UNIFORM MEAN ERGODICITY OF C_0–SEMIGROUPS IN A CLASS OF FRÉCHET SPACES

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Abstract. Let $(T(t))_{t\geq 0}$ be a strongly continuous C_0–semigroup of bounded linear operators on a Banach space X such that $\lim_{t\to\infty}\|T(t)/t\| = 0$. Characterizations of when $(T(t))_{t\geq 0}$ is uniformly mean ergodic, i.e., of when its Cesàro means $r^{-1}\int_0^r T(s)\,ds$ converge in operator norm as $r\to\infty$, are known. For instance, this is so if and only if the infinitesimal generator A has closed range in X if and only if $\lim_{\lambda\downarrow 0^+}\lambda R(\lambda, A)$ exists in the operator norm topology (where $R(\lambda, A)$ is the resolvent operator of A at λ). These characterizations, and others, are shown to remain valid in the class of quojection Fréchet spaces, which includes all Banach spaces, countable products of Banach spaces, and many more. It is shown that the extension fails to hold for all Fréchet spaces. Applications of the results to concrete examples of C_0–semigroups in particular Fréchet function and sequence spaces are presented.

1. Introduction.

Let $(T(t))_{t\geq 0}$ be a 1–parameter C_0–semigroup of continuous linear operators in a Banach space X. Ergodic theorems have a long tradition and are usually formulated via existence of the limits of the Cesàro averages $C(r)x = \frac{1}{r}\int_0^r T(t)x\,dt$, $r > 0$, or of the Abel averages $\lambda R_\lambda x = \lambda\int_0^\infty e^{-\lambda t}T(t)x\,dt$, $\lambda > 0$, for each $x \in X$, when $r \to \infty$ and $\lambda \to 0^+$, respectively. In the former case one speaks of the mean ergodicity of $(T(t))_{t\geq 0}$ and in the latter case of its Abel mean ergodicity; for the general theory and applications see [11, Ch.4], [20, Ch.VIII], [23, Ch.V], [24, Ch.XVIII], [31] and the references therein. Of course, the above convergence is relative to the *strong operator topology* τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators on X. The following fundamental result characterizing the mean ergodicity (resp. Abel mean ergodicity) of $(T(t))_{t\geq 0}$ for the *operator norm convergence* in $\mathcal{L}(X)$, in which case one speaks of uniform mean ergodicity (resp. uniform Abel mean ergodicity), is due to M. Lin; see [33, Theorem & Corollary 1], [34, Theorem 12].

Theorem 1.1. *Let X be a Banach space and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a strongly continuous C_0–semigroup with $T(0) = I$ satisfying $\lim_{t\to\infty}\left\|\frac{T(t)}{t}\right\| = 0$. The following assertions are equivalent.*

1. *$\lim_{r\to\infty} C(r)$ exists for the operator norm topology in $\mathcal{L}(X)$.*
2. *The range $\mathrm{Im}A$ of the infinitesimal generator A of $(T(t))_{t\geq 0}$ is a closed subspace of X.*
3. *$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N R_1^n$ exists for the operator norm topology in $\mathcal{L}(X)$.*
4. *There exists a projection $P \in \mathcal{L}(X)$ with $\mathrm{Im}P = \{x \in X : T(t)x = x\ \forall t \geq 0\}$ such that $\lim_{\lambda\downarrow 0^+}\|\lambda R_\lambda - P\| = 0$.*
5. *$\lim_{n\to\infty}(\lambda R_\lambda)^n$ exists for the operator norm topology in $\mathcal{L}(X)$ for (some) all $\lambda > 0$.*

Key words and phrases. C_0-semigroup, Quojection Fréchet space, Prequojection Fréchet space, Uniform mean ergodicity.

Mathematics Subject Classification 2000: Primary 46A04, 47A35, 47D03; Secondary 46A11.

* Research partially supported by MEC and FEDER Project MTM 2007-62643, GV Project Prometeo/2008/101 and the net MTM 2007–30904–E (Spain).

(6) *There exists* $\lambda_0 > 0$ *such that*

$$\sup_{0<\lambda\le\lambda_0} \|R_\lambda y\| < \infty, \quad y \in \overline{\mathrm{Im} A}.$$

Much of modern analysis occurs in locally convex Hausdorff spaces (briefly, lcHs) which are *non–normable*. The notions of a C_0–semigroup $(T(t))_{t\ge 0} \subseteq \mathcal{L}(X)$ (with X a Banach space) being mean ergodic or Abel mean ergodic relative to τ_s are purely topological and so carry over immediately to the setting when X is a lcHs. The natural analogue of the operator norm topology in $\mathcal{L}(X)$ is the topology τ_b of uniform convergence on the bounded subsets of the lcHs X. Accordingly, the notions of $(T(t))_{t\ge 0}$ being uniformly mean ergodic (resp. uniformly Abel mean ergodic), i.e., relative to τ_b, are also defined. For certain classical aspects of the theory of mean ergodic semigroups of operators, relative to τ_s, in the non–normable setting we refer to [21], [31, Ch.2], [44, Ch.III, §7] and the references therein. Further recent results on this topic occur in [6], [7], [9]. The aim of this paper is to clarify the role of Theorem 1.1 in the setting of lcHs. Some relevant comments in this respect are appropriate.

Leaving the Banach space setting brings with it various inherent (unpleasant) features. For instance, given any strongly continuous C_0–semigroup $(T(t))_{t\ge 0}$ in a Banach space X there always exists $\omega > 0$ such that the semigroup $(e^{-\omega t}T(t))_{t\ge 0}$ is uniformly bounded, i.e., $\sup_{t\ge 0} e^{-\omega t}\|T(t)\| < \infty$, [23, Ch.I Proposition 5.5]. Already in non–normable (lc–)Fréchet spaces X this need *not* be the case (cf. [6], [7], [27], [28], [41]), i.e., $(e^{-\omega t}T(t))_{t\ge 0}$ may fail to be an equicontinuous subset of $\mathcal{L}(X)$ for every $\omega > 0$. So, the general theory of C_0–semigroups in Fréchet spaces is more involved than in Banach spaces. The infinitesimal generator A of $(T(t))_{t\ge 0}$ is always a closed linear operator (not necessarily everywhere defined). In the Banach space setting the resolvent set $\rho(A)$ of A is always non–empty and open, [23, Ch.II Theorem 1.10 & Ch. IV. Proposition 1.3]; not necessarily so if X is a Fréchet space, [7, Example 3.5(vii)]. It can even happen, for X a non–normable Fréchet space, that $\rho(A) = \emptyset$; see Propostion 4.6 below! Moreover, some of the basic techniques for Banach spaces which are crucial for establishing various uniform mean ergodic theorems (eg. if the resolvent operators of A satisfy $\|R(\lambda, A)\| \to 0$ as $\lambda \to 0^+$, then $I - R(\lambda, A)$ is invertible in $\mathcal{L}(X)$ for all λ small enough, or the inequality $\mathrm{dist}(\lambda, \sigma(A)) \ge \frac{1}{\|R(\lambda,A)\|}$ for $\lambda \in \rho(A)$, or that $\rho(A)$ is the natural (open) domain in which $R(\cdot, A)$ is holomorphic) are not always available in non–normable Fréchet spaces. So, one cannot expect Theorem 1.1 to carry over to general lcHs X. In fact, it does not even extend to general Fréchet spaces; see Example 3.7 below.

Despite the negative comments made above it turns out, nevertheless, that Theorem 1.1 *does* have a natural extension (cf. Theorem 3.2) to an important and non–trivial class of Fréchet spaces, namely the *quojections*; see Section 3 for the definition of this class. All Banach spaces, all countable products of Banach spaces, and many more Fréchet spaces are quojections. Concrete examples of quojections include the sequence space $\omega = \mathbb{C}^{\mathbb{N}}$, the function spaces $L^p_{\mathrm{loc}}(\Omega)$, with $1 \le p \le \infty$ and $\Omega \subseteq \mathbb{R}^N$ and open set, and $C^{(m)}(\Omega)$ with $m \in \mathbb{N}_0$ and $\Omega \subseteq \mathbb{R}^N$ an open set, when equipped with their canonical lc–topology. As alluded to above, Theorem 3.2 is the main result of the paper. A further version of Theorem 3.2 is also presented in Section 3, namely to the class of *prequojection* Fréchet spaces (which properly contains the quojections). Section 2 is devoted to establishing various preliminary results needed in the sequel, many of interest in their own right. The final Section 4 presents some examples of concrete C_0–semigroups acting in particular quojection Fréchet spaces, with the aim of determining whether (or not) they are mean ergodic/uniformly mean ergodic.

2. Preliminaries.

Let X be a lcHs and Γ_X a system of continuous seminorms determining the topology of X. The strong operator topology τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators from X into itself (from X into another lcHs Y we write $\mathcal{L}(X,Y)$) is determined by the family of seminorms $q_x(S) := q(Sx)$, for $S \in \mathcal{L}(X)$, for each $x \in X$ and $q \in \Gamma_X$, in which case we write $\mathcal{L}_s(X)$. Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X. The topology τ_b of uniform convergence on bounded sets is defined in $\mathcal{L}(X)$ via the seminorms $q_B(S) := \sup_{x\in B} q(Sx)$, for $S \in \mathcal{L}(X)$, for each $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$; in this case we write $\mathcal{L}_b(X)$. For X a Banach space, τ_b is the operator norm topology in $\mathcal{L}(X)$. If Γ_X is countable and X is complete, then X is called a Fréchet space. The identity operator on a lcHs X is denoted by I.

By X_σ we denote X equipped with its weak topology $\sigma(X, X')$, where X' is the topological dual space of X. The strong topology in X (resp. X') is denoted by $\beta(X, X')$ (resp. $\beta(X', X)$) and we write X_β (resp. X'_β); see [29, §21.2] for the definition. The strong dual space $(X'_\beta)'_\beta$ of X'_β is denoted simply by X''. By X'_σ we denote X' equipped with its weak–star topology $\sigma(X', X)$. Given $T \in \mathcal{L}(X)$, its *dual operator* $T' : X' \to X'$ is defined by $\langle x, T'x'\rangle = \langle Tx, x'\rangle$ for all $x \in X$, $x' \in X'$. It is known that $T' \in \mathcal{L}(X'_\sigma)$ and $T' \in \mathcal{L}(X'_\beta)$, [30, p.134].

Definition 2.1. Let X be a lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a 1–parameter family of operators. The map $t \mapsto T(t)$, for $t \in [0,\infty)$, is denoted by $T : [0,\infty) \to \mathcal{L}(X)$.

We say that $(T(t))_{t\geq 0}$ is a *semigroup* if it satisfies

(i) $T(s)T(t) = T(s + t)$ for all $s, t \geq 0$, with $T(0) = I$.

A semigroup $(T(t))_{t\geq 0}$ is *locally equicontinuous* if, for fixed $K > 0$, the set $\{T(t) : 0 \leq t \leq K\}$ is equicontinuous, i.e., given $p \in \Gamma_X$ there exist $q \in \Gamma_X$ and $M > 0$ (depending on p and K) such that

$$p(T(t)x) \leq Mq(x)\,, \quad x \in X, t \in [0, K]. \tag{2.1}$$

A semigroup $(T(t))_{t\geq 0}$ is said to be a C_0*–semigroup* if it satisfies

(ii) $\lim_{t\to 0^+} T(t) = I$ in $\mathcal{L}_s(X)$.

If the C_0–semigroup $(T(t))_{t\geq 0}$ satisfies the additional condition that

(iii) $\lim_{t\to t_0} T(t) = T(t_0)$ in $\mathcal{L}_s(X)$, for each $t_0 \geq 0$,

then it is called a *strongly continuous* C_0–semigroup.

A semigroup $(T(t))_{t\geq 0}$ is said to be *exponentially equicontinuous* if there exists $a \geq 0$ such that $(e^{-at}T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is equicontinuous, i.e.,

$$\forall p \in \Gamma_X\, \exists q \in \Gamma_X, M_p > 0 \text{ with } p(T(t)x) \leq M_p e^{at} q(x)\ \forall t \geq 0, x \in X. \tag{2.2}$$

If $a = 0$, then we simply say *equicontinuous*. Finally, a semigroup $(T(t))_{t\geq 0}$ is said to be a *uniformly continuous* C_0*–semigroup* if $T : [0,\infty) \to \mathcal{L}_b(X)$ is continuous, i.e.,

(iv) $\lim_{t\to t_0} T(t) = T(t_0)$ in $\mathcal{L}_b(X)$, for each $t_0 \geq 0$ (with $t \to 0^+$ if $t_0 = 0$).

Given any locally equicontinuous C_0–semigroup $(T(t))_{t\geq 0}$ (resp. any locally equicontinuous, uniformly continuous C_0–semigroup) on a lcHs X, observe that condition (iii) (resp. condition (iv)) in Definition 2.1 is equivalent to $T(t) \to I$ in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$) as $t \to 0^+$, [6, Remark 1(iii)].

Remark 2.2. (i) Let X be a lcHs and $(T(t))_{t\geq 0}$ be an *equicontinuous* C_0–semigroup on X. For $p \in \Gamma_X$ define $\tilde{p}(x) := \sup_{t\geq 0} p(T(t)x)$, for $x \in X$. By Definition 2.1(i)–(iii) $\tilde{p}$ is well-defined, is a seminorm and satisfies

$$p(x) \leq \tilde{p}(x) \leq M_p q(x) \leq M_p \tilde{q}(x), \quad x \in X. \tag{2.3}$$

Hence, $\tilde{\Gamma}_X := \{\tilde{p} : \; p \in \Gamma_X\}$ also generates the given lc–topology of X. Moreover, for $\tilde{p} \in \tilde{\Gamma}_X$, we have

$$\tilde{p}(T(t)x) = \sup_{s\geq 0} p(T(t)T(s)x) = \sup_{s\geq 0} p(T(t+s)x) \leq \tilde{p}(x), \quad x \in X,\, t \geq 0. \tag{2.4}$$

(ii) In [28, Prop. 1.1] it is shown that in a barrelled lcHs X every strongly continuous C_0–semigroup $(T(t))_{t\geq 0}$ is *locally equicontinuous*.

(iii) Every C_0–semigroup of operators in a Banach space, being strongly continuous, [23, Ch. I, Proposition 5.3], is necessarily exponentially equicontinuous, [20, p.619], [23, Ch. I, Proposition 5.5]. For Fréchet spaces this need not be so. Indeed, in the sequence space $\omega = \mathbb{C}^{\mathbb{N}}$ (topology of coordinate convergence), $T(t)x := (e^{nt}x_n)_{n=1}^\infty$, for $t \geq 0$ and $x = (x_n)_{n=1}^\infty \in \omega$, defines a strongly continuous C_0–semigroup which is not exponentially equicontinuous. As ω is a Montel space, $(T(t))_{t\geq 0}$ is also uniformly continuous.

If X is a *sequentially complete* lcHs and $(T(t))_{t\geq 0}$ is a locally equicontinuous C_0–semigroup on X, then the linear operator A defined by

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t},$$

for $x \in D(A) := \{x \in X : \; \lim_{t\to 0^+} \frac{T(t)x-x}{t}$ exists in $X\}$, is closed with $\overline{D(A)} = X$, [28, Propositions 1.3 & 1.4]. The operator $(A, D(A))$ is called the *infinitesimal generator* of $(T(t))_{t\geq 0}$. Moreover, A and $(T(t))_{t\geq 0}$ commute, [28, Proposition 1.2(1)], i.e., for each $t \geq 0$ we have $\{T(t)x : x \in D(A)\} \subseteq D(A)$ and $AT(t)x = T(t)Ax$, for all $x \in D(A)$. Also known, [28, Proposition 1.2(2)], is that

$$T(t)x - x = \int_0^t T(s)Ax\, ds = \int_0^t AT(s)x\, ds, \quad x \in D(A), \tag{2.5}$$

and, [28, Corollary p.261], that

$$T(t)x - x = A\int_0^t T(s)x\, ds, \quad x \in X. \tag{2.6}$$

For each $x \in D(A)$ (resp. $x \in X$), the integrals occuring in (2.5) (resp. (2.6)) are Riemann integrals of an X–valued, continuous function on $[0,t]$; see [6, Appendix]. The closedness of A ensures that $\operatorname{Ker} A := \{x \in D(A) : Ax = 0\}$ is a closed subspace of X. The range of A is the subspace $\operatorname{Im} A := \{Ax : x \in D(A)\}$.

Let $A\colon D(A) \subseteq X \to X$ be a linear operator on a lcHs X. Whenever $\lambda \in \mathbb{C}$ is such that $(\lambda I - A)\colon D(A) \to X$ is injective, the linear operator $(\lambda I - A)^{-1}$ is understood to have domain $\operatorname{Im}(\lambda I - A)$. The *resolvent set* of A is defined by

$$\rho(A) := \{\lambda \in \mathbb{C} : (\lambda I - A)\colon D(A) \to X \text{ is bijective and } (\lambda I - A)^{-1} \in \mathcal{L}(X)\}$$

and the *spectrum* of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. For $\lambda \in \rho(A)$ we also write $R(\lambda, A) := (\lambda I - A)^{-1}$. For λ, $\mu \in \rho(A)$ it is routine to check that the *resolvent equation*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A)$$

is valid. The spectral theory for closed linear operators A in a (non–normable) lcHs X is not as well developed as in Banach spaces and many features depart from the well known theory in Banach spaces; see [7, Section 3], for example, where those aspects that we require in this paper can be found.

The following general results, also of interest in their own right, play a crucial role in later sections.

Proposition 2.3. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}\subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$ so that $\left\{\frac{T(t))}{t} : t\geq t_0\right\}$ is equicontinuous for some $t_0>0$. Then*

$$\mathbb{C}_{0+} := \{z\in\mathbb{C} : \operatorname{Re}(z)>0\}\subseteq \rho(A)$$

and $\{R(\lambda, A) : \operatorname{Re}(\lambda)\geq a\}$ is equicontinuous for every $a>0$. Moreover, for every $x\in X$ and $n\in\mathbb{N}$ we have

$$R(\lambda,A)^n x = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}R(\lambda,A)x}{d\lambda^{n-1}} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}T(t)x\,dt,\quad \lambda\in\mathbb{C}_{0+}.$$

Proof. According to [6, Remark 1(iii)] the C_0–semigroup $(T(t))_{t\geq 0}$ is strongly continuous. Let $p\in\Gamma_X$. Then there exist A_p and $r\in\Gamma_X$ such that $p\left(\frac{T(t)x}{t}\right)\leq A_p r(x)$ for $x\in X$ and $t\geq t_0$, that is,

$$p(T(t)x)\leq A_p t r(x),\quad x\in X,\ t\geq t_0.$$

By local equicontinuity of $(T(t))_{t\geq 0}$ the set $\{T(t) : t\in[0,t_0]\}\subseteq\mathcal{L}(X)$ is equicontinuous and so there exist $q\in\Gamma_X$ with $q\geq r$ and $B_p>0$ such that

$$p(T(t)x)\leq B_p q(x),\quad x\in X,\ t\in[0,t_0].$$

Fix any $a>0$. Since $\max\{1,t\}\leq c_a e^{at}$ for $t\geq 0$ (with $c_a:=\max\{1,\frac{1}{a}\}$) it follows that

$$p(T(t)x)\leq c_a M_p e^{at} q(x),\quad x\in X,\ t\geq 0, \tag{2.7}$$

where $M_p:=\max\{A_p,B_p\}$, i.e., the semigroup $(T(t))_{t\geq 0}$ is *a-exponentially equicontinuous*. Then Lemma 5.2 and Remark 5.3 of [7] imply that $\{z\in\mathbb{C} : \operatorname{Re}(z)>a\}\subseteq\rho(A)$ and

$$R(\lambda,A)x=\int_0^\infty e^{-\lambda t}T(t)x\,dt,\quad x\in X,\ \operatorname{Re}(\lambda)>a, \tag{2.8}$$

with the integral existing as an improper X–valued Riemann integral. Since $a>0$ is arbitrary, it follows that $\mathbb{C}_{0+}\subseteq\rho(A)$.

Inequalities (2.7) and (2.8) ensure that

$$p(R(\lambda,A)x)\leq c_a M_p q(x)\int_0^\infty e^{-(\operatorname{Re}(\lambda)-a)t}\,dt = \frac{c_a M_p}{(\operatorname{Re}(\lambda)-a)}q(x),\quad x\in X,$$

whenever $\operatorname{Re}(\lambda)>a$. For $\varepsilon>0$ we have $\frac{1}{(\operatorname{Re}(\lambda)-a)}\leq\frac{1}{\varepsilon}$ whenever λ satisfies $\operatorname{Re}(\lambda)\geq a+\varepsilon$. Accordingly,

$$p(R(\lambda,A)x)\leq\frac{c_a M_p}{\varepsilon}q(x),\quad x\in X,\ \operatorname{Re}(\lambda)\geq a+\varepsilon,$$

which shows that $\{R(\lambda,A) : \operatorname{Re}(\lambda)\geq a+\varepsilon\}$ is equicontinuous. Since a and ε are arbitrary, it follows that $\{R(\lambda,A) : \operatorname{Re}(\lambda)\geq b\}$ is equicontinuous for every $b>0$.

