Convergence of arithmetic means of operators on Fréchet spaces

José Bonet

Instituto Universitario de Matemática Pura y Aplicada IUMPA

Universitat Politècnica de València

Integration, Vector Measures and Related Topics V,

Palermo (Italy), August 2012

Joint work with A.A. Albanese (Lecce, Italy) and W.J. Ricker (Eichstaett, Germany)

Problem

Study the convergence of arithmetic means of iterates T^n of operators T defined on Fréchet spaces

We are mainly interested in possible extensions of a result due to Lin (1974).

A result of Lin

Theorem. Lin. 1974.

Let T a (continuous) operator on a Banach space X which satisfies $\lim_{n\to\infty} ||T^n/n|| = 0$. The following conditions are equivalent:

(1) T is uniformly mean ergodic, i.e., there is $P \in \mathcal{L}(X)$ with $\lim_{n\to\infty} \|\frac{1}{n}\sum_{m=1}^{n} T^m - P\| = 0$.

(2) The range $(I-T)(X)$ is closed and $X = \mathrm{Ker}(I-T) \oplus (I-T)(X)$.

(3) $(I-T)^2(X)$ is closed.

(4) $(I-T)(X)$ is closed.

Definitions

X **is a Hausdorff locally convex space (lcs).**

$\mathcal{L}(X)$ **is the space of all continuous linear operators on** X**.**

$\mathcal{L}_b(X)$ **is the space of all continuous linear operators on** X **endowed with the topology of uniform convergence on the bounded sets. If** X **is Banach, this is the operator norm.**

For $T \in \mathcal{L}(X)$**, we set** $T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m$.

Definitions

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Banach space, an operator T is power bounded if and only if $\sup_n ||T^n|| < \infty$.

If X is a Fréchet space, an operator T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded. This is a consequence of the uniform boundedness principle.

Definitions

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X, \tag{1}$$

exist in X.

Uniformly mean ergodic operators

If $\{T_{[n]}\}_{n=1}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$ to $P \in \mathcal{L}(X)$, then T is called *uniformly mean ergodic*.

Lin's result in action. The discrete Cesàro operator on Banach sequence spaces

The *Cesàro operator* C is defined for a sequence $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ of complex numbers by

$$C(x) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_n , \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

We examine properties of the operator C in the classical sequence spaces c_0, c and ℓ^p $(1 < p \leq \infty)$.

Clearly C is not continuous on ℓ_1, since $C(e_1) = (1, 1/2, 1/3, ...)$.

Lin's result in action. The discrete Cesàro operator on Banach sequence spaces

Let $1 < p < \infty$. The Cesàro operator maps the Banach space ℓ^p continuously into itself, which we denote by $C^{(p)}\colon \ell^p \to \ell^p$, and $\|(C^{(p)})^n\| = (p')^n$, where $\frac{1}{p} + \frac{1}{p'} = 1$, for all $n \in \mathbb{N}$ (Hardy).

Proposition

The Cesàro operator $C^{(p)}\colon \ell^p \to \ell^p$ fails to be power bounded and is not mean ergodic. Moreover, $\mathrm{Ker}(I - C^{(p)}) = \{0\}$ and $(I - C^{(p)})(\ell^p) = \overline{\mathrm{span}}\{e_r\}_{r \geq 2} = \{x \in \ell^p : \ x_1 = 0\}$ is closed.

Observe that $C^{(p)}$ fails the assumption $\sup_n \|(C^{(p)})^n/n\| < \infty$ in Lin's theorem.

Lin's result in action. The discrete Cesàro operator on Banach sequence spaces

The Cesàro operators $C^{(\infty)}\colon \ell^\infty \to \ell^\infty$, $C^{(c)}\colon c \to c$ and $C^{(0)}\colon c_0 \to c_0$ are continuous, and $\|C^{(\infty)}\| = \|C^{(c)}\| = \|C^{(0)}\| = 1$.

Proposition

The Cesàro operators $C^{(\infty)}\colon \ell^\infty \to \ell^\infty$, $C^{(c)}\colon c \to c$ and $C^{(0)}\colon c_0 \to c_0$ are power bounded but not mean ergodic. Moreover,

$$\mathrm{Ker}(I - C^{(0)}) = \{0\},\ \mathrm{Ker}(I - C^{(\infty)}) = \mathrm{Ker}(I - C^{(c)}) = \mathrm{span}\{\mathbf{1}\},$$

and $(I - C^{(0)})(c_0)$ is not closed with

$$\overline{(I - C^{(0)})(c_0)} = \overline{\mathrm{span}}\{e_r\}_{r\geq 2} = \{x \in c_0 :\ x_1 = 0\}. \tag{2}$$

Lin's result in action. The discrete Cesàro operator on Banach sequence spaces

Idea of the proof for $C^{(0)}$.

