Mean ergodic properties of the continuous Cesàro operator

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Aim of the talk

Investigate the dynamics of the continuous Cesàro operator C **on several Banach or Fréchet function spaces.**

Let f be a $\mathbb{C}$-valued, locally integrable function defined on $\mathbb{R}^+ := [0, \infty)$. Then the Cesàro average $\mathsf{C}f$ of f is the function defined by

$$\mathsf{C}f(x) := \frac{1}{x} \int_0^x f(t)\, dt, \quad x \in (0, \infty). \tag{1}$$

The linear map $f \mapsto \mathsf{C}f$ is called the *continuous Cesàro operator*

The boundedness of C on the Banach spaces $L^p([0,1])$ and $L^p(\mathbb{R}^+)$, for $1 < p < \infty$, is due to G.H. Hardy, who showed that the operator norm $\|\mathsf{C}\|_{op} = q$ in both spaces, $\frac{1}{p} + \frac{1}{q} = 1$. **Hardy inequality**.

Cesàro operator on spaces of continuous functions

- The continuous Cesàro operator $Cf(x) := \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0,$ acts also continuously on the Banach spaces of continuous functions $C([0,1])$ and $C_l([0,\infty])$.
- $C_l([0,\infty])$ is the space of all $\mathbb{C}$-valued, continuous functions f on $\mathbb{R}^+$ for which $f(\infty) := \lim_{x\to\infty} f(x)$ exists in $\mathbb{C}$.
- In this case, we set $\mathrm{C}f(0) := \lim_{x\to 0^+} \mathrm{C}f(x) = f(0)$ for every $f \in C([0,1])$ or $f \in C_l([0,\infty])$.
- If $f \in C_l([0,\infty])$, then also $\lim_{x\to\infty} \mathrm{C}f(x)$ exists and equals $f(\infty) := \lim_{x\to\infty} f(x)$, i.e., $\mathrm{C}f(\infty) = f(\infty)$.
- The linear maps $\mathrm{C}\colon C([0,1]) \to C([0,1])$ and $\mathrm{C}\colon C_l([0,\infty]) \to C_l([0,\infty])$ are well defined with $\|\mathrm{C}\|_{op} = 1$ and satisfy $\mathrm{C}\mathbf{1} = \mathbf{1}$.

For a Banach space X, we write

$$\mathcal{L}(X) := \{ T : X \to X \text{ linear and continuous } \}.$$

Given $T \in \mathcal{L}(X)$, the pair (X, T) is a *linear dynamical system*.

Definitions.

- Let $x \in X$. The *orbit of x under T* is the set

$$\mathcal{O}rb(x, T) := \{x, Tx, T^2x, ...\} = \{ T^n x : n \geq 0\}.$$

- $x \in X$ is a *periodic point* if $\exists n \in \mathbb{N}$ such that $T^n x = x$.

Dynamics of linear operators. Definitions.

For a Banach space X and $T \in \mathcal{L}(X)$, we say

Definitions.

- T *topologically mixing* $\Leftrightarrow \forall U, V \neq \emptyset$ open, $\exists n_0 : T^n U \cap V \neq \emptyset$ $\forall n \geq n_0$.
- T *hypercyclic* $\Leftrightarrow \exists x \in X,\ \mathcal{O}rb(T, x) := \{x, Tx, T^2x, \dots\}$ is dense in $X \Rightarrow X$ separable.
- T *supercyclic* $\Leftrightarrow \exists x \in X,\ \mathcal{P}\mathcal{O}rb(T, x) := \{\lambda T^n x | n \in \mathbb{N}, \lambda \in \mathbb{C}\}$ is dense in X.

Definition (Godefroy, Shapiro).

T is *chaotic* if

- T has a dense set of periodic points,
- T is hypercyclic.

Dynamics of linear operators. Definitions.

For a Banach space X and $T \in \mathcal{L}(X)$, we define

Definitions

- T *power bounded* $\Leftrightarrow \sup_n \|T^n\| < \infty$.
- T *Cesàro power bounded* $\Leftrightarrow \sup_n \|\frac{1}{n}\sum_{k=1}^n T^k\| < \infty$.
- T *mean ergodic* $\Leftrightarrow$

$$\forall x \in X, \ \exists Px := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k x \in X.$$

- T *uniformly mean ergodic* $\Leftrightarrow \left\{\frac{1}{n}\sum_{k=1}^n T^k\right\}_n$ converges in the operator norm.

It is clear how to extend these concepts for locally convex spaces, in particular for Fréchet spaces.

Spectrum and resolvent set of an operator

Let X be a Fréchet space and $T \in \mathcal{L}(X)$.