Fix $\lambda\in\mathbb{C}_{0+}$. Then there exists $\eta>0$ such that the closure of $V(\lambda,\eta):=\{z\in\mathbb{C} : |z-\lambda|<\eta\}$ is contained in $\mathbb{C}_{0+}\subseteq\rho(A)$. Moreover, $\{R(\mu,A) : \mu\in V(\lambda,\eta)\}\subseteq\{R(\mu,A) : \operatorname{Re}(\mu)\geq\eta\}$ and so $\{R(\mu,A) : \mu\in V(\lambda,\eta)\}\subseteq\mathcal{L}(X)$ is equicontinuous. Setting $\mathcal{U}:=\mathbb{C}_{0+}$ it follows from [7, Proposition 3.4(i)] that $R(\cdot,A)\colon\mathbb{C}_{0+}\to\mathcal{L}(X)$ is holomorphic from $\mathbb{C}_{0+}$ into $\mathcal{L}_b(X)$ with

$$R(\lambda,A)^n=\frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}R(\lambda,A)}{d\lambda^{n-1}},\quad n\in\mathbb{N}.$$

It remains to establish, for $x\in X$ and $n\in\mathbb{N}$, that

$$\frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}R(\lambda,A)x}{d\lambda^{n-1}}=\frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}T(t)x\,dt,\quad\lambda\in\mathbb{C}_{0+}. \tag{2.9}$$

The case $n=1$ is given by (2.8). Consider now $n=2$. Let $\mathrm{Re}(\lambda)>0$ and set $a:=\frac{\mathrm{Re}(\lambda)}{2}>0$. Given $p\in\Gamma_X$ choose $q\in\Gamma_X$ and $M_p>0$ such that (2.7) holds. Then

$$p(te^{-\lambda t}T(t)x)\le c_aM_pte^{-(\mathrm{Re}(\lambda)-a)t}q(x)=c_aM_pte^{-\frac{\mathrm{Re}(\lambda)t}{2}}q(x), \tag{2.10}$$

for $x\in X$ and $t\ge 0$, with $\int_0^\infty te^{-\frac{\mathrm{Re}(\lambda)t}{2}}\,dt<\infty$. Sequential completeness of X ensures that the improper X–valued Riemann integral $\int_0^\infty te^{-\lambda t}T(t)x\,dt$ exists. Moreover, (2.10) and [6, Proposition 11(vii)] imply that the operator $x\mapsto\int_0^\infty te^{-\lambda t}T(t)x\,dt$, for $x\in X$, belongs to $\mathcal{L}(X)$. Fix any $\eta\in\left(0,\frac{\mathrm{Re}(\lambda)}{3}\right)$, in which case $V(\lambda,\eta)\subseteq\{\mu\in\mathbb{C}:\mathrm{Re}(\mu)>\eta\}$. Then it follows from (2.7) and (2.8) that, for every $\mu\in V(\lambda,\eta)$ with $\mu\ne\lambda$ and $x\in X$,

$$\begin{aligned}
p&\left(\frac{R(\mu,A)x-R(\lambda,A)x}{\mu-\lambda}-\left(-\int_0^\infty te^{-\lambda t}T(t)x\,dt\right)\right)\\
&=p\left(\int_0^\infty\left[\frac{e^{-\mu t}-e^{-\lambda t}}{\mu-\lambda}+te^{-\lambda t}\right]T(t)x\,dt\right)\\
&\le c_aM_pq(x)\int_0^\infty\left|\frac{e^{-(\mu-\lambda)t}-1}{\mu-\lambda}+t\right|e^{-\frac{\mathrm{Re}(\lambda)t}{2}}\,dt.
\end{aligned}\tag{2.11}$$

Considering the power series for the exponential function we have

$$\begin{aligned}
\left|\frac{e^{-(\mu-\lambda)t}-1}{\mu-\lambda}+t\right|&\le\sum_{k=2}^\infty\frac{|\mu-\lambda|^{k-1}t^k}{k!}\le\sum_{k=2}^\infty\frac{\eta^{k-1}t^k}{k!}\\
&<te^{\eta t}\le\max\{1,t\}e^{\eta t}\le c_{\eta/2}e^{\frac{3\eta}{2}t},
\end{aligned}$$

where we have used the fact that $\max\{1,t\}\le c_{\eta/2}e^{\frac{\eta t}{2}}$ for $t\ge 0$. Accordingly, for $t\ge 0$ and μ satisfying $0<|\mu-\lambda|<\eta$ we have

$$\left|\frac{e^{-(\mu-\lambda)t}-1}{\mu-\lambda}+t\right|e^{-\frac{\mathrm{Re}(\lambda)t}{2}}\le c_{\eta/2}e^{\left(\frac{3}{2}\eta-\frac{\mathrm{Re}(\lambda)}{2}\right)t}. \tag{2.12}$$

Since $\left(\frac{3}{2}\eta-\frac{\mathrm{Re}(\lambda)}{2}\right)<0$ (as $0<\eta<\frac{\mathrm{Re}(\lambda)}{3}$), the function $t\mapsto c_{\eta/2}e^{\left(\frac{3}{2}\eta-\frac{\mathrm{Re}(\lambda)}{2}\right)t}$ is integrable on $[0,\infty)$. So, the Dominated Convergence Theorem, the estimates (2.11) and (2.12), and the fact that the pointwise limit

$$\lim_{\mu\to\lambda}\left(\frac{e^{-(\mu-\lambda)t}-1}{\mu-\lambda}+t\right)e^{-\frac{\mathrm{Re}(\lambda)t}{2}}=0,\quad t\in[0,\infty),$$

imply that $\lim_{\mu\to\lambda}\frac{R(\mu,A)-R(\lambda,A)}{\mu-\lambda}=-\int_0^\infty te^{-\lambda t}T(t)\,dt$ in $\mathcal{L}_s(X)$. This is precisely (2.9) for $n=2$.

This argument can be adapted, together with induction, to verify that (2.9) holds for all $n\in\mathbb{N}$ and for $\mathrm{Re}(\lambda)>0$. □

Remark 2.4. Suppose that X is sequentially complete and barrelled and that $(T(t))_{t\ge0}\subseteq\mathcal{L}(X)$ is a strongly continuous C_0–semigroup satisfying $\lim_{t\to\infty}\frac{T(t)}{t}=0$ in $\mathcal{L}_s(X)$.

(i) Under the above conditions the hypotheses of Proposition 2.3 are satisfied for every $t_0>0$. Indeed, according to Remark 2.2(ii) the semigroup $(T(t))_{t\ge0}$ is locally equicontinuous. Fix now any $t_0>0$. Given $x\in X$ we have $\lim_{t\to\infty}\frac{T(t)x}{t}=0$ in X and so there exists $\tau_x>t_0$ such that $\left\{\frac{T(t)x}{t}:t>\tau_x\right\}$ is bounded in X. By local equicontinuity $\{T(t)x:t\in[0,\tau_x]\}$ is bounded in X and hence, so is $\left\{\frac{T(t)x}{t}:t\in[t_0,\tau_x]\right\}$. It follows that

$\left\{\frac{T(t)x}{t}\,:\, t\geq t_0\right\}$ is also bounded. Since $x\in X$ is arbitrary and X is barrelled, we can conclude that $\left\{\frac{T(t)}{t}\,:\, t\geq t_0\right\}$ is equicontinuous in $\mathcal{L}(X)$.

(ii) The above hypotheses also imply that $(T(t))_{t\geq 0}$ is exponentially equicontinuous. Indeed, by part (i) we have $\left\{\frac{T(t)}{t}\,:\, t\geq t_0\right\}$ is equicontinuous. Moreover, $(T(t))_{t\geq 0}$ is locally equicontinuous by Remark 2.2(ii). In particular, $\{T(t)\,:\, t\in[0,1]\}$ is equicontinuous. Let $p\in\Gamma_X$. Then there exist $q_1,q_2\in\Gamma_X$ such that

$$p(T(t)x)\leq M_1q_1(x)\leq M_1e^tq_1(x),\quad x\in X,\, t\in[0,1],$$

$$p(T(t)x)\leq tM_2q_2(x)\leq M_2e^tq_2(x),\quad x\in X,\, t\geq 1,$$

for constants M_1, $M_2>0$. For some $q\geq\max\{q_1,q_2\}$ with $q\in\Gamma_X$ and $M\geq\max\{M_1,M_2\}$ we have $p(T(t)x)\leq Me^tq(x)$, for $x\in X$, $t\geq 0$, i.e., $(T(t))_{t\geq 0}$ is exponentially equicontinuous.

Corollary 2.5. *Let X be a barrelled, sequentially complete lcHs and $(T(t))_{t\geq 0}\subseteq\mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A,D(A))$ and satisfying τ_s-$\lim_{t\to\infty}\frac{T(t)}{t}=0$. Then $(0,\infty)\subseteq\rho(A)$ and $\lambda\mapsto\lambda R(\lambda,A)$ is continuous from $(0,\infty)$ into $\mathcal{L}_b(X)$.*

Proof. It suffices to show that $\lambda\mapsto\lambda R(\lambda,A)$ is continuous from $(0,\infty)$ into $\mathcal{L}_b(X)$. Remark 2.4(i) and Proposition 2.3 imply that $(0,\infty)\subseteq\rho(A)$ and that $\{R(\lambda,A)\,:\, a\leq\lambda<\infty\}$ is equicontinuous for every $a>0$.

Fix $\mu>0$. Let $p\in\Gamma_X$ and $B\in\mathcal{B}(X)$. By equicontinuity of $\{R(\lambda,A)\,:\,\lambda\in[\mu/2,(3\mu)/2]\}$ there exist $M_p>0$ and $q\in\Gamma_X$ such that

$$p(R(\lambda,A)x)\leq M_pq(x),\quad x\in X,\ \lambda\in\left[\frac{\mu}{2},\frac{3\mu}{2}\right].$$

For every $\lambda\in\left[\frac{\mu}{2},\frac{3\mu}{2}\right]$ it follows from the resolvent equation that

$$p_B(R(\lambda,A)-R(\mu,A))=|\lambda-\mu|\sup_{x\in B}p(R(\lambda,A)R(\mu,A)x)\leq M_p|\lambda-\mu|q_B(R(\mu,A)).$$

For $\lambda\in\left[\frac{\mu}{2},\frac{3\mu}{2}\right]$ it follows that τ_b-$\lim_{\lambda\to\mu}R(\lambda,A)=R(\mu,A)$, i.e., $R(\cdot,A)$ is continuous at μ. Since $\mu\in(0,\infty)$ is arbitrary, we are done. □

Proposition 2.6. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}\subseteq\mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A,D(A))$ so that $\left\{\frac{T(t))}{t}\,:\, t\geq t_0\right\}$ is equicontinuous for some $t_0>0$ and τ_b-$\lim_{t\to\infty}\frac{T(t)}{t}=0$. Then, for every real $\lambda>0$, we have*

$$\tau_b-\lim_{n\to\infty}\frac{(\lambda R(\lambda,A))^n}{n}=0.$$

Proof. Fix a real number $\lambda>0$. According to Proposition 2.3 the set $\mathbb{C}_{0+}\subseteq\rho(A)$ and, for every $x\in X$ and $n\in\mathbb{N}$, we have

$$R(\lambda,A)^nx=\frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}T(t)x\,dt,\quad\lambda>0.\tag{2.13}$$

Fix $p\in\Gamma_X$, $\varepsilon>0$ and $B\in\mathcal{B}(X)$. It follows from τ_b-$\lim_{t\to\infty}\frac{T(t)}{t}=0$ that there exists $t_1>0$ such that

$$\sup_{x\in B}p(T(t)x)<\varepsilon\lambda t,\quad t\geq t_1.\tag{2.14}$$

The local equicontinuity of $(T(t))_{t\geq 0}$ ensures that $\{T(t) : t \in [0, t_1]\} \subseteq \mathcal{L}(X)$ is equicontinuous and so there exist $M_p > 0$ and $q \in \Gamma_X$ with

$$p(T(t)x) \leq M_p q(x), \quad x \in X,\, t \in [0, t_1]. \tag{2.15}$$

It follows from (2.13), (2.14) and (2.15) and [6, Proposition 11(vii)] that, for every $x \in B$ and $n \in \mathbb{N}$, we have

$$p\left(\frac{(\lambda R(\lambda, A))^n x}{n}\right) \leq \frac{M_p q(x)}{n} \frac{\lambda^n}{(n-1)!} \int_0^{t_1} e^{-\lambda t} t^{n-1}\, dt + \varepsilon \frac{\lambda^{n+1}}{n!} \int_{t_1}^{\infty} e^{-\lambda t} t^n\, dt.$$

Since $\frac{\mu^{k+1}}{k!} \int_0^\infty e^{-\mu t} t^k\, dt = 1$ for every $k \in \mathbb{N}_0$ and real $\mu > 0$, it follows that

$$p\left(\frac{(\lambda R(\lambda, A))^n x}{n}\right) \leq \frac{M_p q(x)}{n} + \varepsilon, \quad x \in B,\, n \in \mathbb{N}.$$

Hence, with $K := \sup_{x\in B} q(x)$, we can conclude that

$$\sup_{x\in B} p\left(\frac{(\lambda R(\lambda, A))^n x}{n}\right) \leq \frac{M_p K}{n} + \varepsilon, \quad n \in \mathbb{N},$$

from which it follows that $\limsup_{n\to\infty} \sup_{x\in B} p\left(\frac{(\lambda R(\lambda,A))^n x}{n}\right) \leq \varepsilon$. Letting $\varepsilon \to 0^+$, the proof is complete. $\square$

Lemma 2.7. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$ such that $\rho(A) \neq \emptyset$. Then*

$$\mathrm{Fix}(T(\cdot)) = \mathrm{Ker}(I - \lambda R(\lambda, A)), \quad \lambda \in \rho(A), \tag{2.16}$$

where $\mathrm{Fix}(T(\cdot)) := \{x \in X : T(t)x = x\ \forall t \geq 0\}$. In particular, $x \in \mathrm{Fix}(T(\cdot))$ precisely when $x = \lambda R(\lambda, A)x$ for some (all) $\lambda \in \rho(A)$.

Proof. Fix any $\lambda \in \rho(A)$. Let $x \in \mathrm{Ker}(I - \lambda R(\lambda, A))$. Then $x = \lambda R(\lambda, A)x \in D(A)$ and so the identity (2.5) holds for this particular x. On the other hand, $x \in D(A)$ also implies that $R(\lambda, A)(\lambda I - A)x = x = \lambda R(\lambda, A)x$ and so, by the injectivity of $R(\lambda, A)$, we obtain $(\lambda I - A)x = \lambda x$, i.e., $Ax = 0$. Then (2.5) reveals that $T(t)x - x = 0$ for all $t \geq 0$, i.e., $x \in \mathrm{Fix}(T(\cdot))$. So, $\mathrm{Ker}(I - \lambda R(\lambda, A)) \subseteq \mathrm{Fix}(T(\cdot))$.

Conversely, if $x \in \mathrm{Fix}(T(\cdot))$, then $\lim_{t\to 0^+} \frac{T(t)x - x}{t} = 0$. Hence, $x \in D(A)$ and $Ax = 0$. So,

$$x = R(\lambda, A)(\lambda I - A)x = R(\lambda, A)(\lambda x - Ax) = \lambda R(\lambda, A)x.$$

Accordingly, $x \in \mathrm{Ker}(I - \lambda R(\lambda, A))$ and so $\mathrm{Fix}(T(\cdot)) \subseteq \mathrm{Ker}(I - \lambda R(\lambda, A))$. This completes the proof of (2.16). $\square$

Remark 2.8. In the setting of Lemma 2.7 if $0 \in \rho(A)$, then $\mathrm{Fix}(T(\cdot)) = \mathrm{Ker}\, I = \{0\}$ and so $\mathrm{Ker}(I - \lambda R(\lambda, A)) = \{0\}$ for every $\lambda \in \rho(A)$.

Lemma 2.9. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$ such that $\rho(A) \neq \emptyset$. Then*

$$\mathrm{Im} A = (\lambda R(\lambda, A) - I)(X), \quad \lambda \in \rho(A).$$

Proof. Fix any $\lambda \in \rho(A)$. Then $(\lambda I - A)R(\lambda, A) = I$ on X and $R(\lambda, A)(\lambda I - A) = I$ on $D(A)$. It follows that $\lambda R(\lambda, A) = I + AR(\lambda, A)$ on X and so

$$(\lambda R(\lambda, A) - I)(X) = A(R(\lambda, A)(X)) \subseteq \mathrm{Im} A.$$

On the other hand, for every $y \in \mathrm{Im}A$ there is $x \in D(A)$ with $Ax = y$. Accordingly, $x = R(\lambda, A)(\lambda I - A)x = \lambda R(\lambda, A)x - R(\lambda, A)y$, i.e., $R(\lambda, A)y = (\lambda R(\lambda, A) - I)x$. Consequently, we have

$$\begin{aligned} y &= (\lambda I - A)R(\lambda, A)y = (\lambda I - A)(\lambda R(\lambda, A) - I)x \\ &= (\lambda R(\lambda, A) - I)(\lambda I - A)x \in (\lambda R(\lambda, A) - I)(X). \end{aligned}$$

The arbitrariness of y in $\mathrm{Im}A$ implies the reverse inclusion

$$\mathrm{Im}A \subseteq (\lambda R(\lambda, A) - I)(X).$$

□

Let $(T(t))_{t\geq 0}$ be a locally equicontinuous C_0–semigroup on a sequentially complete lcHs X. The linear operators

$$C(0) := I \text{ and } C(r)x := \frac{1}{r}\int_0^r T(t)x\,dt, \quad x \in X,\, r > 0, \tag{2.17}$$

are called the *Cesáro means* of $(T(t))_{t\geq 0}$. The integrals in (2.17) are X–valued Riemann integrals with respect to the locally convex topology of X; see [6], [27], [46], for example. The Cesáro means $\{C(r)\}_{r\geq 0}$ are well defined and belong to $\mathcal{L}(X)$, [6, Section 3]. If $(T(t))_{t\geq 0}$ is equicontinuous, then $\{C(r)\}_{r\geq 0}$ is also equicontinuous, [6, Section 3]. In case X is barrelled the Cesáro means exist in $\mathcal{L}(X)$ whenever the semigroup $(T(t))_{t\geq 0}$ is strongly continuous (via Remark 2.2(ii)). Since the interval $[0, \infty)$ is a directed set relative to the usual order $\geq$ induced from $\mathbb{R}$, it is meaningful to speak about convergence of the net $\{C(r)\}_{r\geq 0}$ in $\mathcal{L}_s(X)$ or $\mathcal{L}_b(X)$ as $r \to \infty$.

Lemma 2.10. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$ and satisfying τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$. If $A\colon D(A) \to X$ is bijective with $A^{-1}\colon X \to D(A)$ continuous, then τ_b-$\lim_{r\to\infty} C(r) = 0$.*

Proof. Let $y \in X$. As A is surjective there is $x \in D(A)$ such that $y = Ax$, namely $x = A^{-1}y$. According to (2.5), for every $r > 0$, we have that

$$(T(r) - I)x = \int_0^r T(s)Ax\,ds = \int_0^r T(s)y\,ds$$

and so, $C(r)y = \frac{(T(r)-I)x}{r}$. Now, fix any $p \in \Gamma_X$ and $B \in \mathcal{B}(X)$. Then

$$\begin{aligned} \sup_{y\in B} p(C(r)y) &= \frac{1}{r} \sup_{x\in A^{-1}(B)} p((T(r) - I)x) \\ &\leq \sup_{x\in A^{-1}(B)} p\left(\frac{T(r)}{r}x\right) + \frac{1}{r} \sup_{x\in A^{-1}(B)} p(x), \quad r > 0, \end{aligned}$$

where the set $A^{-1}(B) \in \mathcal{B}(X)$ and is contained in $D(A)$ as $A^{-1}\colon X \to D(A)$ is continuous. Since $\frac{T(r)}{r} \to 0$ in $\mathcal{L}_b(X)$ as $r \to \infty$, the previous inequality implies that $\sup_{y\in B} p(C(r)y) \to 0$ as $r \to \infty$. By the arbitrariness of p and B the desired claim follows. □

Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup. Then $(T(t))_{t\geq 0}$ is called *mean ergodic* if $P := \lim_{r\to\infty} C(r)$ exists in $\mathcal{L}_s(X)$. By [6, Remarks 4(ii) & 5(iii)] if τ_s-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$, then P is a projection with

$$\mathrm{Im}P = \mathrm{Fix}(T(\cdot)) = \mathrm{Ker}\, A$$

and

$$\mathrm{Ker}\, P = \overline{\mathrm{span}\{x - T(t)x : t \geq 0, \ , x \in X\}} = \overline{\mathrm{Im}A},$$

where $(A, D(A))$ is the infinitesimal generator of $(T(t))_{t\geq 0}$. In particular,

$$X = \operatorname{Ker} A \oplus \overline{\operatorname{Im} A}. \tag{2.18}$$

If $\lim_{r\to\infty} C(r)$ exists in $\mathcal{L}_b(X)$, then $(T(t))_{t\geq 0}$ is called *uniformly mean ergodic.* An alternate notion is that $(T(t))_{t\geq 0}$ is called *Abel mean ergodic* (resp. *uniformly Abel mean ergodic*) if the interval $(0,\infty) \subseteq \rho(A)$ and, for some $\lambda_0 > 0$, the net $\{\lambda R(\lambda, A)\}_{0<\lambda\leq\lambda_0}$ is convergent in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$) for $\lambda \to 0^+$, where the interval $(0, \lambda_0]$ is a directed set for the usual order $\leq$ induced from $\mathbb{R}$. Without mentioning λ_0 explicitly we also write (for the sake of simplicity) τ_s-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A)$ (resp. τ_b-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A)$) for the respective limits in $\mathcal{L}_s(X)$ and in $\mathcal{L}_b(X)$.

