Two questions about radial Hörmander algebras of entire functions

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Aim of the talk

Two questions on radial Hörmander algebras $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$ of entire functions on the complex plane:

- Let $q \leq p$ radial, subharmonic weights with the doubling property. Investigate conditions to ensure that the sequence space canonically associated with the interpolation for $A_q(\mathbb{C})$ (resp. $A_q^0(\mathbb{C})$) is contained in the range of the restriction map defined on the bigger space $A_p(\mathbb{C})$ (resp. $A_p^0(\mathbb{C})$).
- Investigate the dynamics of the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$ and the differentiation operator $Df(z) = f'(z)$ on the spaces $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$.

Radial Hörmander algebras. Weights.

Weights.

A function $p : \mathbb{C} \to]0, \infty[$ is called a **weight function** if it satisfies:

- (w1) p is continuous and subharmonic.
- (w2) p is radial, that is, $p(z) = p(|z|), z \in \mathbb{C}$.
- (w3) $\log(1 + |z|^2) = o(p(z))$ as $|z| \to \infty$.
- (w4) p is doubling, i.e. $p(2z) = O(p(z))$ as $|z| \to \infty$.

Most important example: $p(z) = |z|^s, s > 0$.

We use Landau's notation of little o-growth and capital O-growth.

The space of entire functions is denoted by $\mathcal{H}(\mathbb{C})$.

Radial Hörmander algebras. Definition.

Given a weight p, we define the following weighted (LB)-space of entire functions.

$$A_p(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : \text{ there is } A > 0 : \sup_{z \in \mathbb{C}} |f(z)| \exp(-Ap(z)) < \infty\},$$

endowed with the inductive limit topology, for which it is a (DFN)-algebra.

Radial Hörmander algebras. Definition.

Given a weight p, we define the following weighted Fréchet space of entire functions.

$$A_p^0(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : \text{ for all } \varepsilon > 0 : \sup_{z\in\mathbb{C}} |f(z)| \exp(-\varepsilon p(z)) < \infty\},$$

endowed with the projective topology, for which it is a nuclear Fréchet algebra.

Radial Hörmander algebras. Definition.

For a weight p and $\alpha > 0$, setting

$$A(\alpha p) := \{f \in \mathcal{H}(\mathbb{C}) : \sup_{z \in \mathbb{C}} |f(z)| \exp(-\alpha p(z)) < \infty\},$$

we have

$$A_p(\mathbb{C}) = \bigcup_{n \in \mathbb{N}} A(np),$$

and

$$A_p^0(\mathbb{C}) = \bigcap_{n \in \mathbb{N}} A((1/n)p),$$

Radial Hörmander algebras. Comments.

- $A_p^0(\mathbb{C}) \subset A_p(\mathbb{C})$.
- Condition (w3) implies that $A_p^0(\mathbb{C})$ contains the polynomials.
- Condition (w4) implies that the spaces are stable under differentiation.
- **Braun, Meise and Taylor** studied in 1987 the structure of (complemented) ideals in these algebras.

Radial Hörmander algebras. Examples.

Examples:

- When $p(z) = |z|^s$, then $A_p(\mathbb{C})$ consists of all entire functions of order s and finite type or order less than s.
- When $p(z) = |z|^s$, then $A_p^0(\mathbb{C})$ is the space of all entire functions of order at most s and type 0.
- For $s = 1$, $p(z) = |z|$, $A_p(\mathbb{C})$ is the space of all entire functions of exponential type, and $A_p^0(\mathbb{C})$ is the space of entire functions of infraexponential type.

PART 1

Multiplicity varieties

A **multiplicity variety** $V = \{(z_k, m_k) | k \in \mathbb{N}\}$ is a sequence of different points $(z_k)_k$ with $\lim_{k\to\infty} |z_k| = \infty$ and a sequence $(m_k)_k$ of positive integers corresponding to the multiplicities at the points z_k.

By Weierstrass interpolation theorem, the *restriction map*

$$R_V : \mathcal{H}(\mathbb{C}) \to \prod_{k\in\mathbb{N}} \mathbb{C}^{m_k}, \quad R_V(g) := \left(\left(\frac{g^{(l)}(z_k)}{l!} \right)_{0\leq l<m_k} \right)_k ,$$

is surjective.