- The *resolvent set* $\rho(T;X)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.
- The *spectrum* of T is the set $\sigma(T;X) := \mathbb{C} \setminus \rho(T)$.
- The *point spectrum* $\sigma_{pt}(T;X)$ consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective.
- Unlike for Banach spaces, it may happen that $\rho(T) = \emptyset$: The operator $T: x \mapsto (x_2, x_3, x_4, \ldots)$, for $x \in \omega$, belongs to $\mathcal{L}(\omega)$ and, for every $\lambda \in \mathbb{C}$, the element $(1, \lambda, \lambda^2, \lambda^3, \ldots) \in \omega$ is an eigenvector corresponding to λ.

Known results

- The spectra and point spectra of C in the Banach spaces $L^p([0,1])$ and $L^p(\mathbb{R}^+)$, for $1 < p < \infty$, are known since the work of Boyd and Leibowitz in 1968-1973.
- The operator C is hypercyclic and chaotic on $L^p([0,1]), 1 < p < \infty$. León-Saavedra, Piqueras-Lerena, Seoane–Sepúlveda (2009).
- The operator C is not supercyclic on $L^2(\mathbb{R}^+)$. Gónzalez, León-Saavedra (2009).
- Orbits of the operator C on $C([0,1])$ and $C_l([0,\infty])$ were studied by Galaz Fontes and Solís in 2008.
- The operator C is not supercyclic on $C([0,1])$. León-Saavedra, Piqueras-Lerena, Seoane-Sepúlveda (2009).

The Cesàro operator on $C([0,1])$

Lemma.

The closure $\overline{(I - \mathrm{C})(C([0,1]))}$ of the range $(I - \mathrm{C})(C([0,1]))$ of $(I - \mathrm{C})$ is precisely the space $Z := \{f \in C([0,1])\colon f(0) = 0\}$.

Lemma.

Let $g \in C([0,1])$ belong to $(I - \mathrm{C})(C([0,1]))$. Then $g(0) = 0$ and, for each $x \in (0,1)$, the limit $\lim_{\varepsilon \to 0^+} \int_\varepsilon^x \frac{g(t)}{t}\, dt$ exists.

Theorem.

The Cesàro operator $\mathsf{C}\colon C([0,1]) \to C([0,1])$ is power bounded and mean ergodic but, not uniformly mean ergodic. Also, C fails to be hypercyclic.

Proof that $\mathsf{C}\colon C([0,1]) \to C([0,1])$ **is not uniformly mean ergodic:**
Define $g(x) := -1/(\log x)$, for $x \in (0,1/2]$, with $g(0) := 0$ and $g(x) := 1/(\log 2)$, for $x \in [1/2,1]$. The function $g \in \overline{(I-\mathsf{C})(C([0,1]))}$.
On the other hand, for every $\varepsilon \in (0,1/2)$, we have

$$\int_{\varepsilon}^{1/2} \frac{g(t)}{t}\,dt = -\int_{\varepsilon}^{1/2} \frac{dt}{t\log t} = \log(-\log\varepsilon) - \log(\log 2),$$

which tends to ∞ as $\varepsilon \to 0^+$. Thus $g \notin (I-\mathsf{C})(C([0,1]))$, and $(I-\mathsf{C})(C([0,1]))$ is not closed in $C([0,1])$.
Since $\lim_{n\to\infty} \frac{\|\mathsf{C}^n\|_{op}}{n} = 0$, a result of M. Lin yields that C is not uniformly mean ergodic.

The Cesàro operator on $C_l([0,1])$

Theorem.

The Cesàro operator $\mathsf{C}\colon C_l([0,\infty]) \to C_l([0,\infty])$ is power bounded, not hypercyclic and not mean ergodic. Moreover,

$$\overline{(I-\mathsf{C})(C_l([0,\infty]))} = \{f \in C_l([0,\infty])\colon f(0) = f(\infty) = 0\} \tag{2}$$

Proof that C is not mean ergodic in $C_l([0,\infty])$:

If C is mean ergodic, then $C_l([0,\infty]) = \mathrm{Ker}(I - \mathsf{C}) \oplus \overline{(I - \mathsf{C})(C_l([0,\infty]))}$, and so the function $f(x) = (\cos x)/(x+1) \in C_l([0,\infty])$ could be written as $f = c\mathbf{1} + g$ with $g(0) = g(\infty) = 0$ by (2) in the Theorem above.

This implies that $f(0) = c = f(\infty)$. But, $f(0) = 1$ and $f(\infty) = 0$ which gives a contradiction.