This operator was investigated by Leibowitz in 1972.

If $C^{(0)}$ were mean ergodic, then we would have

$$c_0 = \mathrm{Ker}(I - C^{(0)}) \oplus \overline{(I - C^{(0)})(c_0)} = \{0\} \oplus \overline{\mathrm{span}}\{e_r\}_{r\geq 2}$$

which is *not* the case. Accordingly, $C^{(0)}$ is power bounded but not mean ergodic.
Moreover, $(I - C^{(0)})(c_0)$ is not closed. If it were, then the fact that $\lim_{n\to\infty} \|(C^{(0)})^n/n\| = 0$ together with Lin's theorem would imply that $C^{(0)}$ is uniformly mean ergodic and hence, also mean ergodic. But, this is not the case. So, $(I - C^{(0)})(c_0)$ is not closed.

The discrete Cesàro operator on the space $\mathbb{C}^{\mathbb{N}}$ of all sequences

The operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself with

$$C^{-1}(y) = (ny_n - (n-1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^{\mathbb{N}}, \tag{3}$$

where we set $y_{-1} := 0$.

Proposition

The Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded, uniformly mean ergodic and satisfies $\frac{C^n}{n} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. Moreover, $\mathrm{Ker}(I - C) = \mathrm{span}\{\mathbf{1}\}$ and $(I - C)(\mathbb{C}^{\mathbb{N}}) = \{x \in \mathbb{C}^{\mathbb{N}} : \ x_1 = 0\} = \overline{\mathrm{span}}\{e_r\}_{r\geq 2}$ is closed.

The discrete Cesàro operator on the space $\mathbb{C}^{\mathbb{N}}$ of all sequences

Remarks about the proof of the Proposition:

- The seminorms defining the Fréchet topology in $\mathbb{C}^{\mathbb{N}}$ are $q_k(x) = \max_{1 \leq j \leq k} |x_j|$, $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$. Recall that $\mathbb{C}^{\mathbb{N}}$ is Montel. The operator C is continuous and power bounded follows since

$$q_k(C^n x) \leq q_k(x), \quad x \in X, k, \ n \in \mathbb{N}. \tag{4}$$

- **Why is C uniformly mean ergodic?**
- Direct calculations show

$$\mathrm{span}\{e_r\}_{r \geq 2} \subset (I - C)(X) \subset \{x \in X : \ x_1 = 0\}.$$

Why is $(I - C)(\mathbb{C}^{\mathbb{N}})$ closed?

Yosida's mean ergodic Theorem

- **Barrelled locally convex spaces**
- $\mathcal{L}_s(X), \quad \mathcal{L}_b(X)$

Yosida, 1960

Let X be a barrelled lcs. The operator $T \in \mathcal{L}(X)$ is mean ergodic if and only if $\lim_{n\to\infty} \frac{1}{n} T^n = 0$ in $\mathcal{L}_s(X)$ and

$$\{T_{[n]}x\}_{n=1}^{\infty} \text{ is relatively } \sigma(X, X')\text{–compact, } \forall x \in X. \qquad (5)$$

Setting $P := \tau_s\text{-}\lim_{n\to\infty} T_{[n]}$, the operator P is a projection which commutes with T and satisfies $\mathrm{Im}(P) = \mathrm{Ker}(I - T)$ and $\mathrm{Ker}(P) = \overline{\mathrm{Im}(I - T)}$.

Yosida's mean ergodic Theorem

Corollary

- If X is a reflexive lcs, then every power bounded operator on X is mean ergodic.
- If X is a Montel space, then every power bounded operator on X is uniformly mean ergodic.

This explains why $C : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is uniformly mean ergodic.

Results of Albanese, Bonet, Ricker, 2008

Extensions of important results due to **Fonf, Lin, Wojtaszczyk, J. Funct. Anal. 2001** for Banach spaces

- Let X be a complete barrelled lcs with a Schauder basis. Then X is reflexive if and only if every power bounded operator on X is mean ergodic.
- Let X be a complete barrelled lcs with a Schauder basis. Then X is Montel if and only if every power bounded operator on X is uniformly mean ergodic.
- Let X be a sequentially complete lcs which contains an isomorphic copy of the Banach space c_0. Then there exists a power bounded operator on X which is not mean ergodic.

