Mean ergodic semigroups on Fréchet spaces

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Bedlewo, October, 2013

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Project Prometeo II/2013/013

UNIVERSIDAD POLITECNICA DE VALENCIA

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CONSELLERIA D'EDUCACIÓ, CULTURA I ESPORT

Aim of the talk.

Investigate (uniformly) mean ergodic C_0–semigroups in Fréchet spaces.

The basic theory of C_0–semigroups in (always Hausdorff) locally convex spaces (denoted below by lcHs) was developed by Komura and Yosida, 1970.

Semigroups of operators in lcHs.

Let X be a lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a 1–parameter family of operators.

- We say that $(T(t))_{t\geq 0}$ is a *semigroup* if it satisfies $T(s)T(t) = T(s+t)$ for $s, t \geq 0$, with $T(0) = I$.
- A semigroup $(T(t))_{t\geq 0}$ is *locally equicontinuous* if, for each $K > 0$, the set $\{T(t) : 0 \leq t \leq K\}$ is equicontinuous.
- A semigroup $(T(t))_{t\geq 0}$ is said to be a C_0*–semigroup* if it satisfies $\lim_{t\to 0^+} T(t) = I$ in $\mathcal{L}_s(X)$, i.e. $\lim_{t\to 0^+} T(t)x = x$ in X for each $x \in X$.

- If the C_0–semigroup $(T(t))_{t\geq 0}$ satisfies that $\lim_{t\to t_0} T(t) = T(t_0)$ in $\mathcal{L}_s(X)$, for each $t_0 > 0$, then it is called a *strongly continuous* C_0–semigroup.
- A semigroup $(T(t))_{t\geq 0}$ is said to be *exponentially equicontinuous* if there exists $a \geq 0$ such that $(e^{-at}T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is equicontinuous. If $a = 0$, then we simply say *equicontinuous*.
- A semigroup $(T(t))_{t\geq 0}$ is called *uniformly continuous* if $\lim_{t\to t_0} T(t) = T(t_0)$ in $\mathcal{L}_b(X)$, for each $t_0 > 0$ (with $t \to 0^+$ if $t_0 = 0$). This means uniform convergence on the bounded sets.

Semigroups of operators in lcHs.

- If X is barrelled (for example a Fréchet space), then every strongly continuous C_0–semigroup is locally equicontinuous.
- Every exponentially equicontinuous semigroup is locally equicontinuous.
- A strongly continuous C_0–semigroup in a Banach space is always exponentially equicontinuous. For Fréchet spaces this need not be so. Indeed, in the sequence space $\omega = \mathbb{C}^{\mathbb{N}}$ (topology of coordinate convergence), $T(t)x := (e^{nt}x_n)_{n=1}^{\infty}$, for $t \geq 0$ and $x = (x_n)_{n=1}^{\infty} \in \omega$, is a strongly continuous C_0–semigroup which is not exponentially equicontinuous.

  However, it is locally equicontinuous and uniformly continuous.

Cesàro means of $(T(t))_{t\geq 0}$.

Let $T = (T(t))_{t\geq 0}$ be a locally equicontinuous C_0–semigroup on a sequentially complete lcHs X (hence, T is always strongly continuous). The linear operators $C(0) := I$ and $C(r)$, for $r > 0$, given by

Cesàro means

$x \mapsto C(r)x := \frac{1}{r}\int_0^r T(t)x\,dt, \quad x \in X,$

are called the *Cesàro means* of $(T(t))_{t\geq 0}$. The Cesàro means $\{C(r)\}_{r\geq 0}$ belong to $\mathcal{L}(X)$.

(Uniform) Mean ergodic C_0–semigroups of operators in lcHs.

Definition

Let X be a sequentially complete lcHs.

A locally equicontinuous C_0–semigroup $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is said to be **mean ergodic** if the net $\{C(r)\}_{r\geq 0}$ converges to some operator in $\mathcal{L}_s(X)$ as $r \to \infty$.