Remark 2.11. (i) Let X be a barrelled, sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$ such that τ_s-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$ and $(T(t))_{t\geq 0}$ is Abel mean ergodic. Then $\{\lambda R(\lambda, A)\}_{0<\lambda\leq\lambda_0}$ is *necessarily equicontinuous* for some $\lambda_0 > 0$. Indeed, choose any $\lambda_0 > 0$ such that the net $\{\lambda R(\lambda, A)\}_{0<\lambda\leq\lambda_0}$ converges in $\mathcal{L}_s(X)$ for $\lambda \to 0^+$, say to $P \in \mathcal{L}(X)$. By the barrelledness of X it suffices to show that $\{\lambda R(\lambda, A)x\}_{0<\lambda\leq\lambda_0} \in \mathcal{B}(X)$ for every $x \in X$. So, fix $x \in X$ and $p \in \Gamma_X$. Then there exists $\lambda' \in (0, \lambda_0]$ such that $p(\lambda R(\lambda, A)x - Px) \leq 1$ for all $\lambda \in (0, \lambda')$ and hence, $\sup_{0<\lambda<\lambda'} p(\lambda R(\lambda, A)x) < \infty$. Since $[\lambda', \lambda_0]$ is compact and $\lambda \mapsto \lambda R(\lambda, A)x$ is continuous from $[\lambda', \lambda_0]$ into X (cf. Corollary 2.5), it follows that $\sup_{\lambda'\leq\lambda\leq\lambda_0} p(\lambda R(\lambda, A)x) < \infty$. Consequently, $\{\lambda R(\lambda, A)x\}_{0<\lambda\leq\lambda_0} \in \mathcal{B}(X)$.

If X is a *Banach space*, then the Abel mean ergodicity of $(T(t))_{t\geq 0}$ by itself suffices to ensure that $\sup_{0<\lambda\leq\lambda_0} \|\lambda R(\lambda, A)\| < \infty$, i.e., the condition τ_s-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$ can be omitted. To see this fix $\mu \in (0, \lambda_0]$. Since $\mu \in \rho(A)$, with $\rho(A)$ open in $\mathbb{C}$, and the function $z \mapsto R(z, A)$ (hence, also $z \mapsto zR(z, A)$) is holomorphic, for the operator norm in $\mathcal{L}(X)$, in a neighbourhood of μ (in $\mathbb{C}$), [23, Ch. IV, Proposition 1.3], it follows that $\lim_{\lambda\to\mu,\, \lambda\in\mathbb{R}} \lambda R(\lambda, A)$ exists relative to $\|\cdot\|$ and equals $\mu R(\mu, A)$. Hence, $\lambda \mapsto \lambda R(\lambda, A)$ is operator norm continuous in any interval $[a, \lambda_0]$ with $0 < a < \lambda_0$. The argument of the previous paragraph then applies to show that $\{\lambda R(\lambda, A)\}_{0<\lambda\leq\lambda_0}$ is bounded in $\mathcal{L}_s(X)$ and hence, by the Principle of Uniform Boundedness, that $\sup_{0<\lambda\leq\lambda_0} \|\lambda R(\lambda, A)\| < \infty$.

(ii) Let X be a lcHs and $(T(t))_{t\geq 0}$ be a semigroup as in part (i). Then the equicontinuity of $\{\lambda R(\lambda, A)\}_{0<\lambda\leq\lambda_0}$ (by part (i)) and [7, Lemma 3.8(ii)] imply that

$$\overline{\operatorname{Im} A} = \{x \in X : \lim_{\lambda\to 0^+} \lambda R(\lambda, A)x = 0\}. \tag{2.19}$$

Moreover, since $R(\lambda, A)(X) \subseteq D(A)$ for each $\lambda \in \rho(A)$, it follows from [7, Lemma 3.6] that

$$\operatorname{Ker} A = \{x \in D(A) : \lambda R(\lambda, A)x = x\} = \{x \in X : \lambda R(\lambda, A)x = x\}, \tag{2.20}$$

for each $\lambda \in \rho(A) \setminus \{0\}$. In particular, via (2.19) and (2.20) we have

$$\overline{\operatorname{Im} A} \cap \operatorname{Ker} A = \{0\}.$$

Proposition 2.12. *Let X be a barrelled, sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$ such that τ_s-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$ and $(T(t))_{t\geq 0}$ is Abel mean ergodic. Then $P := \tau_s$-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A)$ is a projection with $\operatorname{Im} P = \operatorname{Ker} A = \operatorname{Fix}(T(\cdot))$ and $\operatorname{Ker} P = \overline{\operatorname{Im} A}$, i.e., P is a projection of X onto $\operatorname{Ker} A$ along $\overline{\operatorname{Im} A}$.*

Proof. Let $\lambda_0 > 0$ be as in Remark 2.11. Let $x \in X$. Then $Px = \lim_{\mu\to 0^+} \mu R(\mu, A)x$. Fix any $\lambda \in (0, \lambda_0]$. For each $0 < \mu < \lambda$ the resolvent equation yields

$$\lambda\mu R(\lambda, A)R(\mu, A)x = \frac{\lambda\mu}{\mu - \lambda} R(\lambda, A)x + \frac{\lambda}{\lambda - \mu}(\mu R(\mu, A)x).$$

Let $\mu \to 0^+$ to deduce that $\lambda R(\lambda, A)Px = 0 + Px = Px$. It follows from (2.20) that $Px \in \operatorname{Ker} A$, i.e., $\operatorname{Im}P \subseteq \operatorname{Ker} A$. In the proof of Lemma 2.9 it was noted that

$$R(\lambda, A)Ax = \lambda R(\lambda, A)x - x, \quad x \in D(A),\ \lambda \in \rho(A) \setminus \{0\}. \tag{2.21}$$

Since $\operatorname{Im}P \subseteq \operatorname{Ker} A \subseteq D(A)$, we conclude that

$$0 = R(\lambda, A)AP = \lambda R(\lambda, A)P - P,$$

i.e., $\lambda R(\lambda, A)P = P$ for all $\lambda > 0$. Let $\lambda \to 0^+$ yields $P^2 = P$ and so P is a projection. Moreover, (2.21) implies if $x \in \operatorname{Ker} A$, then $\lambda R(\lambda, A)x = x$ for all $\lambda \in \rho(A)\setminus\{0\}$ and so, for $\lambda \to 0^+$, we can conclude that $Px = x$, i.e., $x \in \operatorname{Im}P$. This establishes that $\operatorname{Im}P = \operatorname{Ker} A$.

The definition of P and (2.19) imply that $\operatorname{Ker} P = \overline{\operatorname{Im}A}$.

Finally, that $\operatorname{Ker} A = \operatorname{Fix}(T(\cdot))$ is known, [6, Remark 5(iii)]. □

We point out that the formulation of condition (4) in Theorem 1.1 (as given in [33, Theorem]) is not optimal. One merely needs to assume that $P := \lim_{\lambda\to 0^+} \lambda R_\lambda = \lim_{\lambda\to 0^+} \lambda R(\lambda, A)$ *exists* in the operator norm topology. The limit P is then automatically a projection onto $\operatorname{Im}P = \operatorname{Fix}(T(\cdot))$; see Proposition 2.12.

Lemma 2.13. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$ satisfying τ_s-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$. Set $Y := \overline{\operatorname{Im}A}$ and define $A_1x := Ax$ for each $x \in D(A_1) := D(A) \cap Y$. If $(T(t))_{t\geq 0}$ is mean ergodic, then $Y = \overline{\operatorname{Im}A_1}$.*

Proof. Let $y \in \operatorname{Im}A$. Then there is $x \in D(A)$ with $y = Ax$. Via (2.18) $x = x_1 + x_2$ with $x_1 \in \operatorname{Ker} A$ and $x_2 \in Y$ (hence, $x_2 = x - x_1 \in D(A)$ and so $x_2 \in D(A_1)$). So, $y = Ax = A(x_1 + x_2) = Ax_2 = A_1x_2 \in \operatorname{Im}A_1$. Thus, $\operatorname{Im}A \subseteq \operatorname{Im}A_1$ which implies that $Y = \overline{\operatorname{Im}A} \subseteq \overline{\operatorname{Im}A_1} \subseteq Y$. On the other hand if $y \in \operatorname{Im}A_1$, then there is $x \in D(A_1)$ with $y = A_1x = Ax \in \operatorname{Im}A \subseteq Y$. So, $\overline{\operatorname{Im}A_1} \subseteq Y$. Therefore, $\overline{\operatorname{Im}A_1} = Y$. □

Remark 2.14. The space Y defined in Lemma 2.13 is $T(\cdot)$–invariant. Indeed, if $x \in D(A)$, then for each $t \geq 0$ we have $AT(t)x = T(t)Ax$ from which $T(t)(Y) \subseteq Y$ follows. Consequently, the restriction maps $S(t) := T(t)|_Y$, for $t \geq 0$, define a C_0–semigroup on Y. Since $\{p|_Y : p \in \Gamma_X\}$ is a system of continuous seminorms determining the topology of Y, it follows that $(S(t))_{t\geq 0} \subseteq \mathcal{L}(Y)$ is locally equicontinuous. Moreover, it is routine to check that $(A_1, D(A_1))$ is the infinitesimal generator of $(S(t))_{t\geq 0}$ and that $R(\lambda, A_1) = R(\lambda, A)$ for each $\lambda \in \rho(A)$. In particular, $\rho(A) \subseteq \rho(A_1)$ after noting that Y is $R(\cdot, A)$–invariant.

Lemma 2.15. *Let X be a sequentially complete, barrelled lcHs. Let $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a uniformly continuous C_0–semigroup with infinitesimal generator $(A, D(A))$ satisfying τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$. Then $(T(t)')_{t\geq 0}$ is a locally equicontinuous, uniformly continuous C_0–semigroup on X'_β satisfying τ_b-$\lim_{t\to\infty} \frac{T(t)'}{t} = 0$. Moreover, if $(A', D(A'))$ is the infinitesimal generator of $(T(t)')_{t\geq 0}$, then $\lambda \in \rho(A')$ and $R(\lambda, A') = R(\lambda, A)'$ for every $\lambda \in \mathbb{C}_{0^+}$.*

Proof. As already noted, $(T(t)')_{t\geq 0} \subseteq \mathcal{L}(X'_\beta)$. Moreover, it is routine to check that $(T(t)')_{t\geq 0}$ is a semigroup. Since $(T(t))_{t\geq 0}$ is necessarily locally equicontinuous (cf. Remark 2.2(ii)) and X is barrelled, $(T(t)')_{t\geq 0}$ is also locally equicontinuous, [30, §39.3 Theorem (6)]. On the other hand, as $(T(t))_{t\geq 0}$ is a uniformly continuous C_0–semigroup and τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$, we can apply [3, Lemma 2.1] to conclude that $(T(t)')_{t\geq 0}$ is a uniformly continuous C_0–semigroup on X'_β satisfying τ_b-$\lim_{t\to\infty} \frac{T(t)'}{t} = 0$.

Let $(A', D(A'))$ be the infinitesimal generator of $(T(t)')_{t\geq 0}$. Since X is barrelled, X'_β is quasicomplete, [30, §39.6 Theorem (5)]. Moreover, Remark 2.4(i) implies that $\left\{\frac{T(t)}{t} : t \geq t_0\right\}$

is equicontinuous, for every $t_0 > 0$, and hence, also $\left\{\frac{T(t)'}{t} : t \geq t_0\right\} \subseteq \mathcal{L}(X'_\beta)$ is equicontinuous (as X is barrelled), [30, §39.3 Theorem (6)]. Then, by Proposition 2.3 applied to both $(T(t))_{t\geq 0}$ and $(T(t)')_{t\geq 0}$, we can conclude that $\mathbb{C}_{0+} \subseteq \rho(A) \cap \rho(A')$ and, for each $\lambda \in \mathbb{C}_{0+}$, that

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt, \quad x \in X,$$

$$R(\lambda, A')x = \int_0^\infty e^{-\lambda t} T(t)'x'\, dt, \quad x' \in X',$$

So, for every $x \in X$, $x' \in X'$ and $\lambda \in \mathbb{C}_{0+}$, we have

$$\begin{aligned} \langle x, R(\lambda, A')x'\rangle &= \langle x, \int_0^\infty e^{-\lambda t} T(t)'x'\, dt\rangle = \int_0^\infty e^{-\lambda t}\langle x, T(t)'x'\rangle\, dt \\ &= \int_0^\infty e^{-\lambda t}\langle T(t)x, x'\rangle\, dt = \langle \int_0^\infty e^{-\lambda t} T(t)x\, dt, x'\rangle \\ &= \langle R(\lambda, A)x, x'\rangle = \langle x, R(\lambda, A)'x'\rangle. \end{aligned}$$

This implies that $R(\lambda, A')x' = R(\lambda, A)'x'$ for every $x' \in X'$ and $\lambda \in \mathbb{C}_{0+}$, i.e., $R(\lambda, A') = R(\lambda, A)'$ for every $\lambda \in \mathbb{C}_{0+}$. □

In Lemma 2.15, the necessity of the requirement that the C_0–semigroup $(T(t))_{t\geq 0}$ is uniformly continuous, rather than merely strongly continuous, is due to the fact that the dual semigroup $(T(t)')_{t\geq 0}$ may fail to be strongly continuous in X'_β, even for X a Banach space, [23, p.43].

3. Uniform mean ergodicity of C_0–semigroups of operators

The purpose of this section is to extend Theorem 1.1 from Banach spaces to the class of prequojection Fréchet spaces; see Theorem 3.2 and Proposition 3.4. Moreover, in Example 3.7 it is shown that this extension really is confined to this class of Fréchet spaces. First some preliminaries are required.

A Fréchet space X is always a projective limit of continuous linear operators $S_k : X_{k+1} \to X_k$, for $k \in \mathbb{N}$, with each X_k a Banach space. If it is possible to choose X_k and S_k such that each S_k is surjective and X is isomorphic to the projective limit $\mathrm{proj}_j(X_j, S_j)$, then X is called a *quojection*, [12, Section 5]. Banach spaces and countable products of Banach spaces are quojections. Actually, every quojection is the quotient of a countable product of Banach spaces, [14]. In [38] Moscatelli gave the first examples of quojections which are not isomorphic to countable products of Banach spaces. As already mentioned in Section 1, concrete examples of quojections are $\omega = \mathbb{C}^\mathbb{N}$, the spaces $L^p_{loc}(\Omega)$, with $1 \leq p \leq \infty$, and $C^{(m)}(\Omega)$, for all $m \in \mathbb{N}_0$. Indeed, the above function spaces are isomorphic to countable products of Banach spaces. Moreover, the spaces of continuous functions $C(\Lambda)$, with Λ a σ–compact completely regular topological space, endowed with the compact open topology are also examples of quojections. Domański constructed a completely regular topological space Λ such that the Fréchet space $C(\Lambda)$ is a quojection which is not isomorphic to a complemented subspace of a product of Banach spaces, [19, Theorem]. It is known that a Fréchet space X admits a continuous norm if and only if X contains no isomorphic copy of ω, [26, Theorem 7.2.7]. On the other hand, a quojection X admits a continuous norm if and only if it is a Banach space, [12, Proposition 3]. Hence, a quojection is either a Banach space or contains an isomorphic copy of ω, necessarily complemented, [26, Theorem 7.2.7]. For further information on quojections we refer to the survey paper [36] and the references therein; see also [12], [18].

Let X be a quojection Fréchet space and $\{q_j\}_{j=1}^{\infty}$ be *any* fundamental, increasing sequence of seminorms generating the lc–topology of X. For each $j \in \mathbb{N}$, set $X_j := X/q_j^{-1}(\{0\})$ and endow X_j with the quotient lc–topology. Denote by $Q_j \colon X \to X_j$ the corresponding canonical (surjective) quotient map and define the increasing sequence of seminorms $\{(\hat{q}_j)_k\}_{k=1}^{\infty}$ on X_j by

$$(\hat{q}_j)_k(Q_j x) := \inf\{q_k(y) : y \in X \text{ and } Q_j y = Q_j x\}, \quad x \in X, \tag{3.1}$$

for each $k \in \mathbb{N}$. Then

$$(\hat{q}_j)_k(Q_j x) \leq q_k(x), \quad x \in X,\ k,\ j \in \mathbb{N}; \tag{3.2}$$

see (2.4) in [5]. Moreover,

$$(\hat{q}_j)_j(Q_j x) = q_j(x), \quad x \in X,\ j \in \mathbb{N}, \tag{3.3}$$

which implies that $(\hat{q}_j)_j$ is a norm on X_j. Since X is a quojection Fréchet space and since every quotient space (of such a Fréchet space) with a continuous *norm* is necessarily Banach, [12, Proposition 3], it follows that for each $j \in \mathbb{N}$ there exists $k(j) \geq j$ such that the norm $(\hat{q}_j)_{k(j)}$ generates the lc–topology of X_j. Thus, X is isomorphic to the projective limit of the sequence $\{(X_j, (\hat{q}_j)_{k(j)})\}_{j=1}^{\infty}$ of Banach spaces with respect to the continuous, surjective linking maps $Q_{j,j+1} \colon X_{j+1} \to X_j$ defined by

$$Q_{j,j+1} \circ Q_{j+1} = Q_j, \quad j \in \mathbb{N}. \tag{3.4}$$

This particular construction will be used on various occasions in the sequel.

For any sequence $\{x_n\}_{n=1}^{\infty}$ in a lcHs X, its sequence of *arithmetic means* is given by $n^{-1}\sum_{m=1}^{n} x_m$, for $n \in \mathbb{N}$. Given $S \in \mathcal{L}(X)$ we can form its sequence of iterates $S^m := S \circ \ldots \circ S$, for $m \in \mathbb{N}$. Then the arithmetic means

$$S_{[n]} := \frac{1}{n}\sum_{m=1}^{n} S^m, \quad n \in \mathbb{N},$$

of $\{S^m\}_{m=1}^{\infty}$ are called the *Cesàro means* of S. If τ_s-$\lim_{n\to\infty} S_{[n]}$ (resp. τ_b-$\lim_{n\to\infty} S_{[n]}$) exists, then S is called *mean ergodic* (resp. *uniformly mean ergodic*).

Various aspects concerning the mean ergodicity of individual operators in non–normable lcHs can be found in [2], [4], [15], [42], [43] and the references therein.

Remark 3.1. If $\{x_n\}_{n=1}^{\infty}$ is any sequence in a lcHs X for which $x = \lim_{n\to\infty} x_n$ exists, then also its sequence of arithmetic means $\{n^{-1}\sum_{m=1}^{n} x_m\}_{n=1}^{\infty}$ converges to the same limit x. Indeed, by considering each $p \in \Gamma_X$, this can be verified by adapting the standard argument used for scalar sequences; see the proof of Theorem 6b in [25, Ch.5, §6], for example. In particular, if $S \in \mathcal{L}(X)$ and $P := \lim_{m\to\infty} S^m$ exists in $\mathcal{L}_s(X)$ (resp. $\mathcal{L}_b(X)$), then also $\lim_{n\to\infty} S_{[n]} = P$ in $\mathcal{L}_s(X)$ (resp. $\mathcal{L}_b(X)$).

We are now able to formulate the main result of the paper. It should be compared with Theorem 1.1.