Results of Albanese, Bonet, Ricker, 2008

Results about Schauder basis

- A complete, barrelled lcs with a basis is reflexive if and only if every basis is shrinking if and only if every basis is boundedly complete.

  This extends a result of Zippin for Banach spaces and answers positively a problem of Kalton from 1970.

- Every non-reflexive Fréchet space contains a non-reflexive, closed subspace with a basis.

  This is an extension of a result of A. Pelczynski for Banach spaces.

Bonet, de Pagter and Ricker (2011)

A Fréchet lattice X is reflexive if and only if every power bounded operator on X is mean ergodic.

This is an extension of the result of Emelyanov.

Lin's result cannot be extended to Fréchet spaces

Lin's result fails for operators on Köthe echelon spaces:

Every Köthe echelon Fréchet space X with a continuous norm which is a Schwartz space admits a power bounded uniformly mean ergodic continuous linear operator T such that $I - T$ does not have closed range.

Köthe echelon spaces

A sequence $A = (a_n)_n$ of functions $a_n : \mathbb{N} \to [0, \infty)$ is called a **Köthe matrix** on $\mathbb{N}$ if

- $0 \leq a_n(i) \leq a_{n+1}(i)$ for all $i, \ n \in \mathbb{N}$, and
- For each $i \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $a_n(i) > 0$.

To each Köthe matrix A we associate the **Köthe echelon space**

$$\lambda_1(A) := \left\{ x \in \mathbb{C}^{\mathbb{N}} : q_n(x) := \sum_{i \in \mathbb{N}} |a_n(i) x_i| < \infty, \quad \forall n \in \mathbb{N} \right\}. \qquad (6)$$

It is a Fréchet spaces with the continuous seminorms $(q_n)_n$. The space $\lambda_1(A)$ is Schwartz if and only if for each n there is m such that $a_n / a_m \in c_0$.

Lin's result cannot be extended to Köthe spaces

Theorem

Let A be a Köthe matrix on $\mathbb{N}$ such that $a_1(i) > 0$ for each $i \in \mathbb{N}$. If $\lambda_1(A)$ is a Schwartz space, then there is a diagonal operator $T \in \mathcal{L}(\lambda_1(A))$ which is power bounded and uniformly mean ergodic, but $I - T$ is not surjective and has dense range. In particular, $(I - T)(\lambda_1(A))$ is not closed.

Corollary

Let X be a Fréchet space such that $X = X_0 \oplus X_1$ with X_0 topologically isomorphic to a Köthe Schwartz space $\lambda_1(A)$ admitting a continuous norm. Then X admits a power bounded uniformly mean ergodic operator T such that $I - T$ is not surjective and has dense range.

Idea of the proof

- The operator $T\colon \lambda_1(A) \to \lambda_1(A)$ is defined by

$$T((x_i)_i) := ((1-\eta_i)x_i)_i, \;\; x = (x_i)_i \in \lambda_1(A)$$

  for an appropriate $\eta := (\eta_i)_i \in \lambda_1(A)$, $0 < \eta_i < 1, i \in \mathbb{N}$.

- It is power bounded since $q_n(T^m x) \leq q_n(x)$, for $x \in \lambda_1(A)$ and m, $n \in \mathbb{N}$. Since $\lambda_1(A)$ is a Schwartz space, T is uniformly mean ergodic by the corollaries of Yosida's Theorem.

- It is easy to see that $(I - T)(\lambda_1(A))$ contains the canonical basis vectors, hence it is dense.

- If $I - T$ were surjective, then there would be $x \in \lambda_p(A)$ such that $(I - T)x = \eta$. Thus, $x_i - (1-\eta_i)x_i = \eta_i$ for all $i \in \mathbb{N}$, i.e., $x_i = 1$ for all $i \in \mathbb{N}$. This contradicts the fact that $\lambda_1(A)$ is Schwartz.

Lin's result holds for a class of Fréchet spaces. Quojections.

- A **quojection** is a Fréchet space such that every quotient with a continuous norm is a Banach space.
- Banach spaces and countable products of Banach spaces are clearly quojections. Every quojection is the quotient of a countable product of Banach spaces.
- Moscatelli 1980 gave the first examples of quojections which are not isomorphic to countable products of Banach spaces.
- **Examples:** $\mathbb{C}^{\mathbb{N}}$, $L^p_{loc}(\Omega), 1 \leq p \leq \infty$ and $C^{(m)}(\Omega), m \in \mathbb{N}_0$ with Ω an open subset of $\mathbb{R}^n$.
- The spaces of continuous functions $C(X)$, with X a σ–compact completely regular topological space, endowed with the compact open topology, are also examples of quojections.