If $\{C(r)\}_{r\geq 0}$ converges in $\mathcal{L}_b(X)$ as $r \to \infty$, then $(T(t))_{t\geq 0}$ is said to be **uniformly mean ergodic**.

Mean ergodic C_0–semigroups.

Proposition

Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup on X. Suppose that the following three conditions are satisfied:

(i) $\tau_s\text{-}\lim_{r\to\infty} \frac{1}{r} T(r) \int_0^t T(s)\, ds = 0$ for all $t > 0$.

(ii) $\{C(r)\}_{r\geq 0}$ is equicontinuous.

(iii) The net $\{C(r)x\}_{r\geq 0}$ is relatively countably $\sigma(X, X')$–compact for all $x \in X$.

Then $(T(t))_{t\geq 0}$ is mean ergodic and the limit $P = \tau_s\text{-}\lim_{r\to\infty} C(r)$ is a projection.

This is based on a deep Theorem due to Eberlein, 1949.

(Uniform) Mean ergodic C_0–semigroups.

A lcHs X is said to be *semi–reflexive* (resp. *semi–Montel*) if every bounded subset of X is relatively $\sigma(X, X')$–compact (resp. relatively compact).

Corollary

Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be an equicontinuous C_0–semigroup.

(i) If X is semi–reflexive, then $(T(t))_{t\geq 0}$ is mean ergodic.

(ii) If X is semi–Montel, then $(T(t))_{t\geq 0}$ is uniformly mean ergodic.

(Uniform) Mean ergodic C_0–semigroups.

An individual operator $T \in \mathcal{L}(X)$ is *mean ergodic* (resp. *uniformly mean ergodic*) if its discrete Cesàro means $\{T_{[n]}\}_{n=1}^{\infty}$ converge in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$).

Theorem

Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup such that $\{C(r)\}_{r\geq 0}$ is equicontinuous.

(i) If $T(t_0)$ is mean ergodic for some $t_0 > 0$, then $(T(t))_{t\geq 0}$ is mean ergodic.

(ii) If $T(t_0)$ is uniformly mean ergodic for some $t_0 > 0$, then $(T(t))_{t\geq 0}$ is uniformly mean ergodic.

The converse is false, both in Banach spaces and in (non–trivial) non–normable Fréchet spaces.

Reflexivity and Mean ergodic C_0–semigroups.

Theorem

Let X be a complete, barrelled lcHs with a Schauder basis. Then the following assertions are equivalent:

(i) X is reflexive.

(ii) Every equicontinuous C_0–semigroup on X is mean ergodic.

(iii) Every equicontinuous, uniformly continuous C_0–semigroup on X is mean ergodic.

Theorem

Let X be a complete barrelled lcHs with a Schauder basis. Then X is Montel if and only if every equicontinuous, uniformly continuous C_0–semigroup on X is uniformly mean ergodic.

For Banach spaces these results are due to D. Mugnolo, 2004. They are a version for semigroups of deep results due to Fonf, Lin and Wojtaszczek, 2001, for the case of one single operator. Extensions of this work for the case of one single operator were obtained by Albanese, Bonet and Ricker, 2009, and by Piszczek, 2011.

Infinitesimal generator of a C_0–semigroup.

If X is a sequentially complete lcHs and $(T(t))_{t\geq 0}$ is a locally equicontinuous C_0–semigroup on X, then the linear operator A defined by

$Ax := \lim_{t\to 0^+} \frac{T(t)x-x}{t}$, for

$x \in D(A) := \{x \in X : \ \lim_{t\to 0^+} \frac{T(t)x-x}{t} \text{ exists in } X\}$,

is closed with $\overline{D(A)} = X$.

The operator $(A, D(A))$ is called the **infinitesimal generator** of $(T(t))_{t\geq 0}$.