Associated sequence spaces

Given $V = \{(z_k, m_k) | k \in \mathbb{N}\}$ and p, define

$$A_p(V) := \{a = (a_{k,l}) | \text{ there is } B > 0 : \sup_{k \in \mathbb{N}} \sum_{l=0}^{m_k - 1} |a_{k,l}| \exp(-Bp(z_k)) < \infty\},$$

endowed with the inductive limit topology; and

$$A_p^0(V) := \{a = (a_{k,l}) | \text{ for all } \varepsilon > 0 : \sup_{k \in \mathbb{N}} \sum_{l=0}^{m_k - 1} |a_{k,l}| \exp(-\varepsilon p(z_k)) < \infty\},$$

endowed with the projective topology, for which it is a Fréchet space.

Interpolating multiplicity varieties

- $R_V(A_p(\mathbb{C})) \subset A_p(V)$ and $R_V(A_p^0(\mathbb{C})) \subset A_p^0(V)$.
- A multiplicity variety is called *interpolating* for $A_p(\mathbb{C})$ (resp. for $A_p^0(\mathbb{C})$) if $R_V(A_p(\mathbb{C})) = A_p(V)$ (resp. $R_V(A_p^0(\mathbb{C})) = A_p^0(V)$).
- After the seminal work by **Berenstein and Taylor**, a geometric characterization of the interpolating varieties for $A_p(\mathbb{C})$ (resp. for $A_p^0(\mathbb{C})$) was obtained by **Berenstein and Li (resp. Berenstein, Li and Vidras)** in 1995.

Counting functions

The geometric characterizations were formulated in terms of the counting function and the integrated counting function of the multiplicity variety V, that are defined as follows:

For $z \in \mathbb{C}$ and $r > 0$, we set

$$n_V(z, r) := \sum_{|z - z_k| \leq r} m_k,$$

and

$$N_V(z, r) := \int_0^r \frac{n_V(z, t) - n_V(z, 0)}{t} dt + n_V(z, 0) \log r.$$

Characterizations

Theorem. Berenstein, Li, 1995.

V is interpolating for $A_p(\mathbb{C})$ if and only if

(i) $N_V(r, 0) = O(p(r))$ as $r \to \infty$, and

(ii) $N_V(z_k, |z_k|) = O(p(z_k))$ as $k \to \infty$.

Characterizations

Theorem. Berenstein, Li, Vidras, 1995.

V is interpolating for $A_p^0(\mathbb{C})$ if and only if

(i) $N_V(r, 0) = o(p(r))$ as $r \to \infty$, and

(ii) $N_V(z_k, |z_k|) = o(p(z_k))$ as $k \to \infty$.

Comments

- **Massaneda, Ortega-Cerdà and Ounaïes** 2006 gave a geometric description of the interpolating varieties for the algebra of Fourier transforms of distributions and Beurling ultradistributions with compact support on the real line, improving earlier results by **Ehrenpreis, Malliavin and Squires**. This corresponds to spaces of type $A_p(\mathbb{C})$ for non radial weights p. The case of Roumieu ultradistributions was studied by **Ziołło** in 2011.

- The arguments of Berenstein and Li were simplified by **Hartmann and Massaneda in 2000 and by Ounaïes** in 2007 using Hörmander's L^2 estimates for the $\overline{\partial}$ equation, treating also weights that are radial but not doubling.

Problem to be considered. Context

Let q and p two weights such that $q(z) = O(p(z))$ as $|z| \to \infty$. In this case,

$$A_q(\mathbb{C}) \subset A_p(\mathbb{C}), \qquad A_q(V) \subset A_p(V).$$

$$R_V : A_p(\mathbb{C}) \to A_p(V) \supset A_q(V).$$

By Berenstein, Li's theorem, *if V is interpolating for $A_q(\mathbb{C})$ (i.e. $R_V : A_q(\mathbb{C}) \to A_q(V)$ is surjective), then it is interpolating for $A_p(\mathbb{C})$ (i.e. $R_V : A_p(\mathbb{C}) \to A_p(V)$ is also surjective).*

Problem to be considered

Questions

- Assume that $q(z) = o(p(z))$ as $|z| \to \infty$. Is there a multiplicity variety V such that V is interpolating for $A_p(\mathbb{C})$, but not for $A_q(\mathbb{C})$?