Theorem 3.2. *Let X be a quojection Fréchet space and $(T(t))_{t\geq 0}$ be a locally equicontinuous, C_0–semigroup on X satisfying τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$. Then the following assertions are equivalent.*

(1) *The semigroup $(T(t))_{t\geq 0}$ is uniformly mean ergodic.*
(2) *The infinitesimal generator $(A, D(A))$ of $(T(t))_{t\geq 0}$ has closed range.*
(3) *The operator $\lambda R(\lambda, A)$ is uniformly mean ergodic for every $\lambda > 0$.*
(4) *The operator $\lambda R(\lambda, A)$ is uniformly mean ergodic for some $\lambda > 0$.*
(5) *The semigroup $(T(t))_{t\geq 0}$ is uniformly Abel mean ergodic.*

(6) *The sequence of iterates* $\{(\lambda R(\lambda, A))^n\}_{n=1}^\infty$ *converges in* $\mathcal{L}_b(X)$ *for every (some)* $\lambda > 0$.

(7) $\overline{\mathrm{Im}A}$ *is a quojection and there exists* $\lambda_0 > 0$ *such that*

$$\{R(\lambda, A)y : \ y \in (0, \lambda_0]\} \in \mathcal{B}(X), \quad y \in \overline{\mathrm{Im}A}.$$

Proof. Since τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$, the set $\left\{\frac{T(t)}{t} : t \geq t_0\right\}$ is equicontinuous for some $t_0 > 0$; see Remark 2.4(i). Then Proposition 2.3 implies that $\mathbb{C}_{0+} \subseteq \rho(A)$ and, via Proposition 2.6, we can conclude that

$$\tau_b - \lim_{n\to\infty} \frac{[\lambda R(\lambda, A)]^n}{n} = 0, \quad \lambda > 0. \tag{3.5}$$

(2)⇒(3). Let $\lambda > 0$ be arbitrary. By Lemma 2.9 $(\lambda R(\lambda, A) - I)(X)$ is closed in X. As X is a quojection Fréchet space, Theorem 3.4 in [8] applied to $\lambda R(\lambda, A)$ yields that $\{(\lambda R(\lambda, A))_{[n]}\}_{n=1}^\infty$ converges in $\mathcal{L}_b(X)$. Thus, (3) holds.

(3)⇒(4). This is obvious.

(4)⇒(2). Suppose that (4) holds for some $\lambda > 0$. For this λ, since (3.5) holds and X is a quojection Fréchet space, we can apply Theorem 3.4 of [8] to the operator $\lambda R(\lambda, A)$ to conclude that $(I - \lambda R(\lambda, A))(X)$ is closed. On the other hand, Lemma 2.9 yields that $(I - \lambda R(\lambda, A))(X) = \mathrm{Im}A$. Hence, $\mathrm{Im}A$ is closed in X which is precisely (2).

(1)⇒(5). This follows from [7, Theorem 5.5(i) and Remark 5.6(i)].

(2)⇒(1). Consider the closed subspace $Y := \mathrm{Im}A$ of X. Remark 2.14 ensures that Y is $T(\cdot)$–invariant and the restrictions $(T(t)|_Y)_{t\geq 0}$ form a locally equicontinuous C_0–semigroup on Y with infinitesimal generator $(A_1, D(A_1))$ given by

$$D(A_1) := Y \cap D(A) \ \text{ and } \ A_1 x := Ax, \quad x \in D(A_1).$$

It is routine to check that τ_b-$\lim_{t\to\infty} \frac{T(t)|_Y}{t} = 0$. By Lemma 2.9, for any fixed $\lambda > 0$, we have that $Y = (\lambda R(\lambda, A) - I)(X)$. So, $(\lambda R(\lambda, A) - I)(X)$ is closed in X. According to (3.5) and the fact that X is a quojection, Theorem 3.4 and Remark 3.6(1) of [8] can be applied to the operator $\lambda R(\lambda, A)$ to conclude that it is uniformly mean ergodic and that the continuous linear operator $I - \lambda R(\lambda, A) \colon Y \to Y$ is bijective (hence, invertible with a continuous inverse).

If $A_1 y = 0$ for some $y \in D(A_1)$, then $y = R(\lambda, A)(\lambda I - A)y = \lambda R(\lambda, A)y - R(\lambda, A)A_1 y = \lambda R(\lambda, A)y$ and so $(I - \lambda R(\lambda, A))y = 0$, which implies that $y = 0$. Thus, A_1 is one–to–one. On the other hand, if we apply Lemma 2.9 to $(T(t)|_Y)_{t\geq 0}$, then we deduce that $\mathrm{Im}A_1 = (I - \lambda R(\lambda, A_1))(Y) = (I - \lambda R(\lambda, A))(Y) = Y$. Therefore, $A_1 \colon D(A_1) \to Y$ is bijective and so the inverse operator $(A_1)^{-1} \colon Y \to D(A_1)$ exists. Since A_1 is closed, also $(A_1)^{-1}$ is closed. By the Closed Graph Theorem it follows that $(A_1)^{-1}$ is continuous. According to Lemma 2.10 we have that τ_b-$\lim_{r\to\infty} C(r) = 0$.

Via (3.5) and the fact that $(\lambda R(\lambda, A) - I)(X) = Y$ is closed in X (with X a quojection Fréchet space), we can apply [8, Theorem 3.4] to conclude that $X = Y \oplus \mathrm{Ker}(I - \lambda R(\lambda, A))$. Then Lemma 2.7 yields that $X = Y \oplus \mathrm{Fix}(T(\cdot))$. Since $C(r) \to 0$ in $\mathcal{L}_b(Y)$ as $r \to \infty$ and $C(r) = I$ on $\mathrm{Fix}(T(\cdot))$ for all $r > 0$, it follows from the previous identity that τ_b-$\lim_{r\to\infty} C(r)$ exists, i.e., part (1) holds.

(5)⇒(2). Let $P := \tau_b$-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A)$. It follows from (2.19), (2.20) and Proposition 2.12 that P is a projection (hence, $X = \mathrm{Im}P \oplus \mathrm{Ker}\, P$) with

$$\mathrm{Im}P = \mathrm{Fix}(T(\cdot)) = \mathrm{Ker}\, A = \{x \in D(A) : \ \lambda R(\lambda, A)x = x\}, \quad \forall \lambda \in \rho(A),$$

$$\mathrm{Ker}\, P = \overline{\mathrm{Im}A} = \{x \in X : \ \lim_{\lambda\to 0^+} \lambda R(\lambda, A)x = 0\}.$$

Moreover, $Y := \overline{\mathrm{Im} A}$ is invariant for each operator in $\{\lambda R(\lambda, A) :\ \lambda \in \rho(A)\}$, [7, Lemma 3.6], and each operator in $\{T(t) :\ t \geq 0\}$; see Remark 2.14. So, if we define

$$D(A_1) := Y \cap D(A) \quad \text{and} \quad A_1 x := Ax, \quad x \in D(A_1),$$

then $(A_1, D(A_1))$ is the infinitesimal generator of $\{T(t)|_Y\}_{t\geq 0}$ with $R(\lambda, A_1) = R(\lambda, A)|_Y$ for $\lambda \in \rho(A)$; see Remark 2.14. In particular, as $X = \mathrm{Ker}\, A \oplus \overline{\mathrm{Im} A}$ we can proceed as in the proof of Lemma 2.13 to deduce that $Y = \overline{\mathrm{Im} A_1}$. Accordingly, Y is a complemented subspace of the quojection Fréchet space X and so is itself a quojection Fréchet space. Hence, we may assume that $Y = X$ and that $\lambda R(\lambda, A) \to 0$ in $\mathcal{L}_b(X)$ as $\lambda \to 0^+$.

Fix a fundamental, increasing sequence $\{r_j\}_{j=1}^\infty$ of seminorms generating the lc–topology of X. Since τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$ and X is barrelled, $\{T(t)/t :\ t \geq 1\}$ is equicontinuous. Moreover, the local equicontinuity of $\{T(t)\}_{t\geq 0}$ ensures that $\{T(t) :\ t \in [0,1]\}$ is equicontinuous. So, for each $j \in \mathbb{N}$, there is $M_j > 0$ such that

$$r_j(T(t)x) \leq M_j t r_{j+1}(x), \quad t \geq 1,\ x \in X, \tag{3.6}$$
$$r_j(T(t)x) \leq M_j r_{j+1}(x), \quad t \in [0,1],\ x \in X, \tag{3.7}$$

where there is no loss of generality in assuming that (3.6) and (3.7) hold for r_{j+1} as we can pass to a subsequence of $\{r_j\}_{j=1}^\infty$ if necessary.

Fix $j \in \mathbb{N}$ and define q_j on X by setting

$$q_j(x) := \max\{\sup_{t\in[0,1]} r_j(T(t)x), \sup_{t\geq 1} r_j(t^{-1}T(t)x)\}, \quad x \in X.$$

Then q_j is a seminorm on X and, via (3.6) and (3.7), we have

$$r_j(x) \leq q_j(x) \leq M_j r_{j+1}(x), \quad x \in X.$$

Thus, $\{q_j\}_{j=1}^\infty$ is also a fundamental increasing sequence of seminorms generating the lc–topology of X and satisfies

$$q_j(T(t)x) \leq 2q_j(x), \quad t \in [0,1],\ x \in X, \tag{3.8}$$
$$q_j(T(t)x) \leq (1+t)q_j(x), \quad t \geq 1,\ x \in X. \tag{3.9}$$

Indeed, if $t \in [0,1]$ (hence, also $1 - t \in [0,1]$), then (3.8) follows from

$$\begin{aligned}
&q_j(T(t)x) = \max\left\{\sup_{s\in[0,1]} r_j(T(s+t)x), \sup_{s\geq 1} r_j\left(\left(1+\frac{t}{s}\right)\frac{T(s+t)x}{s+t}\right)\right\}\\
&= \max\left\{\sup_{s\in[0,1-t]} r_j(T(s+t)x),\ \sup_{s\in[1-t,1]} r_j((s+t)(s+t)^{-1}T(s+t)x), \sup_{s\geq 1} r_j\left(\left(1+\frac{t}{s}\right)\frac{T(s+t)x}{s+t}\right)\right\}\\
&\leq 2\max\{\sup_{u\in[0,1]} r_j(T(u)x), \sup_{u\geq 1} r_j(u^{-1}T(u)x)\} = 2q_j(x), \quad x \in X.
\end{aligned}$$

On the other hand, if $t \geq 1$, then (3.9) follows from

$$\begin{aligned}
&q_j(T(t)x) = \max\left\{\sup_{s\in[0,1]} r_j((s+t)(s+t)^{-1}T(s+t)x), \sup_{s\geq 1} r_j\left(\left(1+\frac{t}{s}\right)\frac{T(s+t)x}{s+t}\right)\right\}\\
&\leq (1+t)\sup_{u\geq 1} r_j(u^{-1}T(u)x) \leq (1+t)q_j(x), \quad x \in X.
\end{aligned}$$

Moreover, Remark 2.4(i) and Proposition 2.3 imply that $\mathbb{C}_{0+} \subseteq \rho(A)$ and

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt, \quad x \in X,\ \lambda \in \mathbb{C}_{0+}.$$

So, via (3.8) and (3.9) we conclude, for each $j \in \mathbb{N}$, $\lambda > 0$ and $x \in X$, that

$$\begin{aligned} q_j(R(\lambda, A)x) &\leq q_j\left(\int_0^1 e^{-\lambda t}T(t)x\,dt\right) + q_j\left(\int_1^\infty e^{-\lambda t}T(t)x\,dt\right) \\ &\leq 2q_j(x)\int_0^1 e^{-\lambda t}\,dt + q_j(x)\int_1^\infty e^{-\lambda t}(1+t)\,dt \\ &= \frac{2(1-e^{-\lambda})}{\lambda}q_j(x) + \left(\frac{2e^{-\lambda}}{\lambda} + \frac{e^{-\lambda}}{\lambda^2}\right)q_j(x) \\ &= \left(\frac{2}{\lambda} + \frac{e^{-\lambda}}{\lambda^2}\right)q_j(x) =: d_\lambda q_j(x). \end{aligned} \tag{3.10}$$

We now apply the construction (3.1)-(3.4) to the sequence of seminorms $\{q_j\}_{j=1}^\infty$ to yield the corresponding sequence $\{(X_j, (\hat{q}_j)_{k(j)})\}_{j=1}^\infty$ of Banach spaces and the quotient maps $Q_j \in \mathcal{L}(X, X_j)$, for $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$. Define a family of operators $\{R_j(\lambda)\}_{\lambda>0}$ on X_j by setting

$$R_j(\lambda)Q_jx := Q_jR(\lambda, A)x, \quad x \in X,\ \lambda > 0. \tag{3.11}$$

Proceeding as in the proof of Theorems 3.3 and 3.5 in [5] (for the operators $T_j(t)$, $t \geq 0$, there) one shows via (3.11) that each $R_j(\lambda)$ is a well defined linear operator on X_j. Moreover, by (3.2), (3.10) and (3.11) we obtain, for each $\lambda > 0$, that

$$\begin{aligned} (\hat{q}_j)_{k(j)}(R_j(\lambda)\hat{x}) &= (\hat{q}_j)_{k(j)}(R_j(\lambda)Q_jx) = (\hat{q}_j)_{k(j)}(Q_jR(\lambda, A)x) \\ &\leq q_{k(j)}(R(\lambda, A)x) \leq d_\lambda q_{k(j)}(x) \end{aligned}$$

for all $\hat{x} \in X_j$ and $x \in X$ with $Q_jx = \hat{x}$. Taking the infimum with respect to $x \in Q_j^{-1}(\{\hat{x}\})$ it follows that

$$(\hat{q}_j)_{k(j)}(R_j(\lambda)\hat{x}) \leq d_\lambda(\hat{q}_j)_{k(j)}(\hat{x}), \quad \hat{x} \in X_j,\ \lambda > 0,$$

and hence, $R_j(\lambda) \in \mathcal{L}(X_j)$ for every $\lambda > 0$. Moreover, relative to the directed set $(0, \lambda_0)$ for some $\lambda_0 > 0$, we have $\tau_b\text{-}\lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A) = 0$ which implies that $\lambda R_j(\lambda) \to 0$ in $\mathcal{L}_b(X_j)$ as $\lambda \downarrow 0^+$. Indeed, since X is a quojection, if $\hat{B}_j$ denotes the closed unit ball of the Banach space X_j, then by [18, Proposition 1] there is $B_j \in \mathcal{B}(X)$ such that $\hat{B}_j \subseteq Q_j(B_j)$. It follows from (3.11), for the operator norm in $\mathcal{L}(X_j)$, that

$$\begin{aligned} &||\lambda R_j(\lambda)|| := \sup_{\hat{x}\in\hat{B}_j}(\hat{q}_j)_{k(j)}(\lambda R_j(\lambda)\hat{x}) \leq \sup_{\hat{x}\in Q_j(B_j)}(\hat{q}_j)_{k(j)}(\lambda R_j(\lambda)\hat{x}) \\ &= \sup_{x\in B_j}(\hat{q}_j)_{k(j)}(\lambda R_j(\lambda)Q_jx) = \sup_{x\in B_j}(\hat{q}_j)_{k(j)}(Q_j\lambda R(\lambda, A)x) \\ &\leq \sup_{x\in B_j} q_{k(j)}(\lambda R(\lambda, A)x), \end{aligned}$$

where $\sup_{x\in B_j} q_{k(j)}(\lambda R(\lambda, A)x) \to 0$ for $\lambda \downarrow 0^+$ as $(T(t))_{t\geq 0}$ is uniformly Abel mean ergodic. So, $\lim_{\lambda\downarrow 0^+} \|\lambda R_j(\lambda)\| = 0$ for each $j \in \mathbb{N}$. Thus, for each $j \in \mathbb{N}$, there is $\lambda_j \in (0, \lambda_0]$ which can be chosen with $\lambda_j < \lambda_{j-1}$ such that $\|\lambda_jR_j(\lambda_j)\| \leq \frac{1}{2}$. This ensures that each operator $I - \lambda_jR_j(\lambda_j) \in \mathcal{L}(X_j)$, for $j \in \mathbb{N}$, is bijective, hence invertible, [20, Ch. VII, Corollary 6.2], with

$$\frac{1}{2}(\hat{q}_j)_{k(j)}(\hat{x}) \leq (\hat{q}_j)_{k(j)}[(I - \lambda_jR_j(\lambda_j))\hat{x}] \leq \frac{3}{2}(\hat{q}_j)_{k(j)}(\hat{x}), \quad \hat{x} \in X_j. \tag{3.12}$$

We can now show that ImA is closed in X. By (3.2), (3.11) and (3.12) and the fact that $j \leq k(j)$ we have, for each $j \in \mathbb{N}$, that

$$\begin{aligned}\frac{1}{2}(\hat{q}_j)_j(\hat{x}) &\leq \frac{1}{2}(\hat{q}_j)_{k(j)}(\hat{x}) \leq (\hat{q}_j)_{k(j)}[(I-\lambda_j R_j(\lambda_j))\hat{x}] \\ &= (\hat{q}_j)_{k(j)}[Q_j(I-\lambda_j R(\lambda_j, A))x] \leq q_{k(j)}[(I-\lambda_j R(\lambda_j, A))x] \\ &= q_{k(j)}(AR(\lambda_j, A)x)\end{aligned}$$

for all $\hat{x} \in X_j$ and $x \in X$ with $Q_j x = \hat{x}$. Since $(\hat{q}_j)_j(\hat{x}) = q_j(x)$ for all $x \in X$ with $Q_j x = \hat{x}$ (cf. (3.3)), the above inequality yields

$$\frac{1}{2}q_j(x) \leq q_{k(j)}(AR(\lambda_j, A)x), \quad x \in X,\ j \in \mathbb{N}. \tag{3.13}$$

Fix $j \in \mathbb{N}$. Let $y \in D(A)$. Since $R(\lambda_j, A)(X) = D(A)$, there is a unique $x \in X$ with $y = R(\lambda_j, A)x$ and so $(\lambda_j I - A)y = (\lambda_j I - A)R(\lambda_j, A)x = x$. Thus, by (3.13) we obtain that $\frac{1}{2}q_j((\lambda_j I - A)y) \leq q_{k(j)}(Ay)$ and, since $j \leq k(j)$, that

$$\begin{aligned}q_j(y) &\leq \lambda_j^{-1}[q_j((\lambda_j I - A)y) + q_j(Ay)] \leq \lambda_j^{-1}[2q_{k(j)}(Ay) + q_j(Ay)] \\ &\leq 3\lambda_j^{-1}q_{k(j)}(Ay), \quad y \in D(A).\end{aligned} \tag{3.14}$$

Recall we are supposing that $X = Y = \overline{\mathrm{Im}A}$. As $X = \mathrm{Ker}\,A \oplus \overline{\mathrm{Im}A}$, we have $\mathrm{Ker}\,A = \{0\}$ and so A is injective. Thus, from (3.14) it follows that

$$q_j(A^{-1}z) \leq 3\lambda_j^{-1}q_{k(j)}(z), \quad z \in \mathrm{Im}A,\ j \in \mathbb{N}. \tag{3.15}$$

The inequalities (3.15) ensure that $A^{-1}\colon \mathrm{Im}A \to D(A)$ is a continuous linear operator. We claim that the closedness of ImA follows. Indeed, let $y \in X = \overline{\mathrm{Im}A}$. Then there is a sequence $\{y_k\}_{k=1}^\infty \subseteq \mathrm{Im}A$ such that $y_k \to y$ in X as $k \to \infty$. It then follows from (3.15) that the sequence $x_k := A^{-1}y_k$, for $k \in \mathbb{N}$, is Cauchy and so converges to some $z \in X$. On the other hand, each $x_k \in D(A)$ and $Ax_k = y_k \to y$ in X as $k \to \infty$ by assumption. Since A is a closed operator, it follows that $z \in D(A)$ and $A(z) = y$, i.e., $y \in \mathrm{Im}A$. This implies that $X = \mathrm{Im}A$ and so ImA is closed.

(1)⇒(6). Let $P := \tau_b\text{-}\lim_{r\to\infty} C(r)$. According to (2.18) we have $X = \mathrm{Ker}\,A \oplus \overline{\mathrm{Im}A}$ with

$$\mathrm{Im}P = \mathrm{Fix}(T(\cdot)) = \mathrm{Ker}\,A,$$
$$\mathrm{Ker}\,P = \overline{\mathrm{span}\{x - T(t)x :\ t \geq 0,\ x \in X\}} = \overline{\mathrm{Im}A}.$$

Moreover, since (1)⇒(5)⇒(2), ImA is closed and so $\overline{\mathrm{Im}A} = \mathrm{Im}A$.

Note that $Y := \mathrm{Im}A$ is a quojection Fréchet space as it is a complemented subspace of X. By (2.16) we have $(\lambda R(\lambda, A))|_{\mathrm{Fix}(T(\cdot))} = I_{\mathrm{Fix}(T(\cdot))} = P|_{\mathrm{Fix}(T(\cdot))}$ and Lemma 3.6 of [7] implies that ImA is $R(\lambda, A)$–invariant for any $\lambda > 0$. So, we may assume that $Y = X$ and, correspondingly, that $C(r) \to 0$ in $\mathcal{L}_b(X)$ as $r \to \infty$. Therefore, we need to prove that $(\lambda R(\lambda, A))^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ for (some) every $\lambda > 0$.