Proposition.

The Cesàro operator $\mathsf{C}\colon C_l([0,\infty]) \to C_l([0,\infty])$ is not supercyclic.

Spectrum and point spectrum on $C([0,1])$

Proposition

The Cesàro operator $\mathsf{C}\colon C([0,1]) \to C([0,1])$ satisfies

$$\sigma(\mathsf{C}; C([0,1])) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

and

$$\sigma_{pt}(\mathsf{C}; C([0,1])) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\} \setminus \{0\}.$$

Proposition

The Cesàro operator $\mathsf{C}\colon C_l([0,\infty]) \to C_l([0,\infty])$ satisfies

$$\sigma(\mathsf{C}; C_l([0,\infty])) = \left\{ \lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| = \frac{1}{2} \right\}$$

and

$$\sigma_{pt}(\mathsf{C}; C_l([0,\infty])) = \{1\}.$$

The main step in the proof is to show that certain operators constructed by Boyd act continuously in the spaces $C([0,1])$ and $C_l([0,1])$.

The Cesàro operator on the Banach spaces $L^p([0,1])$

Theorem.

The Cesàro operator $\mathsf{C}\colon L^p([0,1]) \to L^p([0,1])$, $1 < p < \infty$, is not power bounded and not mean ergodic. On the other hand, it is hypercyclic, chaotic and satisfies

$$\sigma(\mathsf{C}; L^p([0,1])) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\right\}$$

and

$$\sigma_{pt}(\mathsf{C}; L^p([0,1])) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\}.$$

The statements about the spectrum are due to Leibowitz.

Proof that C is not mean ergodic on $L^p([0,1]), 1 < p < \infty$:

Suppose that C is mean ergodic on $L^p([0,1])$. It follows that $\frac{C^n}{n} f \to 0$ as $n \to \infty$ for each $f \in L^p([0,1])$. So, $\left\{ \frac{C^n}{n} \right\}_{n\in\mathbb{N}}$ is uniformly bounded relative to $\|\cdot\|_{op}$ (by the Principle of Uniform Boundedness). Hence, there exists $M > 0$ such that $\left\| \frac{C^n}{n} \right\|_{op} \leq M$ for all $n \in \mathbb{N}$.

On the other hand, by the spectral mapping theorem $\sigma(C^n; L^p([0,1])) = \{\lambda^n \colon \lambda \in \sigma(C; L^p([0,1]))\}$, for $n \in \mathbb{N}$, and so $q^n \in \sigma(C^n; L^p([0,1]))$, for $n \in \mathbb{N}$. Therefore,

$$q^n \leq r(C^n) \leq \|C^n\|_{op} \leq Mn, \quad n \in \mathbb{N};$$

a contradiction as $q > 1$.

Theorem.

The Cesàro operator $\mathsf{C}\colon L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, $1 < p < \infty$, is not power bounded and not mean ergodic. Moreover,

$$\sigma(\mathsf{C}; L^p(\mathbb{R}^+)) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| = \frac{q}{2}\right\}$$

and

$$\sigma_{pt}(\mathsf{C}; L^p(\mathbb{R}^+)) = \emptyset.$$

The statements about the spectrum are due to Boyd and Leibowitz.

The Cesàro operator on the Fréchet space $C(\mathbb{R}^+)$

The Fréchet space $C(\mathbb{R}^+)$ is endowed with the topology of uniform convergence on the compact subsets of $\mathbb{R}^+$.

Theorem

The Cesàro operator $\mathsf{C}\colon C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ is power bounded and mean ergodic but, not uniformly mean ergodic and not supercyclic (hence, not hypercyclic). Moreover,

$$\sigma(\mathsf{C}; C(\mathbb{R}^+)) = \left\{ \lambda \in \mathbb{C} \colon \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \right\}$$

and

$$\sigma_{pt}(\mathsf{C}; C(\mathbb{R}^+)) = \sigma(\mathsf{C}; C(\mathbb{R}^+)) \setminus \{0\}.$$

The Cesàro operator on the Fréchet space $L^p_{loc}(\mathbb{R}^+)$, $1 < p < \infty$,

$L^p_{loc}(\mathbb{R}^+)$, $1 < p < \infty$, is the Fréchet space of all $\mathbb{C}$-valued, measurable functions f on $\mathbb{R}^+$ such that

$$q_j(f) := \left(\int_0^j |f(x)|^p \, dx \right)^{1/p} < \infty, \quad j \in \mathbb{N}, \tag{3}$$

endowed with the locally convex topology generated by the increasing sequence of seminorms $\{q_j\}_{j\in\mathbb{N}}$.