- Assume that the range $R_V(A_p(\mathbb{C}))$ contains the sequence space $A_q(V)$ associated with the weight q. Is V interpolating for $A_p(\mathbb{C})$?

  In other words, is it true that if every sequence in the space $A_q(V)$ can be interpolated by a function in $A_p(\mathbb{C})$, then every sequence in the larger space $A_p(V)$ can be interpolated by a function in $A_p(\mathbb{C})$?

Results

Theorem

Let $V = \{(z_k, m_k) | k \in \mathbb{N}\}$ a multiplicity variety and let q and p be weights such that $q(z) = O(p(z))$ as $|z| \to \infty$.

(1) If the restriction map R_V satisfies $A_q(V) \subset R_V(A_p(\mathbb{C}))$, then $R_V(A_p(\mathbb{C})) = A_p(V)$, i.e. V is interpolating for $A_p(V)$.

(2) If the restriction map R_V satisfies $A_q^0(V) \subset R_V(A_p^0(\mathbb{C}))$, then $R_V(A_p^0(\mathbb{C})) = A_p^0(V)$, i.e. V is interpolating for $A_p^0(V)$.

The proof uses Grothendieck's factorization theorem in the (LB)-space case, the open mapping theorem and the lifting of compact sets from a quotient in the Fréchet space case, and an argument of Ounaïes that uses Jensen's formula.

Results with a similar flavour

(1) **Bonet, Meise and Taylor (1992)** investigated when the range of the Borel map on a non-quasianalytic class contains a sequence space associated with a quasianalytic weight.

(2) **Bonet, Galbis and Meise (1996, 1998)** investigated the range of non-surjective convolution operators on spaces of non-quasianalytic ultradifferentiable functions.

(3) **Frerick and Wengenroth (2003, 2004)** proved that a convolution operator is surjective in a class of Beurling ultradifferentiable functions if the range contains both the space of real analytic functions and the space of smooth functions with compact support.

Results

Theorem

If q and p are weights such that $q(z) = o(p(z))$ as $|z| \to \infty$, then there are multiplicity varieties V and W such that V is interpolating for $A_p(\mathbb{C})$, but not for $A_q(\mathbb{C})$ and W is interpolating for $A_p^0(\mathbb{C})$, but not for $A_q^0(\mathbb{C})$.

The construction of V and W proceeds by induction and it uses the following result by Berenstein and Li 1995:

Proposition

If a multiplicity variety $V = \{(z_k, m_k) | k \in \mathbb{N}\}$ satisfies $|z_{k+1}| \geq L|z_k|, k \in \mathbb{N}$, for some constant $L > 1$, then V is interpolating for $A_p(\mathbb{C})$ if and only if $N_V(r, 0) = O(p(r))$ as $r \to \infty$ and $m_k \log |z_k| = O(p(z_k))$ as $k \to \infty$.

Further Results

- **Ounaïes** in 2008 used divided differences to characterize those sequences that are in the range $R_V(A_p(\mathbb{C}))$ of the restriction map R_V when the multiplicity variety satisfies the assumption (a) (i) in the Theorem Berenstein and Li.
- This work was continued in 2009 by **Massaneda, Ortega-Cerdà and Ounaïes**. Traces of functions in Bargmann-Fock spaces on lattices of critical density are investigated by **Buckley, Massaneda and Ortega-Cerdà** in 2012.
- A reduction argument of Meise and Taylor can be used to obtain *the description of the range* $R_V(A_p^0(\mathbb{C}))$ *of the restriction operator on* $A_p^0(\mathbb{C})$ *in terms of divided differences*, as a consequence of a theorem of Ounaïes.

PART 2

We need not assume in this part that the weight p is subharmonic.

Definitions

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ is the space of all continuous linear operators on X.

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Fréchet space, or more generally if the uniform boundedness principle is valid for operators defined on X, then T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded.

Definitions

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X, \tag{1}$$

exist in X.

If $\frac{1}{n}\sum_{m=1}^{n} T^m$ converges to P uniformly on the bounded sets, we say that T is *uniformly mean ergodic*.