Lin's result holds for a class of Fréchet spaces. Prequojections.

- A **prequojection** is a Fréchet space X such that X'' is a quojection. Every quojection is a prequojection.
- A Fréchet space X is a prequojection if and only if X has no Köthe nuclear quotient which admits a continuous norm. Bellenot, Dubinski, Önal, Terzioğlu, 1982-90.
- There exist prequojections which are not quojections. Behrends, S. Dierolf, Harmand, Moscatelli, 1986-90.
- (Pre)quojections have been relevant in the splitting of short exact sequences of Fréchet spaces, topological tensor products, spaces of vector valued functions,...

Main positive result

Theorem

Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$ such that $\tau_b\text{-}\lim_{n\to\infty} \frac{T^n}{n} = 0$. The following conditions are equivalent.

(1) T is uniformly mean ergodic.

(2) $(I - T)(X)$ is closed and $X = \mathrm{Ker}(I - T) \oplus (I - T)(X)$.

(3) $(I - T)^2(X)$ is closed.

(4) $(I - T)(X)$ is closed.

This explains why $(I - C)(\mathbb{C}^{\mathbb{N}})$ is closed for the Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$.

Comments about the proof

- The result is first proved for quojections and then, using duality, for prequojections.
- (2)$\Rightarrow$(3) and (3)$\Rightarrow$(4) can be obtained adapting the argument of Lin and hold for arbitrary locally convex spaces.
- The implications (1)$\Rightarrow$(2) and (4)$\Rightarrow$(1) are more difficult. They require writing the quojection X as a projective limit of Banach spaces $(X_k)_k$ with surjective linking maps such that the operator T induces operators $T_k \in \mathcal{L}(X_k)$ with specific properties.

The Browder equality and Uniform Mean Ergodicity

Let X be a locally convex space and $T \in \mathcal{L}(X)$. We denote

$$S(T) := \left\{ x \in X : \left\{ \sum_{k=1}^{n} T^k x \right\}_{n=1}^{\infty} \in \mathcal{B}(X) \right\}.$$

The identity (when it holds)

$$S(T) = (I - T)(X) \qquad (7)$$

is called **Browder's equality**.

Inspired by the work of Fonf, Lin and Wojtaszczyk, we investigate the validity of (**??**) in relation to uniform mean ergodicity of T.

Proposition

Let $T \in \mathcal{L}(X)$ be a power bounded operator. Then

$$(I - T)(X) \subset S(T) \subset \overline{(I - T)(X)}. \tag{8}$$

In particular, if $S(T)$ is closed, then

$$S(T) = \overline{(I - T)(X)} = \{x \in X : \lim_{n\to\infty} T_{[n]}x = 0\}.$$

Lin and Sine 1983 showed the existence of a mean ergodic operator T acting on a Banach space X for which neither of the inclusions in (**??**) is an equality.

The Browder equality and Uniform Mean Ergodicity

Proposition

Let $T \in \mathcal{L}(X)$ be a power bounded operator.

(1) If X is semireflexive, then $S(T) = (I - T)(X)$.

(2) Suppose that there exist a barrelled lcHs Y and $R \in \mathcal{L}(Y)$ such that $X = Y'$ and $T = R'$. If either the topology η of X is compatible with the duality $\langle Y, Y' \rangle$, or $\eta = \beta(Y', Y)$, then $S(T) = (I - T)(X)$.

(3) Let X be a prequojection Fréchet space. T is uniformly mean ergodic if and only if $S(T)$ is a complemented closed subspace of X.

Part (3) does not hold in general for operators on Köthe echelon spaces that are Schwartz. It may happen that T is uniformly mean ergodic, $S(T) = (I - T)(X)$ but $S(T)$ is not closed.

Proposition

A Fréchet space X with a Schauder basis is reflexive if and only if every power bounded operator $T \in \mathcal{L}(X)$ satisfies

$$S(T) = (I - T)[\overline{(I - T)(X)}].$$

References

1. **A. A. Albanese, J. Bonet, W. J. Ricker,** Mean Ergodic Operators in Fréchet Spaces, Anal. Acad. Math. Sci. Fenn. Math. 34 (2009), 401-436.
2. **A. A. Albanese, J. Bonet, W. J. Ricker,** C_0–semigroups and mean ergodic operators in a class of Fréchet spaces, J. Math. Anal. Appl. 365 (2010), 142–157.
3. **A. A. Albanese, J. Bonet, W. J. Ricker,** Convergence of arithmetic means of operators in Fréchet spaces, Preprint 2012.