The closedness of A ensures that $\mathrm{Ker}\, A := \{x \in D(A) : Ax = 0\}$ is a closed subspace of X. The range of A is the subspace $\mathrm{Im} A := \{Ax : x \in D(A)\}$.

These results are due to T. Komura, 1968.

Infinitesimal generator of a C_0–semigroup.

Theorem. Frerick, Jordá, Kalmes, Wengenroth, 2013

Let X be a Fréchet space that is the countable projective limit of a sequence of Banach spaces with surjective linking maps (e.g. a countable product of Banach spaces). If $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup on X, then its generator is continuous and everywhere defined, and for all $x \in X$ and $t \geq 0$ we have

$$T(t)(x) = exp(tA)(x) = \sum_{k=0}^{\infty}(t^k/k!)A^k x.$$

This answers positively a question asked by Conejero, 2010.

Theorem

Every Fréchet space X which contains a complemented copy of an infinite dimensional nuclear Köthe echelon space admits an equicontinuous, uniformly continuous semigroup on X whose infinitesimal generator is not defined everywhere.

Resolvent of a C_0–semigroup.

Let $A\colon D(A) \subseteq X \to X$ be a linear operator on a lcHs X. Whenever $\lambda \in \mathbb{C}$ is such that $(\lambda I - A)\colon D(A) \to X$ is injective, the linear operator $(\lambda I - A)^{-1}$ is understood to have domain $\mathrm{Im}(\lambda I - A)$.

The **resolvent set** of A is defined by

$\rho(A) := \{\lambda \in \mathbb{C} : (\lambda I - A)\colon D(A) \to X$ is bijective and $(\lambda I - A)^{-1} \in \mathcal{L}(X)\}$.

The **spectrum** of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. For $\lambda \in \rho(A)$ we also write

$R(\lambda, A) := (\lambda I - A)^{-1}$.

Definition

Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}$ be a locally equicontinuous C_0-semigroup on X with infinitesimal generator $(A, D(A))$ such that $(0, +\infty) \subseteq \rho(A)$.

The semigroup $(T(t))_{t\geq 0}$ is said to be **Abel mean ergodic** (resp. *uniformly Abel mean ergodic*) if $\lim_{\lambda\to 0^+} \lambda R(\lambda, A)$ exists in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$).

Mean ergodicity and Abel mean ergodicity.

Theorem

Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}$ be a locally equicontinuous C_0-semigroup on X with infinitesimal generator $(A, D(A))$.

(i) If $(T(t))_{t\geq 0}$ is mean ergodic (resp. uniformly mean ergodic) and $\{C(r)\}_{r\geq 0}$ is equicontinuous, then $\mathbb{C}_{0^+} \subseteq \rho(A)$ and $(T(t))_{t\geq 0}$ is Abel mean ergodic (resp. uniformly Abel mean ergodic).

(ii) If $\mathcal{L}_s(X)$ is sequentially complete, $(T(t))_{t\geq 0}$ is a bounded set in $\mathcal{L}_s(X)$ and $(T(t))_{t\geq 0}$ is Abel mean ergodic (resp. uniformly Abel mean ergodic with $\mathcal{L}_b(X)$ sequentially complete), then $(T(t))_{t\geq 0}$ is mean ergodic (resp. uniformly mean ergodic).

Theorem

Let X be a Fréchet space that is the countable projective limit of a sequence of Banach spaces with surjective linking maps (e.g. a countable product of Banach spaces). Let $(T(t))_{t\geq 0}$ be a locally equicontinuous, C_0–semigroup on X satisfying $\tau_b\text{-}\lim_{t\to\infty} \frac{T(t)}{t} = 0$. Then the following assertions are equivalent:

(1) The semigroup $(T(t))_{t\geq 0}$ is uniformly mean ergodic.

(2) The infinitesimal generator $(A, D(A))$ of $(T(t))_{t\geq 0}$ has closed range.