A power bounded operator T is mean ergodic precisely when

$$X = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Im}(I - T)}, \tag{2}$$

where I is the identity operator, $\mathrm{Im(I-T)}$ denotes the range of $(I - T)$ and the bar denotes the "closure in X".

Definitions

Hypercyclic operators

An operator $T \in \mathcal{L}(X)$ is said to be *hypercyclic* if there is a vector $x \in X$ whose orbit $\{T^m(x)\}_{m=1}^{\infty}$ is dense in X.

Transitive operators

An operator $T \in \mathcal{L}(X)$ is said to be *topologically transitive* if for every pair of non-empty open subsets U, V in X there is n such that $T^n(U) \cap V \neq \emptyset$.

Proposition

If $T \in \mathcal{L}(X)$ is an operator on a separable complete metrizable lcs X, T is hypercyclic if and only if it is topologically transitive. This is a consequence of Baire category theorem.

Definitions

For $T \in \mathcal{L}(X)$, we say

Definitions

- T *topologically mixing* $\Leftrightarrow \forall U, V \neq \emptyset$ open, $\exists n_0 : T^n U \cap V \neq \emptyset$ $\forall n \geq n_0$.

  Every topologically mixing operator is transitive.

Definition (Godefroy, Shapiro)

T is *chaotic* if

- T has a dense set of periodic points,
- T is hypercyclic.

Definitions. Comments

- If an operator T is power bounded, then it is not transitive.
- Every power bounded operator on a Fréchet or (LB)-space in which the bounded sets are compact is uniformly mean ergodic, as a consequence of Yosida's mean ergodic theorem.
- **A. Peris** has constructed mixing, hence hypercylcic, operators on a Banach space that are even uniformly mean ergodic. The example is a variation of a recent example due to **F. Martínez, Oprocha and Peris** that appeared in 2013 in Math. Z.

A description of $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$

Let $v : \mathbb{C} \to]0,1]$ be a continuous function such that $v(z) = v(|z|)$, $z \in \mathbb{C}$, $v(r)$ is non-increasing and $\lim_{r\to\infty} v(r) r^k = 0 \ \forall k \in \mathbb{N}$.

The **weighted Banach space of entire functions**

$$H_v^0(\mathbb{C}) = \{f \in H(\mathbb{C}) : \ v|f| \text{ vanishes at infinity}\}$$

endowed by the norm $\|f\|_v = \sup_{z\in\mathbb{C}} v(z)|f(z)|$, $f \in H_v^0(\mathbb{C})$.

For $v(r) = e^{-p(r)}$, $r \geq 0$, p a weight function,

$$A_p(\mathbb{C}) = \text{ind}_n H^0_{v^n}(\mathbb{C}) \text{ and } A_p^0(\mathbb{C}) = \text{proj}_n H^0_{v^{1/n}}(\mathbb{C}).$$

Continuity of D and J on $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$

Proposition

D and J are continuous on $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$.

Remark:

Given $v(r) = e^{-\alpha p(r)}$, $p(r) = r^a$, $a > 0$, $\alpha > 0$:

- D is not continuous on $H_v^0(\mathbb{C})$ for $a < 1$.
- J is not continuous on $H_v^0(\mathbb{C})$ for $a > 1$.

The differentiation operator

Lemma

Let E be a locally convex space continuously included in $H(\mathbb{C})$ and assume that there is $a > 1$ such that $e^{az} \in E$. If $D : E \to E$ is continuous, then it is not mean ergodic.

Theorem

(i) If $r = O(p(r))$ as $r \to \infty$, then D is not mean ergodic on $A_p(\mathbb{C})$.

(ii) If $r = o(p(r))$ as $r \to \infty$, then D is not mean ergodic on $A_p^0(\mathbb{C})$.

(iii) If $p(r) = o(r)$ as $r \to \infty$, then D is power bounded, hence uniformly mean ergodic and not hypercyclic on $A_p(\mathbb{C})$ and on $A_p^0(\mathbb{C})$.