The Cesàro operator on the Fréchet space $L^p_{loc}(\mathbb{R}^+)$, $1 < p < \infty$,

Theorem

Let $1 < p < \infty$. The Cesàro operator $\mathsf{C}\colon L^p_{loc}(\mathbb{R}^+) \to L^p_{loc}(\mathbb{R}^+)$ is not power bounded and not mean ergodic but, it is hypercyclic, chaotic and satisfies

$$\sigma(\mathsf{C}; L^p_{loc}(\mathbb{R}^+)) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\right\}$$

and

$$\sigma_{pt}(\mathsf{C}; L^p_{loc}(\mathbb{R}^+)) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\}.$$

Lemmas about operators on Fréchet spaces

Lemma.

Let X be a Fréchet space and $S \in \mathcal{L}(X)$. Suppose that $X = \mathrm{proj}_{j\in\mathbb{N}}(X_j, Q_{j,j+1})$, with X_j a Banach space (having norm $\| \|_j$) and linking maps $Q_{j,j+1} \in \mathcal{L}(X_{j+1}, X_j)$ which are surjective for all $j \in \mathbb{N}$, and suppose, for each $j \in \mathbb{N}$, that there exists $S_j \in \mathcal{L}(X_j)$ satisfying

$$S_j Q_j = Q_j S, \tag{4}$$

where $Q_j \in \mathcal{L}(X, X_j)$, $j \in \mathbb{N}$, denotes the canonical projection of X onto X_j (i.e., $Q_{j,j+1} \circ Q_{j+1} = Q_j$). Then

$$\sigma(S) \subseteq \cup_{j=1}^{\infty} \sigma(S_j) \subseteq \sigma(S) \cup \cup_{j=1}^{\infty} \sigma_{pt}(S_j). \tag{5}$$

Moreover,

$$\sigma_{pt}(S) \subseteq \cup_{j=1}^{\infty} \sigma_{pt}(S_j). \tag{6}$$

Lemma.

Let $X = \operatorname{proj}_{j\in\mathbb{N}}(X_j, Q_{j.j+1})$ be a Fréchet space and operators $S \in \mathcal{L}(X)$ and $S_j \in \mathcal{L}(X_j)$, for $j \in \mathbb{N}$, be given which satisfy the assumptions of Lemma above (with $Q_j \in \mathcal{L}(X, X_j)$, $j \in \mathbb{N}$, denoting the canonical projection of X onto X_j and $\| \|_j$ being the norm in the Banach space X_j).

(i) $S \in \mathcal{L}(X)$ is power bounded if and only if each $S_j \in \mathcal{L}(X_j)$, $j \in \mathbb{N}$, is power bounded.

(ii) $S \in \mathcal{L}(X)$ is uniformly mean ergodic if and only if each $S_j \in \mathcal{L}(X_j)$, $j \in \mathbb{N}$, is uniformly mean ergodic.

(iii) $S \in \mathcal{L}(X)$ is mean ergodic if and only if each $S_j \in \mathcal{L}(X_j)$, $j \in \mathbb{N}$, is mean ergodic.

How to apply the lemmas to $C(\mathbb{R}^+)$?

- For each $j \in \mathbb{N}$, we set $X_j := C([0,j])$ the Banach space of all $\mathbb{C}$-valued, continuous functions on $[0,j]$ endowed with the sup-norm $\|.\|_j$
- For each $j \in \mathbb{N}$, let $Q_j \colon C(\mathbb{R}^+) \to C([0,j])$ and $Q_{j,j+1} \colon C([0,j+1]) \to C([0,j])$ be the respective restriction maps.
- $Q_{j,j+1} \circ Q_{j+1} = Q_j$ and $\|Q_{j,j+1} g\|_j = \|g\|_j \leq \|g\|_{j+1}$ for every $g \in C([0,j+1])$ and $j \in \mathbb{N}$.
- $C(\mathbb{R}^+) = \mathrm{proj}_{\,j \in \mathbb{N}}(C([0,j]), Q_{j,j+1})$. Observe that all of the operators $Q_{j,j+1}$ and Q_j, for $j \in \mathbb{N}$, are *surjective*.
- If $\mathsf{C}_j \colon C([0,j]) \to C([0,j])$ is the Cesàro operator defined by the same formula but, now for $f \in C([0,j])$, $j \in \mathbb{N}$, then $\mathsf{C}_j Q_j = Q_j \mathsf{C}$ and $Q_{j,j+1}\mathsf{C}_{j+1} = \mathsf{C}_j Q_{j,j+1}$ for every $j \in \mathbb{N}$.

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