Let $\{r_j\}_{j=1}^\infty$ be any fundamental, increasing sequence of seminorms generating the lc-topology of X. Fix $a > 0$. Then Remark 2.4(i) and Proposition 2.3 ensure that $(T(t))_{t\geq 0}$ is a-exponentially equicontinuous. So, for each $j \in \mathbb{N}$, there is $c_j > 0$ such that

$$r_j(T(t)x) \leq c_j e^{at} r_{j+1}(x), \quad t \geq 0,\ x \in X, \tag{3.16}$$

where there is no loss of generality in taking r_{j+1} as we can pass to a subsequence of $\{r_j\}_{j=1}^\infty$ if necessary.

Fix $j \in \mathbb{N}$ and define a seminorm q_j on X by setting

$$q_j(x) := \sup_{t\geq 0} r_j(e^{-at}T(t)x), \quad x \in X. \tag{3.17}$$

Then (3.16) implies that q_j satisfies

$$r_j(x) \leq q_j(x) \leq c_j r_{j+1}(x), \quad x \in X.$$

Therefore, $\{q_j\}_{j=1}^{\infty}$ is also a fundamental, increasing sequence of seminorms generating the lc-topology of X and (via (3.17)) satisfies, for each $j \in \mathbb{N}$,

$$\begin{aligned} q_j(e^{-at}T(t)x) &= \sup_{s\geq 0} r_j(e^{-as}T(s)(e^{-at}T(t)x)) \\ &= \sup_{s\geq 0} r_j(e^{-a(s+t)}T(s+t)x) \leq q_j(x), \end{aligned}$$

for $x \in X$, $t \geq 0$. Accordingly, for each $j \in \mathbb{N}$, we have

$$q_j(T(t)x) \leq e^{at} q_j(x), \quad x \in X, \ t \geq 0. \tag{3.18}$$

We again apply the construction (3.1)–(3.4), now to the seminorms $\{q_j\}_{j=1}^{\infty}$ given by (3.17), to yield the corresponding sequence of Banach spaces $\{(X_j, (\hat{q}_j)_{k(j)})\}_{j=1}^{\infty}$ and the quotient maps $Q_j \in \mathcal{L}(X, X_j)$, for $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$. Define a family of operators $(T_j(t))_{t\geq 0}$ on X_j via

$$T_j(t)Q_j x := Q_j T(t)x, \quad x \in X, \ t \geq 0. \tag{3.19}$$

By (3.18) and (3.19) we can proceed as in the proof of [5, Theorem 3.3] to show that each $T_j(t)$ is a well defined linear operator on X_j with $T_j(0) = I$. Moreover, by (3.2) and (3.18) we also obtain, for each $t \geq 0$, that

$$\begin{aligned} (\hat{q}_j)_{k(j)}(T_j(t)\hat{x}) &= (\hat{q}_j)_{k(j)}(T_j(t)Q_j x) = (\hat{q}_j)_{k(j)}(Q_j T(t)x) \\ &\leq q_{k(j)}(T(t)x) \leq e^{at} q_{k(j)}(x), \end{aligned}$$

for all $\hat{x} \in X_j$ and $x \in X$ with $Q_j x = \hat{x}$. Taking the infimum with respect to $x \in Q_j^{-1}(\hat{x})$ it follows that

$$(\hat{q}_j)_{k(j)}(T_j(t)\hat{x}) \leq e^{at} (\hat{q}_j)_{k(j)}(\hat{x}), \quad \hat{x} \in X_j, \tag{3.20}$$

and hence, since $(\hat{q}_j)_{k(j)}$ is the norm of X_j, that $T_j(t) \in \mathcal{L}(X_j)$. In particular, $(T_j(t))_{t\geq 0} \subseteq \mathcal{L}(X_j)$ is a–exponentially equicontinuous. Moreover, for each $t > 0$ and $\hat{x} \in X_j$ with $\hat{x} = Q_j x$, we have via (3.2) and (3.19) that

$$\begin{aligned} &(\hat{q}_j)_{k(j)}(T_j(t)\hat{x} - \hat{x}) = (\hat{q}_j)_{k(j)}(T_j(t)Q_j x - Q_j x) \\ &= (\hat{q}_j)_{k(j)}(Q_j(T(t)x - x)) \leq q_{k(j)}(T(t)x - x), \end{aligned}$$

and so, $(\hat{q}_j)_{k(j)}(T_j(t)\hat{x} - \hat{x}) \to 0$ as $t \to 0^+$ as $(T(t))_{t\geq 0}$ is a C_0–semigroup. Since $\hat{x}$ is arbitrary, $(T_j(t))_{t\geq 0}$ is also a C_0–semigroup. According to [6, Remark 1(iii)] the C_0–semigroup $(T_j(t))_{t\geq 0}$ is strongly continuous at every point $t \geq 0$. Moreover, $\frac{T_j(t)}{t} \to 0$ for the operator norm in $\mathcal{L}_b(X_j)$ as $t \to \infty$. Indeed, since X is a quojection, if $\hat{B}_j$ denotes the closed unit ball of the Banach space X_j, then by [18, Proposition 1] there is $B_j \in \mathcal{B}(X)$ so that $\hat{B}_j \subseteq Q_j(B_j)$. It follows from (3.19), for the operator norm in $\mathcal{L}_b(X_j)$, that for each $t > 0$ we have

$$\begin{aligned} &\left\| \frac{T_j(t)}{t} \right\| = \sup_{\hat{x}\in \hat{B}_j} \frac{1}{t} (\hat{q}_j)_{k(j)}(T_j(t)\hat{x}) \\ &\leq \sup_{\hat{x}\in Q_j(B_j)} \frac{1}{t} (\hat{q}_j)_{k(j)}(T_j(t)\hat{x}) = \sup_{x\in B_j} \frac{1}{t} (\hat{q}_j)_{k(j)}(T_j(t)Q_j x) \\ &= \sup_{x\in B_j} \frac{1}{t} (\hat{q}_j)_{k(j)}(Q_j T(t)x) \leq \sup_{x\in B_j} \frac{1}{t} q_{k(j)}(T(t)x). \end{aligned}$$

Since $\sup_{x\in B_j} \frac{1}{t} q_{k(j)}(T(t)x) \to 0$ as $t \to \infty$, this implies that $\left\| \frac{T_j(t)}{t} \right\| \to 0$ as $t \to \infty$.

Denote by $(A_j, D(A_j))$ the infinitesimal generator of $(T_j(t))_{t\geq 0}$. It follows from (3.19) that the family $\{R(\lambda, A_j)\}_{\lambda\in\mathbb{C},\mathrm{Re}(\lambda)>0} \subseteq \mathcal{L}(X_j)$ of resolvent operators of $(A_j, D(A_j))$ exists and satisfies

$$R(\lambda, A_j)Q_j = Q_j R(\lambda, A), \quad \lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > 0 \tag{3.21}$$

in $\mathcal{L}(X, X_j)$. Next, for each $r > 0$, define

$$C_j(r)\hat{x} := \frac{1}{r}\int_0^r T_j(s)\hat{x}\,ds, \quad \hat{x} \in X_j,$$

and observe that by (3.19), the continuity of $Q_j\colon X \to X_j$ and [6, Proposition 11(vi)], we have

$$\begin{aligned} C_j(r)Q_j x &= \frac{1}{r}\int_0^r T_j(s)Q_j x\,ds = \frac{1}{r}\int_0^r Q_j T(s)x\,ds \\ &= Q_j\left(\frac{1}{r}\int_0^r T(s)x\,ds\right) = Q_j C(r)x, \quad x \in X. \end{aligned} \tag{3.22}$$

To see that $C_j(r) \to 0$ in $\mathcal{L}_b(X_j)$ as $r \to \infty$, choose $B_j \in \mathcal{B}(X)$ such that $\hat{B}_j \subseteq Q_j(B_j)$. For each $r > 0$, it follows via (3.2) and (3.22) that

$$\begin{aligned} \sup_{\hat{x}\in\hat{B}_j} (\hat{q}_j)_{k(j)}(C_j(r)\hat{x}) &\leq \sup_{\hat{x}\in Q_j(B_j)} (\hat{q}_j)_{k(j)}(C_j(r)\hat{x}) = \sup_{x\in B_j} (\hat{q}_j)_{k(j)}(C_j(r)Q_j x) \\ &= \sup_{x\in B_j} (\hat{q}_j)_{k(j)}(Q_j C(r)x) \leq \sup_{x\in B_j} q_{k(j)}(C(r)x). \end{aligned}$$

But, $\sup_{x\in B_j} q_{k(j)}(C(r)x) \to 0$ as $r \to \infty$ by assumption. So, it follows that

$$\|C_j(r)\| = \sup_{\hat{x}\in\hat{B}_j} (\hat{q}_j)_{k(j)}(C_j(r)\hat{x}) \to 0$$

as $r \to \infty$, i.e., $C_j(r) \to 0$ in $\mathcal{L}_b(X_j)$ as $r \to \infty$. According to [34, Theorem 12] we have that $\|(\lambda R(\lambda, A_j))^n\| \to 0$ in $\mathcal{L}_b(X_j)$ as $n \to \infty$ for (some) every $\lambda > 0$. Since $j \in \mathbb{N}$ is arbitrary, it follows that $(\lambda R(\lambda, A))^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ for (some) every $\lambda > 0$. Indeed, by (3.21) we have (in $\mathcal{L}(X, X_j)$) that

$$Q_j(\lambda R(\lambda, A))^n = \lambda R(\lambda, A_j)Q_j(\lambda R(\lambda, A))^{n-1} = \ldots = (\lambda R(\lambda, A_j))^n Q_j,$$

for $j, n \in \mathbb{N}$ and (some) all $\lambda > 0$. Fix any $j \in \mathbb{N}$ and $B \in \mathcal{B}(X)$. The previuos identity and (3.3) yield

$$\begin{aligned} \sup_{x\in B} q_j((\lambda R(\lambda, A))^n x) &= \sup_{x\in B} (\hat{q}_j)_j(Q_j(\lambda R(\lambda, A))^n x) = \sup_{x\in B} (\hat{q}_j)_j((\lambda R(\lambda, A_j))^n Q_j x) \\ &\leq \sup_{\hat{x}\in Q_j(B)} (\hat{q}_j)_{k(j)}((\lambda R(\lambda, A_j))^n \hat{x}), \quad n \in \mathbb{N}, \end{aligned}$$

for (some) all $\lambda > 0$, where $\sup_{\hat{x}\in Q_j(B)} (\hat{q}_j)_{k(j)}((\lambda R(\lambda, A_j))^n \hat{x}) \to 0$ as $n \to \infty$ for (some) every $\lambda > 0$ as $Q_j(B) \subseteq d_j \hat{B}_j$ for some $d_j > 0$ by the continuity of $Q_j\colon X \to X_j$.

As $j \in \mathbb{N}$ and $B \in \mathcal{B}(X)$ are arbitrary, we obtain that $(\lambda R(\lambda, A))^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ for (some) every $\lambda > 0$.

(6)$\Rightarrow$(3). Let $\lambda > 0$ be such that $P := \tau_b\text{-}\lim_{n\to\infty}(\lambda R(\lambda, A))^n$ exists. By Remark 3.1 also $P = \tau_b\text{-}\lim_{n\to\infty}(R(\lambda, A))_{[n]}$. This is precisely condition (3).

(1)$\Rightarrow$(7). Because of (1)$\Rightarrow$(2) we have $\mathrm{Im}A = \overline{\mathrm{Im}A}$. Then (2.18) implies that $\mathrm{Im}A$ is a complemented subspace of X and hence, $\mathrm{Im}A$ is a quojection Fréchet space (as X is a quojection Fréchet space).

Fix $y \in \mathrm{Im}A$. Then, for each $\lambda > 0$, we can write $R(\lambda, A)y = -x + \lambda R(\lambda, A)x$ for some $x \in D(A)$ satisfying $Ax = y$. Since (1)$\Rightarrow$(5), the limit of the net $\{\lambda R(\lambda, A)\}_{0<\lambda\leq\lambda_0}$ exists in $\mathcal{L}_b(X)$ as $\lambda \downarrow 0^+$ (for some $\lambda_0 > 0$). In particular, $z := \lim_{\lambda\to 0^+} \lambda R(\lambda, A)x$ exists in X.

So, for a given $p \in \Gamma_X$, there is $\lambda' = \lambda'(x,p) \in (0,\lambda_0]$ such that $p(\lambda R(\lambda,A)x - z) < 1$ for all $0 < \lambda < \lambda'$. It follows that

$$\begin{aligned} p(R(\lambda,A)y) &= p(-x + \lambda R(\lambda,A)x) \leq p(x) + p(\lambda R(\lambda,A)x - z) + p(z) \\ &< p(x) + p(z) + 1, \quad 0 < \lambda < \lambda'. \end{aligned} \tag{3.23}$$

On the other hand, by the equicontinuity of $\{R(\lambda,A) : \lambda \geq \lambda'\}$ (cf. Remark 2.4(i) and Proposition 2.3) there exist $M_p > 0$ and $q \in \Gamma_X$ such that $p(R(\lambda,A)u) \leq M_p q(u)$, for $u \in X$ and $\lambda \geq \lambda'$. In particular,

$$p(R(\lambda,A)y) \leq M_p q(y), \ \lambda' \leq \lambda \leq \lambda_0. \tag{3.24}$$

By (3.23) and (3.24) we see that $\sup_{\lambda\in(0,\lambda_0]} p(R(\lambda,A)y) < \infty$. As p is arbitrary, this implies that $\{R(\lambda,A)y : \lambda \in (0,\lambda_0]\} \in \mathcal{B}(X)$.

(7)⇒(2). By assumption $Y := \overline{\mathrm{Im}A}$ is a quojection Fréchet space and $\{R(\lambda,A)y : \lambda \in (0,\lambda_0]\} \in \mathcal{B}(X)$ for every $y \in Y$ and some fixed $\lambda_0 > 0$. Then $(T(t)|_Y)_{t\geq 0}$ is a locally equicontinuous C_0–semigroup on Y whose infinitesimal generator $(\tilde{A}_1, D(\tilde{A}_1))$ is given by $\tilde{A}_1 x := Ax$ for $x \in D(\tilde{A}_1) := D(A) \cap Y$ and with the resolvent operator $R(\lambda,\tilde{A}_1) = R(\lambda,A)|_Y$ for every $\lambda \in \rho(A)$; see Remark 2.14. Moreover, τ_b-$\lim_{t\to\infty} \frac{T(t)|_Y}{t} = 0$.

Since $\{R(\lambda,\tilde{A}_1) : \lambda \geq \lambda_0\}$ is equicontinuous (apply Remark 2.4(i) and Proposition 2.3 to $(T(t)|_Y)_{t\geq 0}$), it follows that $\{R(\lambda,A)y : \lambda > 0\} = \{R(\lambda,\tilde{A}_1)y : \lambda > 0\}$, being the union of two bounded sets, belongs to $\mathcal{B}(Y) \subseteq \mathcal{B}(X)$ for every $y \in Y$. This implies that the net $\{\lambda R(\lambda,\tilde{A}_1)\}_{0<\lambda\leq\lambda_0}$ converges to 0 in $\mathcal{L}_b(Y)$ for $\lambda \downarrow 0^+$. Indeed, fix $B \in \mathcal{B}(Y)$ and $p \in \Gamma_Y$. Then p_B is a continuous seminorm in $\mathcal{L}_b(Y)$ and $C := \cup_{\lambda>0} R(\lambda,\tilde{A}_1)(B)$ is bounded in Y. Set $M := \sup_{c\in C} p(c) < \infty$. Given any $\varepsilon > 0$, select $\lambda' > 0$ such that $\lambda' < \min\{\lambda_0, \varepsilon/M\}$. Then, for any $\lambda \in (0,\lambda')$ it follows that

$$p_B(\lambda R(\lambda,\tilde{A}_1)) = \sup_{z\in B} p(\lambda R(\lambda,\tilde{A}_1)z) \leq \lambda \sup_{c\in C} p(c) \leq \lambda M < \varepsilon.$$

The arbitrariness of B, p and ε implies that τ_b-$\lim_{\lambda\downarrow 0^+} \lambda R(\lambda,\tilde{A}_1) = 0$ in $\mathcal{L}_b(Y)$, i.e., $(T(t)|_Y)_{t\geq 0}$ is uniformly Abel mean ergodic in Y.

Since Y is a quojection Fréchet space, we can apply (5)⇒(2) to $(T(t)|_Y)_{t\geq 0}$ to conclude that $\mathrm{Im}\tilde{A}_1$ is closed in Y and so, $\mathrm{Im}\tilde{A}_1 = Y$. Thus, we have $Y = \mathrm{Im}\tilde{A}_1 \subseteq \mathrm{Im}A \subseteq Y$, i.e., $Y = \mathrm{Im}A$. This means that $\mathrm{Im}A$ is closed in X, which is precisely condition (2). □

Remark 3.3. In the proof of (1)⇒(6) in Theorem 3.2, with $P := \tau_b$-$\lim_{r\to\infty} C(r)$, it was shown, for each $\lambda > 0$, that $(\lambda R(\lambda,A))^n|_{\mathrm{Fix}(T(\cdot))} = P|_{\mathrm{Fix}(T(\cdot))}$, for all $n \in \mathbb{N}$, and so $(\lambda R(\lambda,A))^n \to P$ for $n \to \infty$ (relative to τ_b) on $\mathrm{Fix}(T(\cdot)) = \mathrm{Im}P$. It was also proved that $(\lambda R(\lambda,A))^n \to 0$ for $n \to \infty$ (relative to τ_b) on $\mathrm{Im}A = \overline{\mathrm{Im}A} = \mathrm{Ker}\,P$. Hence, for each $\lambda > 0$, the limit of $\{(\lambda R(\lambda,A))^n\}_{n=1}^\infty$ in $\mathcal{L}_b(X)$ is actually the projection $P \in \mathcal{L}(X)$.

A *prequojection* is a Fréchet space X such that X'' is a quojection. Every quojection is a prequojection. A prequojection is called *non–trivial* if it is not itself a quojection. It is known that X is a prequojection if and only if X'_β is a strict (LB)–space. An alternative characterization is that X is a prequojecton if and only if X has no Köthe nuclear quotient which admits a continuous norm; see [12, 17, 40, 45]. This implies that a quotient of a prequojection is again a prequojection. In particular, every complemented subspace of a prequojection is again a prequojection. The problem of the existence of non–trivial prequojections arose in a natural way in [12]; it has been solved, in the positive sense, in various papers, [13], [17], [39]. All of these papers employ the same method, which consists in the construction of the dual of a prequojection, rather than the prequojection itself, which is often difficult to describe (see the survey paper [36] for further information). However, in [37] an alternative method for constructing prequojections is presented which has the advantage of being direct. For an example of a concrete space (i.e., a space of continuous functions on a suitable topological space), which is a non–trivial prequojection, see [1].

The following extension of Theorem 3.2 is relevant for *non–trivial* prequojection Fréchet spaces.

Proposition 3.4. *Let X be a prequojection Fréchet space and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a uniformly continuous C_0–semigroup satisfying τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$. Then the infinitesimal generator A of $(T(t))_{t\geq 0}$ belongs to $\mathcal{L}(X)$. Moreover, the following assertions are equivalent.*

(1) *The semigroup $(T(t))_{t\geq 0}$ is uniformly mean ergodic.*
(2) $\mathrm{Im}A$ *is a closed subspace of X.*
(3) *The operator $\lambda R(\lambda, A)$ is uniformly mean ergodic for every $\lambda > 0$.*
(4) *The operator $\lambda R(\lambda, A)$ is uniformly mean ergodic for some $\lambda > 0$.*
(5) *The semigroup $(T(t))_{t\geq 0}$ is uniformly Abel mean ergodic.*
(6) *The sequence of iterates $\{(\lambda R(\lambda, A))^n\}_{n=1}^\infty$ converges in $\mathcal{L}_b(X)$ for (some) every $\lambda > 0$.*
(7) $\overline{\mathrm{Im}A}$ *is a prequojection and there exists $\lambda_0 > 0$ such that*

$$\{R(\lambda, A)y : \ y \in (0, \lambda_0]\} \in \mathcal{B}(X), \quad y \in \overline{\mathrm{Im}A}.$$

Proof. According to Remark 2.2(ii) the semigroup $(T(t))_{t\geq 0}$ is locally equicontinuous. Furthermore, Remark 2.4(ii) implies that $(T(t))_{t\geq 0}$ is exponentially equicontinuous. Then [5, Proposition 3.4] yields that $A \in \mathcal{L}(X)$.