(3) The operator $\lambda R(\lambda, A)$ is uniformly mean ergodic for every $\lambda > 0$.

Extension of a Theorem of Lin continued.

(4) The operator $\lambda R(\lambda, A)$ is uniformly mean ergodic for some $\lambda > 0$.

(5) The semigroup $(T(t))_{t\geq 0}$ is uniformly Abel mean ergodic.

(6) The sequence of iterates $\{(\lambda R(\lambda, A))^n\}_{n=1}^{\infty}$ converges in $\mathcal{L}_b(X)$ for every (some) $\lambda > 0$.

(7) $\overline{\mathrm{Im} A}$ is a quojection and there exists $\lambda_0 > 0$ such that

$$\{R(\lambda, A)y : \ y \in (0, \lambda_0]\} \in \mathcal{B}(X), \quad y \in \overline{\mathrm{Im} A}.$$

An example.

Let $X = C(\mathbb{R})$ be the space of all $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ with the compact open topology.

Let $\varphi \in X, \varphi \neq 0$, be $\mathbb{R}$–valued and consider the (continuous) multiplication operator $A\colon X \to X$ defined by

$$Af := \varphi f, \quad f \in X.$$

Recall that $S \in \mathcal{L}(X)$ is **power bounded** if $\{S^n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}(X)$ is equicontinuous.

An example.

Lemma

(1) A is power bounded if and only if $\varphi(\mathbb{R}) \subseteq [-1, 1]$.

(2) If $\varphi(x) \neq 0$ for every $x \in \mathbb{R}$, then A is surjective.

(3) The resolvent operator $R(\lambda, A)$ exists in $\mathcal{L}(X)$ if and only if $\lambda \notin \varphi(\mathbb{R})$. Equivalently, $\rho(A) = \mathbb{C} \setminus \varphi(\mathbb{R})$.

(4) (A, X) is the infinitesimal generator of the uniformly continuous C_0–semigroup $(T(t))_{t\geq 0}$ on X given by $T(t)f = e^{t\varphi} f$ for all $t \geq 0$ and $f \in X$.

(5) $(T(t))_{t\geq 0}$ is equicontinuous if and only if $\varphi(\mathbb{R}) \subseteq (-\infty, 0]$.

Proposition

If $\varphi(x_0) > 0$ for some $x_0 \in \mathbb{R}$, then $(T(t))_{t\geq 0}$ is not mean ergodic.

An example.

Proposition

Suppose that $\varphi(x) \leq 0$ for all $x \in \mathbb{R}$, i.e., $(T(t))_{t\geq 0}$ is equicontinuous. Then the following conditions are equivalent:

(1) $\sigma(A) \subseteq (-\infty, 0)$.

(2) $\varphi(x) < 0$ for all $x \in \mathbb{R}$.

(3) $(T(t))_{t\geq 0}$ is uniformly mean ergodic.

(4) $(T(t))_{t\geq 0}$ is mean ergodic.

(5) $\mathrm{Im} A = X$.

(6) $\mathrm{Im} A$ is closed.

(7) $(T(t))_{t\geq 0}$ is uniformly Abel mean ergodic.

(8) $(T(t))_{t\geq 0}$ is Abel mean ergodic.

References

- **A.A. Albanese, J. Bonet and W.J. Ricker,** C_0–semigroups and mean ergodic operators in a class of Fréchet spaces. J. Math. Anal. Appl. 365 (2010), 142–157.
- **A.A. Albanese, J. Bonet and W.J. Ricker,** Mean ergodic semigroups of operators, RACSAM 106, 299-319.
- **A.A. Albanese, J. Bonet and W.J. Ricker,** Montel resolvents and uniformly mean ergodic semigroups of linear operators, Quaestiones Math. 36, 253-290.
- **A.A. Albanese, J. Bonet and W.J. Ricker,** Uniform mean ergodicity of C_0-semigroups in a class of Fréchet spaces, Functiones Approx. (to appear).