varphi(-1) = 1$. Moreover $1 - (1 - a_0)^{1/2} \in [0, 1)$ is a fixed point. Thus φ is elliptic and, by [7, Theorem 2.8], the operator $C_\varphi : \mathscr{A}((-1, 1)) \to \mathscr{A}((-1, 1))$ is power bounded and mean ergodic.

Case 2. $0 \le a_0 < 1$ and $-1 + a_0^2 < a_1 < (1 - a_0)^2$.

In this case $-1 < \varphi(-1) < 1$, $-1 < \varphi(1) < 1$ and φ maps the closed unit disc $\overline{\mathbb{D}}$ into the open unit disc; see page 5778 in [12]. Therefore C_φ is power bounded on $H(\mathbb{D})$ and on $\mathscr{A}((-1, 1))$ by [6] and by [7, Theorems 2.1 and 2.2].

Case 3. $0 \le a_0 < 1$ and $a_1 = (1 - a_0)^2$.

If $a_0 = 0$, then $\varphi(z) = z, z \in \mathbb{D}$, and C_φ coincides with the identity on $\mathscr{A}((-1, 1))$. In case $0 < a_0 < 1$, we get $\varphi'(x) > 0$ for each $x \in (-1, 1)$, φ is not an automorphism of the disc, $\varphi(1) = 1, -1 < \varphi(-1) < 1$, there are no fixed points in $(-1, 1)$ and $C_\varphi : \mathscr{A}((-1, 1)) \to \mathscr{A}((-1, 1))$ is sequentially hypercyclic, as we mentioned above.

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Authors' Addresses:

J. Bonet
Instituto Universitario de Matemática Pura y Aplicada IUMPA
Universitat Politècnica de València
E-46071 Valencia, SPAIN
e-mail: jbonet@mat.upv.es

P. Domański
Faculty of Mathematics and Comp. Sci.
A. Mickiewicz University Poznań
Umultowska 87
61-614 Poznań, POLAND
e-mail: domanski@amu.edu.pl