The proofs of (2)⇔(3)⇔(4), (1)⇒(5), (2)⇒(1) and (6)⇒(3) are exactly the same as in Theorem 3.2 after taking into account that Theorem 3.4 of [8] is also valid in prequojection Fréchet spaces.

In order to establish (5)⇒(2) and (1)⇒(6) we first observe, since X is a prequojection Fréchet space, that X'_β is a barrelled strict (LB)–space (being the strong dual of a quasinormable Fréchet space) and X'' is a quojection Fréchet space. Moreover, X'_β is complete, [29, §28, 5(1), p.385]. Applying twice Lemma 2.15, we conclude that $(T(t)'')_{t\geq 0} \subseteq \mathcal{L}(X'')$ is a uniformly continuous C_0-semigroup satisfying τ_b-$\lim_{t\to\infty} \frac{T(t)''}{t} = 0$. It follows from Lemma 2.15 (which ensures that $(T(t)')_{t\geq 0}$ is a locally equicontinuous, uniformly continuous C_0-semigroup on X'_β satisfying τ_b-$\lim_{t\to\infty} \frac{T(t)'}{t} = 0$), the formula (2.17) and a standard duality argument (based on properties of the Riemann integral, [5, Proposition 11]) that the Cesàro means of $(T(t)')_{t\geq 0}$ are precisely the dual operators $\{C(r)'\}_{r\geq 0}$ of $\{C(r)\}_{r\geq 0}$. Repeating the argument it follows that the bidual operators $\{C(r)''\}_{r\geq 0}$ form the family of Cesáro means of $(T(t)'')_{t\geq 0}$. Of course, $A'' \in \mathcal{L}(X'')$ is the infinitesimal generator of $(T(t)'')_{t\geq 0}$. Applying Proposition 2.3, Remark 2.4(i) and Lemma 2.15 to $(T(t))_{t\geq 0}$ and $(T(t)'')_{t\geq 0}$ yields $\mathbb{C}_{0+} \subseteq \rho(A) \cap \rho(A'')$ and $R(\lambda, A)'' = R(\lambda, A'')$ for every $\lambda \in \mathbb{C}_{0+}$. Now we can proceed with the proof of further equivalences.

(5)⇒(2). Let $P := \tau_b$-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A)$. Since X'_β is barrelled, it follows by applying Lemma 2.1 of [3] twice that $\lambda R(\lambda, A'') = \lambda R(\lambda, A)'' \to P''$ in $\mathcal{L}_b(X'')$ as $\lambda \to 0^+$. Hence, $(T(t)'')_{t\geq 0}$ is uniformly Abel mean ergodic. Proposition 2.12 applied to $(T(t)'')_{t\geq 0}$ shows that P'' is the projection of X'' onto $\mathrm{Ker}\, A'' = \mathrm{Im}P'' = \mathrm{Fix}(T(\cdot)'')$.

Since X'' is a quojection Fréchet space, we can apply Theorem 3.2 to conclude that the Cesáro means $\{(R(1, A''))_{[n]}\}_{n=1}^\infty$ converge in $\mathcal{L}_b(X''_\beta)$. As X is an invariant subspace for $R(1, A)'' = R(1, A'')$ and bounded subsets of X are bounded in X'', it follows that $\{(R(1, A))_{[n]}\}_{n=1}^\infty$ converges in $\mathcal{L}_b(X)$, i.e., condition (4) holds. But, (4)⇔(2) and so (2) holds.

(1)⇒(6). Let $Q := \tau_b\text{-}\lim_{r\to\infty} C(r)$. Again by Lemma 2.1 of [3], applied twice, it follows that $C(r)'' \to Q''$, in $\mathcal{L}_b(X'')$ as $r \to \infty$, i.e., $(T(t)'')_{t\geq 0}$ is uniformly mean ergodic. Since X'' is a quojection Fréchet space, we can apply Theorem 3.2 and so Remark 3.3 to conclude that $(\lambda R(\lambda, A''))^n \to Q''$ in $\mathcal{L}_b(X'')$ as $n \to \infty$ for every $\lambda > 0$. As X is an invariant subspace of $R(\lambda, A)'' = R(\lambda, A'')$, for every $\lambda > 0$, it follows that $\{(\lambda R(\lambda, A))^n\}_{n=1}^\infty$ converges in $\mathcal{L}_b(X)$ to Q, i.e., condition (6) holds.

So, we have established that all equivalences (1)⇔(2)⇔ ... ⇔(6) are available for $(T(t))_{t\geq 0}$.

(1)⇒(7). Using the availability of all equivalences just mentioned for $(T(t))_{t\geq 0}$ and the fact that a complemented subspace of a prequojection Fréchet space is again a prequojection Fréchet space (in place of the same fact for quojection Fréchet spaces), the *same proof* as for (1)⇒(7) in Theorem 3.2 applies again.

(7)⇒(2). By assumption $Y = \overline{\mathrm{Im} A}$ is a prequojection Fréchet space. The same proof as for (7)⇒(2) in Theorem 3.2 shows that $(T(t)|_Y)_{t\geq 0}$ is uniformly Abel mean ergodic in Y. Now, apply (5)⇒(2), which is available in the prequojection Fréchet space setting, to conclude that (2) holds (as in the proof of (7)⇒(2) in Theorem 3.2). □

Remark 3.5. The assumption that $(T(t))_{t\geq 0}$ is a *uniformly* continuous C_0–semigroup is needed to guarantee that the dual and bidual semigroups $(T(t)')_{t\geq 0}$ and $(T(t)'')_{t\geq 0}$ are also (uniformly continuous) C_0–semigroups on X'_β and X'' resp. Recall that in general the dual semigroup of a strongly continuous C_0–semigroup need not be a C_0–semigroup, even in Banach spaces.

A lcHs X is a *Grothendieck space* if sequences in X' which are convergent for $\sigma(X', X)$ also converge for $\sigma(X', X'')$. Reflexive lcHs' are Grothendieck spaces. A lcHs X has the *Dunford–Pettis property* (briefly, DP) if every element of $\mathcal{L}(X, Y)$, for Y any quasicomplete lcHs, which transforms elements of $\mathcal{B}(X)$ into relatively $\sigma(Y, Y')$–compact subsets of Y, also transforms $\sigma(X, X')$–compact subsets of X into relatively compact subsets of Y, [22, pp.633-634]. It suffices if Y runs through all Banach spaces, [16, p.79]. A reflexive lcHs has the DP property if and only if it is Montel, [22, p.634]. A Grothendieck lcHs X with the DP property is called a GDP-space. Every Montel lcHs is a GDP-space, [16, Remark 2.2], [3, Corollary 3.8]. For further information on non–normable GDP-spaces we refer to [3], [10], [16].

Corollary 3.6. *Let X be a prequojection GDP-Fréchet space and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup satisfying $\tau_b\text{-}\lim_{t\to\infty} \frac{T(t)}{t} = 0$. Then its infinitesimal generator $A \in \mathcal{L}(X)$. Moreover, all seven assertions in Proposition 3.4 are equivalent.*

Proof. Since X is a GDP–Fréchet space, the semigroup $(T(t))_{t\geq 0}$ is necessarily uniformly continuous, [6, Theorem 7]. So, the result follows from Proposition 3.4. □

Example 3.7. The validity of Theorem 3.2 and Proposition 3.4 remains confined to the setting of prequojection Fréchet spaces. Indeed, consider the semigroup $(T(t))_{t\geq 0}$ constructed in [5, Example 3.1] and acting in the nuclear Fréchet space $\lambda_1(B)$. More precisely, let $B = (a_n(i))_{i,n\in\mathbb{N}}$ be a Köthe matrix, i.e., $1 \leq a_n(i) \leq a_{n+1}(i)$ for all $i, n \in \mathbb{N}$. Then the space

$$\lambda_1(B) := \left\{ x = (x_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \ p_n(x) = \sum_{i\in\mathbb{N}} a_n(i)|x_i| < \infty, \ \forall n \in \mathbb{N} \right\}$$

is Fréchet relative to the lc–topology generated by the sequence of norms $\{p_n\}_{n=1}^\infty$. Choose B such that $\lambda_1(B)$ is nuclear, i.e., $\left(\frac{a_n(i)}{a_{n+1}(i)}\right)_{i\in\mathbb{N}} \in \ell^1$ for all $n \in \mathbb{N}$ (pass to a subsequence if necessary), in which case $\lambda_1(B)$ is not a prequojection. Let $\mu = (\mu_i)_{i\in\mathbb{N}}$ be a sequence of real numbers with each $\mu_i > 0$ and $\lim_{i\to\infty} \mu_i = 0$. For each $t \geq 0$ let $T(t) \in \mathcal{L}(\lambda_1(B))$ defined by $T(t)x := (e^{-\mu_i t} x_i)_{i\in\mathbb{N}}$ for $x \in \lambda_1(B)$. Then $(T(t))_{t\geq 0}$ is an equicontinuous (in particular, τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$), uniformly continuous C_0–semigroup on $\lambda_1(B)$ with infinitesimal generator $(A, D(A))$ given by $Ax := (-\mu_i x_i)_{i\in\mathbb{N}}$ for $x \in D(A) = \{x \in \lambda_1(B) : \mu\cdot x := (\mu_i x_i)_{i\in\mathbb{N}} \in \lambda_1(B)\}$. Moreover, A is clearly injective and $\mathrm{Im}A$ is a dense subspace of $\lambda_1(B)$. Indeed, $\{e_j\}_{j=1}^\infty \subseteq \mathrm{Im}A$ where e_j denotes the element of $\lambda_1(B)$ with a 1 in the j-th coordinate and 0's elsewhere and so $\mathrm{span}\{e_j\}_{j=1}^\infty$ is dense in $\lambda_1(B)$. So, there exists the closed, densely defined linear operator $A^{-1}\colon \mathrm{Im}A \to D(A)$ given by $A^{-1}x = \left(-\frac{1}{\mu_i} x_i\right)_{i\in\mathbb{N}}$ for $x \in \mathrm{Im}A$. In particular, if μ grows fast enough (eg., $\mu_i = \sum_{n=1}^i a_n(i)$ for $i \in \mathbb{N}$), then $D(A)$ is a proper dense subspace of $\lambda_1(B)$ because $(1/\mu_i)_{i\in\mathbb{N}} \in \lambda_1(B) \setminus D(A)$.

The semigroup $(T(t))_{t\geq 0}$ is mean ergodic (hence, uniformly mean ergodic as $\lambda_1(B)$ is nuclear and thus Montel) because τ_s-$\lim_{r\to\infty} C(r) = 0$ via equicontinuity of $\{C(r)\}_{r\geq 0}$ (as $(T(t))_{t\geq 0}$ is equicontinuous), [6, Remark 4(ii), Remark 5(i), (iii), (v)]. So, condition (1) of Theorem 3.2 holds. By [7, Theorem 5.5(i)] also condition (5) of Theorem 3.2 holds.

We claim that τ_b-$\lim_{n\to\infty}(\lambda R(\lambda, A))^n = 0$ for every $\lambda > 0$, i.e., condition (6) of Theorem 3.2 holds. Indeed, fix any $\lambda > 0$. Then $\lambda R(\lambda, A)x = \left(\frac{\lambda}{\lambda+\mu_i} x_i\right)_{i\in\mathbb{N}}$, for $x \in \lambda_1(B)$, and so $[\lambda R(\lambda, A)]^n x = \left(\left(\frac{\lambda}{\lambda+\mu_i}\right)^n x_i\right)_{i\in\mathbb{N}}$ for each $x \in \lambda_1(B)$ and $n \in \mathbb{N}$. Now, fix $x \in \lambda_1(B)$. Given $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that $\sum_{i>i_0} a_k(i)|x_i| < \varepsilon/2$ and so also $\sum_{i>i_0} a_k(i) \left|\left(\frac{\lambda}{\lambda+\mu_i}\right)^n x_i\right| < \varepsilon/2$ for every $n \in \mathbb{N}$ as $0 < \frac{\lambda}{\lambda+\mu_i} < 1$ for each $i \in \mathbb{N}$. On the other hand, there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{i_0} a_k(i) \left|\left(\frac{\lambda}{\lambda+\mu_i}\right)^{n_0} x_i\right| < \varepsilon/2$ because $\lim_{n\to\infty} \left(\frac{\lambda}{\lambda+\mu_i}\right)^n = 0$ for all $1 \leq i \leq i_0$. Since the sequence $\left\{\left(\frac{\lambda}{\lambda+\mu_i}\right)^n\right\}_{n=1}^\infty$ is decreasing, for each $1 \leq i \leq i_0$, it follows that $\sum_{i=1}^{i_0} a_k(i) \left|\left(\frac{\lambda}{\lambda+\mu_i}\right)^n x_i\right| < \varepsilon/2$ for all $n \geq n_0$. So, $p_k((\lambda R(\lambda, A))^n x) < \varepsilon$ for all $n \geq n_0$. The arbitrariness of k and $\varepsilon > 0$ yields that τ_s-$\lim_{n\to\infty}(\lambda R(\lambda, A))^n = 0$ and hence, that τ_b-$\lim_{n\to\infty}(\lambda R(\lambda, A))^n = 0$ as $\lambda_1(B)$ is Montel.

On the other hand, $\mathrm{Im}A$ is dense in $\lambda_1(B)$ but *not* closed, i.e., condition (2) of Theorem 3.2 *fails* to hold. Indeed, in case $\mathrm{Im}A$ is closed, we have $\mathrm{Im}A = \lambda_1(B)$ and so $A^{-1}\colon \lambda_1(B) \to D(A)$ (with A^{-1} continuous by the Closed Graph Theorem). Thus, $A^{-1}(1/\mu) = (-1)_{i\in\mathbb{N}} \in D(A) \subseteq \lambda_1(B)$ which is *not* the case.

4. Applications

The purpose of this section is to present some relevant examples of semigroups acting in quojection Fréchet spaces and to determine whether or not they are mean ergodic/uniformly mean ergodic.

4.1. **A semigroup of multiplication operators in $C(\mathbb{R})$.** Let $X = C(\mathbb{R})$ be the space of all $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ with the compact open topology. Then X is a quojection Fréchet space and its lc–topology is generated by the increasing sequence of seminorms defined by

$$q_k(f) := \sup_{|x|\leq k} |f(x)|, \quad f \in X,$$

for $k \in \mathbb{N}$. Let $\varphi \in X\backslash\{0\}$ be $\mathbb{R}$–valued and consider the multiplication operator $A\colon X \to X$ defined by

$$Af := \varphi f, \quad f \in X.$$

Recall that $S \in \mathcal{L}(X)$, with X any lcHs, is *power bounded* if $\{S^n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}(X)$ is equicontinuous.

Proposition 4.1. *The following properties hold for A.*

(1) $A \in \mathcal{L}(X)$.
(2) $A^n f = \varphi^n f$ *for all* $n \in \mathbb{N}$ *and* $f \in X$.
(3) A *is power bounded if and only if* $\varphi(\mathbb{R}) \subseteq [-1, 1]$.
(4) *If* $\varphi(x) \neq 0$ *for every* $x \in \mathbb{R}$, *then* A *is surjective.*
(5) *The resolvent operator* $R(\lambda, A)$ *exists in* $\mathcal{L}(X)$ *if and only if* $\lambda \notin \varphi(\mathbb{R})$. *Equivalently,* $\rho(A) = \mathbb{C} \setminus \varphi(\mathbb{R})$.
(6) (A, X) *is the infinitesimal generator of the uniformly continuous* C_0*–semigroup* $(T(t))_{t\geq 0}$ *on* X *given by* $T(t)f = e^{t\varphi} f$ *for all* $t \geq 0$ *and* $f \in X$.
(7) $(T(t))_{t\geq 0}$ *is equicontinuous if and only if* $\varphi(\mathbb{R}) \subseteq (-\infty, 0]$.
(8) $(T(t))_{t\geq 0}$ *is exponentially equicontinuous if and only if there exists* $L \geq 0$ *such that* $\varphi(x) \leq L$ *for every* $x \in \mathbb{R}$.

Proof. It is routine to verify that (1) and (2) are valid.

(3) Suppose that $|\varphi| \leq 1$. By part (2) we have

$$q_k(A^n f) = \sup_{|x|\leq k} |(\varphi(x))^n f(x)| \leq \sup_{|x|\leq k} |f(x)| = q_k(f), \quad f \in X,\ n \in \mathbb{N},$$

for each $k \in \mathbb{N}$. Hence, $\{A^n\}_{n\in\mathbb{N}}$ is equicontinuous, i.e., A is power bounded.

On the other hand, suppose there is some $x_0 \in \mathbb{R}$ such that $|\varphi(x_0)| > 1$. Choose $k_0 \in \mathbb{N}$ such that $x_0 \in [-k_0, k_0]$ and let $f_0 \equiv 1 \in X$. Then $|(\varphi(x_0))^n| \leq q_{k_0}(A^n f_0)$ for all $n \in \mathbb{N}$ and so $\sup_n q_{k_0}(A^n f_0) = \infty$, i.e., $\{A^n f_0 :\ n \in \mathbb{N}\} \notin \mathcal{B}(X)$. Accordingly, $\{A^n\}_{n\in\mathbb{N}}$ is not equicontinuous, i.e., A is not power bounded.

(4) Fix any $g \in X$. Since $\varphi(x) \neq 0$ for every $x \in \mathbb{R}$, we can define $f := g/\varphi$ pointwise on $\mathbb{R}$. Then $f \in X$ and satisfies $Af = g$.

(5) Let $\lambda \in \mathbb{C}$. Suppose that $\lambda \notin \varphi(\mathbb{R})$. Then the operator of multiplication by $1/(\lambda-\varphi)$, namely

$$R(\lambda, A)f := \frac{f}{\lambda - \varphi}, \quad f \in X, \tag{4.1}$$

is clearly linear and satisfies $R(\lambda, A)(\lambda I - A) = (\lambda I - A)R(\lambda, A) = I$ on X. Continuity follows from $q_k(R(\lambda, A)f) \leq M_k(\lambda, A)q_k(f)$, for $f \in X$ and $k \in \mathbb{N}$, with $M_k(\lambda, A) := \max_{|x|\leq k} 1/|\lambda - \varphi(x)| < \infty$.

On the other hand, if $R(\lambda, A) \in \mathcal{L}(X)$ exists, i.e., $R(\lambda, A)(\lambda I-A) = (\lambda I-A)R(\lambda, A) = I$ on X, then for the constant function $f_0 \equiv 1$ on $\mathbb{R}$ we have $(\lambda I - A)f_0 = (\lambda - \varphi)$ and so $(\lambda - \varphi)R(\lambda, A)f_0 = f_0$. Consequently, $\lambda \notin \varphi(\mathbb{R})$.

(6) We first show that $(T(t))_{t\geq 0}$ is a locally equicontinuous, uniformly continuous C_0–semigroup on X. Clearly, it is a semigroup.

Fix $k \in \mathbb{N}$ and $B \in \mathcal{B}(X)$. Then $T(t)f - f = (e^{t\varphi} - 1)f$, for $t \geq 0$ and $f \in B$, and $\alpha_k(B) := \sup_{f\in B} q_k(f) < \infty$. Moreover, for $t > 0$ we have

$$\begin{aligned}\sup_{f\in B} q_k(T(t)f - f) &= \sup_{f\in B} \sup_{|x|\leq k} |e^{t\varphi(x)} - 1| \cdot |f(x)| \\ &\leq \alpha_k(B) q_k(e^{t\varphi} - 1) \leq \alpha_k(B) \cdot (e^{tq_k(\varphi)} - 1).\end{aligned}$$

Since $\lim_{t\to 0^+}(e^{tq_k(\varphi)} - 1) = 0$, this ensures that $\sup_{f\in B} q_k(T(t)f - f) \to 0$ as $t \to 0^+$. By the arbitrariness of k and B we conclude that τ_b-$\lim_{t\to 0^+} T(t) = I$.

Fix $R > 0$. Then, for every $k \in \mathbb{N}$, $f \in X$ and all $t \in [0, R]$, we have

$$q_k(T(t)f) = \sup_{|x|\leq k} |e^{t\varphi(x)} f(x)| \leq q_k(f) \sup_{|x|\leq k} e^{t\varphi(x)} \leq e^{Rq_k(\varphi)} q_k(f).$$

This implies that the semigroup $(T(t))_{t\geq 0}$ is locally equicontinuous. Since τ_b-$\lim_{t\to 0^+} T(t) = I$, it follows from the discussion prior to Remark 2.2 that $(T(t))_{t\geq 0}$ is uniformly continuous.

Now if $f \in X$, then $\frac{T(t)f-f}{t} = \frac{e^{t\varphi}-1}{t} f$ for all $t > 0$ and so,

$$\frac{T(t)f-f}{t} - Af = \left(\frac{e^{t\varphi} - 1 - t\varphi}{t} \right) f, \quad t > 0.$$

Let $k \in \mathbb{N}$. Then, for every $f \in X$ and $t > 0$, we have

$$q_k \left(\frac{T(t)f-f}{t} - Af \right) = \sup_{|x| \le k} \left| \left(\frac{e^{t\varphi(x)} - 1 - t\varphi(x)}{t} \right) f(x) \right|$$

$$\le q_k(f) \sup_{|x| \le k} \left| \frac{e^{t\varphi(x)} - 1 - t\varphi(x)}{t} \right| \le t[q_k(\varphi)]^2 e^{tq_k(\varphi)} q_k(f).$$

This implies that $q_k \left(\frac{T(t)f-f}{t} - Af \right) \to 0$ as $t \to 0^+$. Hence, (A, X) is the infinitesimal generator of $(T(t))_{t \ge 0}$.

(7) Let $\varphi(x) \le 0$ for all $x \in \mathbb{R}$. Then $e^{t\varphi(x)} \le 1$ for all $t \ge 0$ and $x \in \mathbb{R}$. It follows, for every $k \in \mathbb{N}$ and $f \in X$, that

$$q_k(T(t)f) = \sup_{|x| \le k} |e^{t\varphi(x)} f(x)| \le q_k(f) \sup_{|x| \le k} e^{t\varphi(x)} \le q_k(f).$$

So, $(T(t))_{t \ge 0}$ is equicontinuous.

Conversely, suppose that $(T(t))_{t \ge 0}$ is equicontinuous. Then $\{T(t)f : t \ge 0\} \in \mathcal{B}(X)$ for every $f \in X$ and hence, $\sup_{t \ge 0} |(T(t)f)(x)| < \infty$ for every $f \in X$ and $x \in \mathbb{R}$. If $\varphi(x_0) > 0$ for some $x_0 \in \mathbb{R}$, then the choice $f_0 \equiv 1 \in X$ yields

$$\sup_{t \ge 0} e^{t\varphi(x_0)} = \sup_{t \ge 0} |(T(t)f_0)(x_0)| < \infty$$

which is not the case. Therefore, $\varphi \le 0$ for all $x \in \mathbb{R}$.

(8) Let $\varphi(x) \le L$ for all $x \in \mathbb{R}$ and some $L \ge 0$. Then $e^{t\varphi(x)} \le e^{Lt}$ for all $t \ge 0$ and $x \in \mathbb{R}$. It follows, for every $k \in \mathbb{N}$ and $f \in X$, that

$$q_k(T(t)f) = \sup_{|x| \le k} |e^{t\varphi(x)} f(x)| \le q_k(f) \sup_{|x| \le k} e^{t\varphi(x)} \le e^{Lt} q_k(f).$$

So, $(T(t))_{t \ge 0}$ is exponentially equicontinuous; see Definition 2.1.

Suppose now that $(T(t))_{t \ge 0}$ is exponentially equicontinuous in which case there is $a \ge 0$ such that, for each $h \in \mathbb{N}$ there exist $k \in \mathbb{N}$ with $k \ge h$ and $M_h > 0$ for which

$$q_h(T(t)f) \le M_h e^{at} q_k(f), \quad f \in X, \ t \ge 0.$$

Suppose that φ is not bounded from above, i.e., there is a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that $\varphi(x_n) > n$ for all $n \in \mathbb{N}$. Let $n_0 := [a] + 1$, $h_0 := [|x_{n_0}|] + 1$ and $f_0 \equiv 1 \in X$. Then

$$q_{h_0}(T(t)f_0) = \sup_{|x| \le h_0} |e^{t\varphi(x)}| \le M_{h_0} e^{at} q_{k_0}(f_0) = M_{h_0} e^{at}, \quad t \ge 0.$$

Since $|x_{n_0}| \le h_0$, we would have $e^{t\varphi(x_{n_0})} \le M_{h_0} e^{at}$, for $t \ge 0$, which is not the case as $\varphi(x_{n_0}) > n_0 > a$. So, φ must be bounded from above. $\square$

Remark 4.2. In the setting of Proposition 4.1 the semigroup $(T(t))_{t \ge 0}$ is equicontinuous if and only if $\tau_b\text{-}\lim_{t \to \infty} \frac{T(t)}{t} = 0$ if and only if $\tau_s\text{-}\lim_{t \to \infty} \frac{T(t)}{t} = 0$.

Indeed, if $(T(t))_{t \ge 0}$ is equicontinuous, then it is routine to verify that $\tau_b\text{-}\lim_{t \to \infty} \frac{T(t)}{t} = 0$ (hence, also in $\mathcal{L}_s(X)$).

On the other hand, assume that $\tau_s\text{-}\lim_{t \to \infty} \frac{T(t)}{t} = 0$, in which case $\lim_{t \to \infty} \frac{T(t)f_0}{t} = 0$ in X (with $f_0 \equiv 1 \in X$) and hence, also pointwise on $\mathbb{R}$. That is, $\lim_{t \to \infty} \frac{e^{t\varphi(x)}}{t} = 0$ for each $x \in \mathbb{R}$. This implies that $\varphi(x) \leq 0$ for every $x \in \mathbb{R}$ and hence, that $(T(t))_{t\geq 0}$ is equicontinuous; see Proposition 4.1(7).

Since the point evaluations $f \mapsto f(u)$, $f \in X$, belong to X' for each $u \in \mathbb{R}$, it follows that the vector–valued Riemann integral $\frac{1}{r}\int_0^r T(t)f\,dt$ in X is precisely the function $x \mapsto \frac{1}{r}\int_0^r (T(t)f)(x)\,dt = f(x)\frac{1}{r}\int_0^r e^{t\varphi(x)}\,dt$, for $x \in \mathbb{R}$, i.e., the Cesáro means of the semigroup $(T(t))_{t\geq 0}$ are given by

$$(C(r)f)(x) = f(x)\frac{1}{r}\int_0^r e^{t\varphi(x)}\,dt = \begin{cases} f(x)\frac{e^{r\varphi(x)}-1}{r\varphi(x)}, & \varphi(x) \neq 0, \\ f(x), & \varphi(x) = 0, \end{cases} \tag{4.2}$$

for each $f \in X$ and $r > 0$.

Proposition 4.3. *If $\varphi(x_0) > 0$ for some $x_0 \in \mathbb{R}$, then $(T(t))_{t\geq 0}$ is not mean ergodic.*

Proof. Suppose that $(T(t))_{t\geq 0}$ is mean ergodic. Then for $f_0 \equiv 1 \in X$, the limit $g := \lim_{r\to\infty} C(r)f$ exists in X. In particular, (4.2) implies that

$$g(x_0) = \lim_{r\to\infty}(C(r)f)(x_0) = \lim_{r\to\infty}\frac{e^{r\varphi(x_0)}-1}{r\varphi(x_0)} = \frac{1}{\varphi(x_0)}\lim_{r\to\infty}\frac{e^{r\varphi(x_0)}-1}{r} = \infty,$$

which is a contradiction. So, $(T(t))_{t\geq 0}$ cannot be mean ergodic. $\square$

Proposition 4.4. *Suppose that $\varphi(x) \leq 0$ for all $x \in \mathbb{R}$, i.e., $(T(t))_{t\geq 0}$ is equicontinuous. Then the following conditions are equivalent.*

(1) $\sigma(A) \subseteq (-\infty, 0)$.
(2) $\varphi(x) < 0$ *for all* $x \in \mathbb{R}$.
(3) $(T(t))_{t\geq 0}$ *is uniformly mean ergodic.*
(4) $(T(t))_{t\geq 0}$ *is mean ergodic.*
(5) $\mathrm{Im}A = X$.
(6) $\mathrm{Im}A$ *is closed.*
(7) $(T(t))_{t\geq 0}$ *is uniformly Abel mean ergodic.*
(8) $(T(t))_{t\geq 0}$ *is Abel mean ergodic.*

Proof. (1)⇔(2). This follows from the assumption $\varphi \leq 0$, i.e., $\varphi(\mathbb{R}) \subseteq (-\infty, 0]$, and the identity $\sigma(A) = \varphi(\mathbb{R})$; see Proposition 4.1(5).

(2)⇒(5). This is immediate from Proposition 4.1(4).

(5)⇒(6). This is obvious.

(6)⇒(3). Since $\varphi(x) \leq 0$ for all $x \in \mathbb{R}$, by Proposition 4.1(6) $(T(t))_{t\geq 0}$ is an equicontinuous C_0–semigroup on X with infinitesimal generator (A, X). Remark 4.2 ensures that τ_b-$\lim_{t\to\infty}\frac{T(t)}{t} = 0$. Hence, Theorem 3.2 implies that $(T(t))_{t\geq 0}$ is uniformly mean ergodic.

(3)⇒(4). This is obvious.

(4)⇒(2). Suppose that there is $x_0 \in \mathbb{R}$ with $\varphi(x_0) = 0$. Since $\varphi \not\equiv 0$ on $\mathbb{R}$, we may assume that x_0 is a boundary point of $\varphi^{-1}(\{0\})$. Hence, there exists $(x_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ such that $\lim_{k\to\infty} x_k = x_0$ in $\mathbb{R}$ and $\varphi(x_k) < 0$ for all $k \in \mathbb{N}$.

Since $(T(t))_{t\geq 0}$ is mean ergodic and $f_0 \equiv 1 \in X$ there is $g \in X$ such that $\lim_{r\to\infty} C(r)f_0 = g$ exists in X and hence, also pointwise on $\mathbb{R}$. Thus, by (4.2) it follows that

$$g(x_k) = \lim_{r\to\infty}(C(r)f_0)(x_k) = \lim_{r\to\infty}\frac{e^{r\varphi(x_k)}-1}{r\varphi(x_k)} = \frac{1}{\varphi(x_k)}\lim_{r\to\infty}\frac{e^{r\varphi(x_k)}-1}{r} = 0,$$

for $k \in \mathbb{N}$, and that

$$g(x_0) = \lim_{r\to\infty}(C(r)f_0)(x_0) = f_0(x_0) = 1.$$

This is a contradiction as g is a continuous function on $\mathbb{R}$ and $\lim_{k\to\infty} x_k = x_0$ in $\mathbb{R}$. So, $\varphi(x) < 0$ for all $x \in \mathbb{R}$.

(6)⇔(7). This follows from (2)⇔(4) in Theorem 3.2.
(7)⇒(8). This is obvious.
(8)⇒(4). See [7, Theorem 5.13]. □

Corollary 4.5. *If $\varphi(x) \le 0$ for all $x \in \mathbb{R}$ and $\varphi(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then* ImA *is not closed in X.*

Proof. This is a consequence of (2)⇔(6) in Proposition 4.4. □

4.2. **The translation (semi)group on $C(\mathbb{R})$.** We now consider, in the quojection Fréchet space $X = C(\mathbb{R})$, the 1–parameter group of translation operators $(T(t))_{t\in\mathbb{R}}$ defined by

$$T(t)f(x) := f(x+t), \quad f \in X,\ x \in \mathbb{R},\ t \in \mathbb{R}.$$

Proposition 4.6. *The following properties hold for $(T(t))_{t\in\mathbb{R}}$.*

1. *$(T(t))_{t\in\mathbb{R}}$ is a strongly continuous C_0–group on X.*
2. *$(T(t))_{t\in\mathbb{R}}$ is not exponentially equicontinuous. In particular, $\frac{T(t)}{t} \not\to 0$ in $\mathcal{L}_s(X)$ as $t \to \infty$.*
3. *For each $f \in C^1(\mathbb{R})$ set $Af := f'$. Then $(A, C^1(\mathbb{R}))$ is the infinitesimal generator of $(T(t))_{t\in\mathbb{R}}$.*
4. *The operator $A : D(A) \to C(\mathbb{R})$ is surjective, but not injective (with $D(A) = C^1(\mathbb{R})$).*
5. *$\sigma(A) = \mathbb{C}$ with every point of $\sigma(A)$ an eigenvalue of A.*

Proof. (1) Clearly $(T(t))_{t\in\mathbb{R}} \subseteq \mathcal{L}(X)$ is a group. Moreover, for each $R > 0$ and $k \in \mathbb{N}$ we have

$$q_k(T(t)f) = \sup_{|x|\le k} |f(x+t)| \le \sup_{|y|\le k+[R]+1} |f(y)| = q_{k+[R]+1}(f), \quad f \in X,\ |t| \le R,$$

which shows that $(T(t))_{t\in\mathbb{R}}$ is locally equicontinuous.

Fix $f \in X$ and $k \in \mathbb{N}$. Then

$$q_k(T(t)f - f) = \sup_{|x|\le k} |f(x+t) - f(x)|, \quad t \in \mathbb{R}.$$

Since f is uniformly continuous in $[-k-1, k+1]$ we have $\sup_{|x|\le k} |f(x+t) - f(x)| \to 0$ as $t \to 0$, from which it follows that $q_k(T(t)f - f) \to 0$ as $t \to 0$. The arbitrariness of k and f now imply that $(T(t))_{t\in\mathbb{R}}$ is a C_0–group on X. The strong continuity of $(T(t))_{t\in\mathbb{R}}$ follows from the discussion prior to Remark 2.2.

(2) Let $f_0(x) = e^{x^2}$ for all $x \in \mathbb{R}$. Then $f_0 \in X$ and

$$q_1(T(t)f_0) = \sup_{|x|\le 1} e^{(x+t)^2} = e^{(1+t)^2}, \quad t \ge 0,$$

with $\sup_{t\ge 0} e^{-at} e^{(1+t)^2} = \infty$ for every $a \ge 0$. So, $(T(t))_{t\in\mathbb{R}}$ is not exponentially equicontinuous. In particular, $\frac{T(t)}{t} \not\to 0$ in $\mathcal{L}_s(X)$ as $t \to \infty$ via Remark 2.4(ii).

(3) Let $f \in C^1(\mathbb{R})$. By the mean value theorem, for each $k \in \mathbb{N}$, $t \ne 0$ and $x \in [-k, k]$ there exists $x_t \in \mathbb{R}$ between with x and $x+t$ such that

$$q_k\left(\frac{T(t)f - f}{t} - f'\right) = \sup_{|x|\le k} \left|\frac{f(x+t) - f(x)}{t} - f'(x)\right| = \sup_{|x|\le k} |f'(x_t) - f'(x)|,$$

with $\sup_{|x|\le k} |f'(x_t) - f'(x)| \to 0$ for $t \to 0$ as f' is uniformly continuous on compact subsets of $\mathbb{R}$. It follows that $q_k\left(\frac{T(t)f-f}{t} - f'\right) \to 0$ as $t \to 0$. Thus, $f' \in D(A)$ and $Af = f'$.

Conversely, let $f \in D(A)$. Then, for a fixed $x_0 \in \mathbb{R}$ and with $k_0 := [x_0] + 1$ we have

$$\left| \frac{f(x_0 + t) - f(x_0)}{t} - Af(x_0) \right| \leq q_{k_0} \left(\frac{T(t)f - f}{t} - Af \right), \quad 0 < |t| < k_0 - x_0.$$

Since $q_{k_0} \left(\frac{T(t)f-f}{t} - Af \right) \to 0$ as $t \to 0$, it follows that $\frac{f(x_0+t)-f(x_0)}{t} - Af(x_0) \to 0$ as $t \to 0$, i.e., $f'(x_0) = (Af)(x_0)$ exists. By the arbitrariness of x_0 we conclude that f' exists and $f' = Af \in X$, i.e., $f \in C^1(\mathbb{R})$.

(4) The operator A is not injective because $\operatorname{Ker} A = \{f \in X : \ f \text{ constant function on } \mathbb{R}\}$. Let $g \in X$. Then the function $f \in C(\mathbb{R})$ defined by

$$f(x) := \int_0^x g(t)\, dt, \quad x \in \mathbb{R},$$

belongs to $C^1(\mathbb{R})$ and $f' = g$ on $\mathbb{R}$. So, $f \in D(A)$ and $Af = g$. Hence, $\operatorname{Im} A = X$, i.e., A is surjective.

(5) Let $\lambda \in \mathbb{C}$. Then the function $f_\lambda(x) := e^{\lambda x}$, for $x \in \mathbb{R}$, belongs to $D(A)$ and $Af = f_\lambda' = \lambda f_\lambda$. So, f_λ is an eigenvector of A. Thus, $\sigma(A) = \mathbb{C}$. □

Since the evaluation functionals at points of $\mathbb{R}$ belong to X', it follows that the Cesáro means of the group $(T(t))_{t\in\mathbb{R}}$ are given by

$$(C(r)f)(x) = \frac{1}{r} \int_0^r f(x + t)\, dt, \quad f \in X, \ r > 0, \ x \in \mathbb{R}. \tag{4.3}$$

As noted in Proposition 4.6(2) the translation group $(T(t))_{t\in\mathbb{R}}$ *fails* to satisfy the condition τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$ and so Theorem 3.2 is not applicable to $(T(t))_{t\in\mathbb{R}}$. According to Proposition 4.6(5) we have $\rho(A) = \emptyset$ and so the notion of Abel mean ergodicity is not available at all! Nevertheless Proposition 4.6(4) shows that $\operatorname{Im} A = X$ is *closed* and from $Af = f'$, for $f \in C^1(\mathbb{R})$, we see (from the proof of (4)) that $\operatorname{Ker} A$ consists of the constant functions. In particular, $\overline{\operatorname{Im} A} \cap \operatorname{Ker} A \neq \{0\}$ and so $X \neq \overline{\operatorname{Im} A} \oplus \operatorname{Ker} A$, i.e., (2.18) fails to hold. On the other hand, $\operatorname{Ker} A = \operatorname{Fix}(T(\cdot))$ is valid. In view of these observations the following result is expected.

Proposition 4.7. *The group $(T(t))_{t\in\mathbb{R}}$ is not mean ergodic.*

Proof. Let $f \in X$ be given by $f(x) = e^x$, $x \in \mathbb{R}$. It follows from (4.3) that $(C(r)f)(x) = \frac{e^x(e^r-1)}{r}$ for $x \in \mathbb{R}$ and $r > 0$ and hence, that $q_1(C(r)f) = \frac{e(e^r-1)}{r}$. Since $\sup_{r>0} q_1(C(r)f) = \infty$, the set $\{C(n)f : \ n \in \mathbb{N}\} \not\in \mathcal{B}(X)$. It follows that $\{C(r)\}_{r\geq 0}$ cannot be convergent in $\mathcal{L}_s(X)$ ar $r \to \infty$, i.e., $(T(t))_{t\in\mathbb{R}}$ is not mean ergodic. □

4.3. **A semigroup of multiplication operators in $L^p_{\text{loc}}(\mathbb{R})$.** Let $X = L^p_{\text{loc}}(\mathbb{R})$, $1 < p < \infty$. Then X is a reflexive quojection Fréchet space with respect to the lc–topology generated by the increasing sequence of seminorms

$$q_k(f) := \left(\int_{-k}^k |f(x)|^p\, dx \right)^{1/p}, \quad f \in X, \ k \in \mathbb{N}.$$

Let $\varphi \colon \mathbb{R} \to (-\infty, 0]$ be a continuous function and consider the linear operator $A \colon D(A) \to X$ defined by

$$Af := \varphi f, \quad f \in D(A) := \{f \in X : \ \varphi f \in X\}.$$

Proposition 4.8. *The following properties hold for $(A, D(A))$.*

(1) *$D(A) = X$ and $A \in \mathcal{L}(X)$.*
(2) *$A^n f = \varphi^n f$ for all $n \in \mathbb{N}$ and $f \in X$.*
(3) *A is power bounded if and only if $\varphi(\mathbb{R}) \subseteq [-1, 0]$.*

(4) *If* $\varphi(\mathbb{R}) \subseteq (-\infty, 0)$, *then* A *is a bijection of* X *onto itself. In particular,* $A^{-1} \in \mathcal{L}(X)$.

(5) (A, X) *generates the equicontinuous, uniformly continuous* C_0*–semigroup* $(T(t))_{t\geq 0}$ *given by*

$$T(t)f = e^{t\varphi} f, \quad t \geq 0, \ f \in X.$$

(6) *The semigroup* $(T(t))_{t\geq 0}$ *is mean ergodic.*

Proof. It is routine to check (1); simply use $q_k(Af) \leq (\sup_{|x|\leq k} |\varphi(x)|) q_k(f)$ for each $f \in X$ and $k \in \mathbb{N}$.

Property (2) is clear.

(3) Suppose that $\varphi(\mathbb{R}) \subseteq [-1, 0]$. Then, for each $k \in \mathbb{N}$, we have

$$q_k(A^n f) \leq q_k(f), \quad f \in X, \, n \in \mathbb{N},$$

and hence, A is power bounded.

Conversely, suppose that $\varphi(x_0) < -1$ for some $x_0 \in \mathbb{R}$. As φ is continuous there exist $\alpha < -1$ and an open interval $J(x_0)$ containing x_0 such that $\varphi(x) \leq \alpha$ for all $x \in J(x_0)$. Choose $k \in \mathbb{N}$ such that $J(x_0) \subseteq [-k, k]$ and let $f_0 \equiv 1 \in X$. Then

$$q_k(A^n f_0) \geq \left(\int_{J(x_0)} |\varphi(x)|^{np} |f_0(x)|^p \right)^{1/p} \geq |\alpha|^n [\mu(J(x_0))]^{1/p}, \ n \in \mathbb{N},$$

with μ denoting the Lebesgue measure. Since $|\alpha| > 1$, it follows that $\sup_{n\in\mathbb{N}} q_k(A^n f_0) = \infty$, i.e., $\{A^n f_0\}_{n\in\mathbb{N}} \notin \mathcal{B}(X)$. Hence, $\{A^n\}_{n\in\mathbb{N}}$ is not power bounded.

(4) Fix $g \in X$. Since $\varphi(x) \neq 0$ for all $x \in \mathbb{R}$, the function $1/\varphi \in C(\mathbb{R})$. Then $f := g/\varphi \in X$ and satisfies $Af = g$. So, A is surjective. Let $f \in X \setminus \{0\}$. Then there is a measurable subset $B \subseteq \mathbb{R}$ with $\mu(B) > 0$ such that $f(x) \neq 0$ for all $x \in B$. Hence, also $\varphi(x) f(x) \neq 0$ for all $x \in B$, i.e., $Af \neq 0$ in X and so A is also injective. Since $A^{-1} \colon X \to X$ is a closed operator (because of part (1)), the Closed Graph Theorem ensures that $A^{-1} \in \mathcal{L}(X)$.

(5) We first show that $(T(t))_{t\geq 0}$ is a C_0–semigroup on X. It is clearly a semigroup. Fix $f \in X$ and $k \in \mathbb{N}$. Then, with $\alpha_k(\varphi) := \max_{|x|\leq k} |\varphi(x)| < \infty$, we have for each $t \geq 0$ that

$$\begin{aligned} q_k(T(t)f - f) &= \left(\int_{-k}^{k} |(e^{t\varphi(x)} - 1) f(x)|^p \, dx \right)^{1/p} \leq (\sup_{|x|\leq k} |e^{t\varphi(x)} - 1|) q_k(f) \\ &\leq t\alpha_k(\varphi) e^{t\alpha_k(\varphi)} q_k(f). \end{aligned} \tag{4.4}$$

This implies that $q_k(T(t)f - f) \to 0$ as $t \to 0^+$. By the arbitrariness of k, it follows that $\lim_{t\to 0+} T(t)f = f$. So, $(T(t))_{t\geq 0}$ is a C_0–semigroup on X.

Moreover, $\varphi(\mathbb{R}) \subseteq (-\infty, 0]$ implies that $e^{t\varphi(x)} \leq 1$ for all $x \in \mathbb{R}$, $t \geq 0$ and so

$$q_k(T(t)f) \leq (\sup_{|x|\leq k} |e^{t\varphi(x)}|) q_k(f) \leq q_k(f), \quad f \in X, \ t \geq 0,$$

i.e., the C_0–semigroup $(T(t))_{t\geq 0}$ is equicontinuous. Since its infinitesimal generator $A \in \mathcal{L}(X)$, it follows from [7, Proposition 2.3] that $(T(t))_{t\geq 0}$ is uniformly continuous.

Now, for a fixed $f \in X$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} q_k\left(\frac{T(t)f - f}{t} - Af \right) &\leq \sup_{|x|\leq k} \left| \frac{e^{t\varphi(x)} - 1 - t\varphi(x)}{t} \right| q_k(f) \\ &\leq t[\alpha_k(\varphi)]^2 e^{t\alpha_k(\varphi)} q_k(f), \quad t > 0. \end{aligned}$$

This implies that $q_k\left(\frac{T(t)f-f}{t} - Af\right) \to 0$ as $t \to 0^+$. Since f and k are arbitrary, it follows that (A, X) is the infinitesimal generator of $(T(t))_{t\geq 0}$.

(6) Since X is reflexive and the C_0–semigroup $(T(t))_{t\geq 0}$ is equicontinuous, the desired conclusion follows from [6, Corollary 2]. □

If $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, then $1/(\lambda - \varphi) \in C(\mathbb{R})$; recall that $\varphi(\mathbb{R}) \subseteq (-\infty, 0]$. Accordingly, the resolvent operators $R(\lambda, A) \in \mathcal{L}(X)$ exist for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and are the multlipication operators given by

$$R(\lambda, A)f = \frac{1}{(\lambda - \varphi)} f, \quad f \in X. \tag{4.5}$$

In particular, $\mathbb{C} \setminus (-\infty, 0] \subseteq \rho(A)$.

Proposition 4.9. *If $\varphi(\mathbb{R}) \subseteq (-\infty, 0)$, then τ_b-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A) = 0$. In particular, the equicontinuous C_0–semigroup $(T(t))_{t\geq 0}$ is uniformly Abel mean ergodic and hence, also uniformly mean ergodic.*

Proof. Since φ is continuous and φ is strictly negative on $\mathbb{R}$, for every $k \in \mathbb{N}$ we have that $\beta_k(\varphi) := \max_{|x|\leq k} \varphi(x) < 0$. Hence, $\frac{1}{\lambda - \varphi(x)} \leq \frac{1}{(\lambda - \beta_k(\varphi))}$ for every $\lambda > 0$ and $x \in [-k, k]$. For a fixed $k \in \mathbb{N}$ and $B \in \mathcal{B}(X)$, it follows via (4.5) that

$$(q_k(\lambda R(\lambda, A)f))^p = \left(q_k\left(\frac{\lambda}{\lambda - \varphi} f\right)\right)^p = \int_{-k}^{k} \left(\frac{\lambda}{\lambda - \varphi(x)}\right)^p |f(x)|^p \, dx$$
$$= \lambda^p \int_{-k}^{k} \frac{|f(x)|^p}{(\lambda - \varphi(x))^p} \, dx \leq \left(\frac{\lambda}{\lambda - \beta_k(\varphi)}\right)^p q_k(f)^p,$$

for all $f \in X$ and $\lambda > 0$. This inequality ensures that

$$\sup_{f\in B} q_k(\lambda R(\lambda, A)f) \leq \frac{\lambda}{(\lambda - \beta_k(\varphi))} \sup_{f\in B} q_k(f), \quad \lambda > 0.$$

Accordingly, $\lim_{\lambda\to 0^+} q_k(\lambda R(\lambda, A)f) = 0$. By the arbitrariness of k and B it follows that τ_b-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A) = 0$, i.e., the semigroup $(T(t))_{t\geq 0}$ is uniformly Abel mean ergodic. That $(T(t))_{t\geq 0}$ is also uniformly mean ergodic follows from Theorem 3.2 above; see also [7, Theorem 5.5(ii)]. □

In view of Proposition 4.9 all the equivalences of Theorem 3.2 apply to $(T(t))_{t\geq 0}$.

Remark 4.10. An alternate proof of Proposition 4.9 is as follows. By parts (1) and (4) of Proposition 4.8 we have $A\colon D(A) = X \to X$ is bijective with $A^{-1}\colon X \to D(A)$ continuous. It is routine to check that the equicontinuity of $(T(t))_{t\geq 0}$ (cf. Propostion 4.8(5)) implies that τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$. Then Lemma 2.10 yields that τ_b-$\lim_{r\to\infty} C(r) = 0$, i.e., $(T(t))_{t\geq 0}$ is uniformly mean ergodic. The uniform Abel mean ergodicity of $(T(t))_{t\geq 0}$ is then a consequence of Theorem 5.5(i) and Remark 5.6(i) in [7].

Proposition 4.11. *If $\varphi^{-1}(\{0\})$ is a Lebesgue null set, then τ_s-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A) = 0$, i.e., $(T(t))_{t\geq 0}$ is Abel mean ergodic and hence, also mean ergodic.*

Proof. Since $\varphi^{-1}(\{0\})$ is a Lebesgue null set and $\varphi \leq 0$ on $\mathbb{R}$, we have $0 \leq \frac{\lambda}{\lambda - \varphi} \leq 1$ a.e. on $\mathbb{R}$ and for all $\lambda > 0$. On the other hand, $\lim_{\lambda\to 0^+} \frac{\lambda}{\lambda - \varphi} = 0$ pointwise a.e. on $\mathbb{R}$. Fix $f \in X$ and $k \in \mathbb{N}$. Given any sequence $\lambda_n \to 0^+$, we can apply the Dominated Convergence Theorem to the sequence $\left\{\left(\frac{\lambda_n}{\lambda_n - \varphi}\right)^p |f|^p\right\}_{n=1}^{\infty} \subseteq L^1([-k, k])$ to obtain that

$$(q_k(\lambda_n R(\lambda_n, A)f))^p = \int_{-k}^{k} \left(\frac{\lambda_n}{\lambda_n - \varphi(x)}\right)^p |f(x)|^p \, dx \to 0$$

as $n \to \infty$, i.e., $\lim_{n\to\infty} q_k(\lambda_n R(\lambda_n, A)f) = 0$. Since k is arbitrary, it follows that $\lim_{n\to\infty} \lambda_n R(\lambda_n, A)f = 0$ in X. On the other hand, the arbitrariness of $f \in X$ and the sequence $\lambda_n \to 0^+$ ensures that τ_s-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A) = 0$, i.e., $(T(t))_{t\geq 0}$ is Abel mean ergodic.

The mean ergodicity of $(T(t))_{t\geq 0}$ follows from [7, Theorem 5.5(ii)]. □

Lemma 4.12. *Let $k \in \mathbb{N}$ and $g \in C([-k,k])$. Then the multiplication operator*

$$M_g \colon L^p([-k,k]) \to L^p([-k,k]), \ \textit{given by } f \mapsto gf,$$

is continuous in the Banach space $L^p([-k,k])$ with operator norm $\|M_g\|_{p,k} = \max_{|x|\leq k} |g(x)|$.

Proof. It is routine to check that M_g is continuous with $\|M_g\|_{p,k} \leq \max_{|x|\leq k} |g(x)|$. On the other hand, $\sigma(M_g) = g([-k,k])$ and so $r(M_g) = \max_{|x|\leq k} |g(x)| \leq \|M_g\|_{p,k}$ by the Spectral Radius Theorem (here $r(M_g)$ denotes the spectral radius of M_g), [23, Ch. IV, Corollary 1.4]. □

Proposition 4.13. *If $\varphi^{-1}(\{0\})$ is a non–empty Lebesgue null set, then $(T(t))_{t\geq 0}$ is not uniformly mean ergodic.*

Proof. Suppose that $(T(t))_{t\geq 0}$ is uniformly mean ergodic. By Theorem 3.2 the limit τ_b-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A)$ exists. Then Proposition 4.11 yields that τ_b-$\lim_{\lambda\to 0^+} \lambda R(\lambda, A) = 0$. Fix any $k \in \mathbb{N}$. Since $\lambda/(\lambda - \varphi) \in L^\infty([-k,k])$ and the unit ball $B(k)$ of the Banach space $L^p([-k,k])$ is (in the natural sense) a subset of $\mathcal{U}(k) := q_k^{-1}([0,1]) \in \mathcal{B}(X)$, it follows from (4.5) that

$$\begin{aligned} \|M_{\lambda/(\lambda-\varphi)}\|_{p,k} &:= \sup_{h\in B(k)} \left(\int_{-k}^{k} \left| \frac{\lambda}{(\lambda - \varphi(x))} h(x) \right|^p dx \right)^{1/p} \\ &\leq \sup_{f\in\mathcal{U}(k)} q_k(\lambda R(\lambda, A) f) \to 0 \quad \text{as } \lambda \to 0^+. \end{aligned}$$

But, $\lambda/(\lambda - \varphi) \in C([-k,k])$ and so Lemma 4.12 implies that

$$\lim_{\lambda\to 0^+} \sup_{|x|\leq k} \left| \frac{\lambda}{(\lambda - \varphi(x))} \right| = \lim_{\lambda\to 0^+} \|M_{\lambda/(\lambda-\varphi)}\|_{p,k} = 0. \tag{4.6}$$

On the other hand, there exists $k_0 \in \mathbb{N}$ and $x_0 \in [-k_0, k_0]$ such that $\varphi(x_0) = 0$. Then $\sup_{|x|\leq k_0} \left| \frac{\lambda}{\lambda - \varphi(x)} \right| \geq \frac{\lambda}{\lambda - \varphi(x_0)} = 1$ for every $\lambda > 0$. This contradicts (4.6) for $k = k_0$. Hence, $(T(t))_{t\geq 0}$ is not uniformly mean ergodic. □

4.4. **A semigroup on $\omega = \mathbb{C}^{\mathbb{N}}$.** Let $X = \mathbb{C}^{\mathbb{N}}$ be the Fréchet space of all sequences with the increasing seminorms $q_k \colon X \to [0,\infty)$, for $k \in \mathbb{N}$, where $q_k(x) = \max_{1\leq j\leq k} |x_j|$, for $x = (x_n)_n \in X$, in which case X is Montel and a quojection. Define $A \in \mathcal{L}(X)$ by $Ax := (\mu_n x_n)_n$, for $x \in X$, where the real numbers $\mu_n < 0$ for every $n \in \mathbb{N}$ are arbitrary and, for each $t \geq 0$, define $T(t) \in \mathcal{L}(X)$ by $T(t)x := (e^{\mu_n t} x_n)_n$, for $x \in X$. Then $A \in \mathcal{L}(X)$ is a topological isomorphism on X and $(T(t))_{t\geq 0}$ is semigroup on X.

Proposition 4.14. *The following properties hold for $(T(t))_{t\geq 0}$.*

(1) *A is power bounded if and only if $-1 \leq \mu_n < 0$ for all $n \in \mathbb{N}$.*
(2) *For every $\lambda \notin \{\mu_n : \ n \in \mathbb{N}\}$ the resolvent operator $R(\lambda, A)$ exists with*
$$R(\lambda, A)x = \left(\frac{1}{\lambda - \mu_n} x_n \right)_n, \quad x \in X.$$
Moreover, $\sigma(A) = \{\mu_n\}_{n\in\mathbb{N}}$ and each point of $\sigma(A)$ is an eigenvalue of A.
(3) *$(T(t))_{t\geq 0}$ is an equicontinuous, uniformly continuous C_0–semigroup on X. In particular, the operator (A, X) defined above is its infinitesimal generator.*
(4) *$(T(t))_{t\geq 0}$ is uniformly mean ergodic.*

Proof. Most of the details are straightforward to verify. We only point out that the uniform mean ergodicity of $(T(t))_{t\geq 0}$ is a consequence of the fact that τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$ follows from the equicontinuity of $(T(t))_{t\geq 0}$ and so Lemma 2.10 can be applied. Moreover, the uniform continuity of $(T(t))_{t\geq 0}$ follows from its strong continuity (which is routine to verify), [7, Proposition 2.3]. □

4.5. **Another semigroup on $\omega = \mathbb{C}^{\mathbb{N}}$.** Let $X = \mathbb{C}^{\mathbb{N}}$ be as in the previous example and consider the unit right shift $A \in \mathcal{L}(X)$ given by $A(x) := (0, x_1, x_2, \ldots)$, for $x = (x_1, x_2, \ldots) \in X$. Clearly, A is power bounded. Moreover, A is injective but not surjective and $\rho(A) = \mathbb{C} \setminus \{0\}$ with the resolvent operators $R(\lambda, A) \in \mathcal{L}(X)$, for $\lambda \neq 0$, given by

$$R(\lambda, A)(x) = \left(\frac{1}{\lambda}x_1,\ \frac{1}{\lambda}x_2 + \frac{1}{\lambda^2}x_1,\ \frac{1}{\lambda}x_3 + \frac{1}{\lambda^2}x_2 + \frac{1}{\lambda^3}x_1, \ldots\right), \quad x \in X. \tag{4.7}$$

The semigroup $T(t) := e^{tA}$, for $t \geq 0$, is given by

$$T(t)x = \left(x_1,\ x_2 + tx_1,\ x_3 + tx_2 + \frac{t^2}{2!}x_1,\ x_4 + tx_3 + \frac{t^2}{2!}x_2 + \frac{t^3}{3!}x_1, \ldots\right), \tag{4.8}$$

for $x \in X$, and is exponentially (hence, also locally) equicontinuous. These facts can be found in [7, Remark 3.5(v)].

Let $\{e_n\}_{n=1}^\infty$ be the standard (absolute) unit basis of X. Via (4.8) we have $T(t)e_1 = \left(1,\ t,\ \frac{t^2}{2!}, \ldots\right)$ for $t \geq 0$ and so $\{T(t)e_1\}_{t\geq 0} \notin \mathcal{B}(X)$, i.e., $(T(t))_{t\geq 0}$ is not equicontinuous. Again from (4.8) we have

$$T(t)x - x = \left(0, tx_1, tx_2 + \frac{t^2}{2!}x_1,\ tx_3 + \frac{t^2}{2!}x_2 + \frac{t^3}{3!}x_1, \ldots\right), \quad x \in X,\ t \geq 0,$$

which implies that $(T(t))_{t\geq 0}$ is a C_0–semigroup and hence, is also strongly continuous by the discussion prior to Remark 2.2. Since its infinitesimal generator $A \in \mathcal{L}(X)$, it follows that $(T(t))_{t\geq 0}$ is also uniformly continuous, [7, Proposition 2.3]. Of course, the uniform continuity of $(T(t))_{t\geq 0}$ also follows from its strong continuity and the fact that X is Montel, [30, §39.5 Theorem (1)]. For each $t > 0$ and $x \in X$ it follows from (4.8) that

$$\frac{T(t)x}{t} = \left(\frac{x_1}{t},\ \frac{x_2}{t} + x_1,\ \frac{x_3}{t} + x_2 + \frac{t}{2!}x_1,\ \frac{x_4}{t} + x_3 + \frac{t}{2!}x_2 + \frac{t^2}{3!}x_1, \ldots\right).$$

In particular, $\frac{T(t)e_1}{t} = \left(\frac{1}{t},\ 1,\ \frac{t}{2!},\ \frac{t^2}{3!}, \ldots\right)$, for $t > 0$, shows that $\left\{\frac{T(t)e_1}{t}\right\}_{t>0} \notin \mathcal{B}(X)$ which implies that $\frac{T(t)}{t} \not\to 0$ in $\mathcal{L}_s(X)$ (hence, also in $\mathcal{L}_b(X)$) as $t \to \infty$. So, again Theorem 3.2 is unavailable.

It is routine to verify that $\operatorname{Ker} A = \operatorname{Fix}(T(\cdot)) = \{0\}$ and that $\operatorname{Im} A = \overline{\operatorname{span}}\{e_n\}_{n=2}^\infty$ is a *proper closed* subspace of X. In particular, $X \neq \overline{\operatorname{Im} A} \oplus \operatorname{Ker} A$, i.e., (2.18) fails to hold.

Proposition 4.15. *The exponentially equicontinuous, uniformly continuous C_0–semigroup $(T(t))_{t\geq 0}$ is neither mean ergodic nor is it Abel mean ergodic.*

Proof. Direct calculation from (4.8) shows that

$$C(r)x = \frac{1}{r}\int_0^r T(t)x\,dt = \left(x_1,\ x_2 + \frac{r}{2!}x_1,\ x_3 + \frac{r}{2!}x_2 + \frac{r^2}{3!}x_1,\ \ldots\right),$$

for each $x \in X$ and $r > 0$. In particular, $C(r)e_1 = \left(1,\ \frac{r}{2!},\ \frac{r^2}{3!},\ \ldots\right)$ for $r > 0$ shows that the sequence $\{C(n)e_1\}_{n=1}^\infty \notin \mathcal{B}(X)$ and so the net $\{C(r)\}_{r\geq 0}$ is not convergent in $\mathcal{L}_s(X)$ for $r \to \infty$, i.e., $(T(t))_{t\geq 0}$ is not mean ergodic.

Direct calculation from (4.7) yields, for each $\lambda \neq 0$ and $x \in X$, that

$$\lambda R(\lambda, A)x = x + \left(0, \frac{1}{\lambda}x_1, \frac{1}{\lambda}x_2 + \frac{1}{\lambda^2}x_1, \frac{1}{\lambda}x_3 + \frac{1}{\lambda^2}x_2 + \frac{1}{\lambda^3}x_1, \ldots\right).$$

In particular, $\lambda R(\lambda, A)e_1 = e_1 + \left(0, \frac{1}{\lambda}, \frac{1}{\lambda^2}, \ldots\right)$, for $\lambda \neq 0$, shows that $\left\{\frac{1}{n}R\left(\frac{1}{n}, A\right)e_1\right\}_{n=1}^{\infty} \notin \mathcal{B}(X)$ and so the net $\{\lambda R(\lambda, A)\}_{0<\lambda\leq 1}$ is not convergent in $\mathcal{L}_s(X)$ for $\lambda \to 0^+$, i.e., $(T(t))_{t\geq 0}$ is not Abel mean ergodic. $\square$

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