The differentiation operator

Idea of the **Proof** of (iii) for $A_p(\mathbb{C})$: Assume $p(r) = o(r)$

The Cauchy inequalities imply

$$\forall m \ \ \forall 0 < \alpha < 1 \ \ \exists k \ \ \exists C > 0 \ \ \forall n \ \ \forall z \in \mathbb{C} :$$

$$e^{-kp(|z|)}|f^{(n)}(z)| \leq C\alpha^n \sup_{z\in\mathbb{C}} e^{-mp(|z|)}|f(z)|.$$

This implies $(D^n(f))_n$ converges to 0 in $A_p(\mathbb{C})$ for each $f \in A_p(\mathbb{C})$.

The differentiation operator

Lemma

For $v(r) = e^{-\alpha r}$, $\alpha > 1$, D is topologically mixing and chaotic on $H_v^0(\mathbb{C})$.

Theorem

(i) If $p(r) = o(r - \frac{1}{2}\log(r))$ as $r \to \infty$, then D is not hypercyclic on $A_p(\mathbb{C})$.

(ii) If $r = O(p(r))$ as $r \to \infty$, then D is topologically mixing and has a dense set of periodic points on $A_p(\mathbb{C})$.

(iii) If $r = o(p(r))$ as $r \to \infty$, then D is topologically mixing and has a dense set of periodic points on $A_p^0(\mathbb{C})$.

Corollary

Let $p_a(r) = r^a$, $a > 0$:

(i) If $a > 1$, then D is topologically mixing, chaotic and not mean ergodic on $A_{p_a}(\mathbb{C})$ and on $A^0_{p_a}(\mathbb{C})$.

(ii) If $a < 1$, then D is power bounded, hence uniformly mean ergodic on $A_{p_a}(\mathbb{C})$ and $A^0_{p_a}(\mathbb{C})$.

(iii) If $a = 1$, then D is topologically mixing, chaotic and not mean ergodic on $A_{p_1}(\mathbb{C})$, and it is power bounded on $A^0_{p_1}(\mathbb{C})$.

The integration operator

Proposition

J is not hypercyclic on the Hörmander algebras $A_p(\mathbb{C})$ nor $A_p^0(\mathbb{C})$.

Theorem

(i) The operator of integration is power bounded and hence uniformly mean ergodic on $A_p(\mathbb{C})$, provided that $r = O(p(r))$ as $r \to \infty$.

(ii) If $p(r) = o(r)$ as $r \to \infty$, then J is not mean ergodic on $A_p(\mathbb{C})$.

(iii) J is power bounded and hence uniformly mean ergodic on $A_p^0(\mathbb{C})$ provided that $r = o(p(r))$ as $r \to \infty$.

(iv) If $p(r) = O(r)$ as $r \to \infty$, then J is not mean ergodic on $A_p^0(\mathbb{C})$.

The integration operator

Corollary

Let $p_a(r) = r^a$, $a > 0$:

(i) J is power bounded on $A_{p_a}(\mathbb{C})$ for $a \geq 1$, and it is not mean ergodic for $a < 1$.

(ii) J is power bounded on $A^0_{p_a}(\mathbb{C})$ for $a > 1$ and it is not mean ergodic for $a \leq 1$.

The spectrum

Fix $T \in \mathcal{L}(X)$.

The **resolvent set** $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

The spectrum of D

Theorem

(i) If $r = O(p(r))$ (resp. $r = o(p(r))$) as $r \to \infty$, then the spectrum of D in $A_p(\mathbb{C})$ (resp. in $A_p^0(\mathbb{C})$) is $\mathbb{C}$.

(ii) If $p(r) = o(r)$ as $r \to \infty$, then the spectrum of D in $A_p(\mathbb{C})$ and in $A_p^0(\mathbb{C})$ reduces to $\{0\}$.

The spectrum of D

Idea of the proof

(i) If $r = O(p(r))$ (resp. $r = o(p(r))$) as $r \to \infty$, then $e^{az} \in A_p$ (resp. $e^{az} \in A_p^0$) for every $a \in \mathbb{C}$, and each complex number is an eigenvalue of the operator.

(ii) Assume now $p(r) = o(r)$ as $r \to \infty$, and the case of $A_p(\mathbb{C})$. It can be shown that, for each $b \in \mathbb{C}$ and every $f \in A_p$, the series

$$\sum_{n=0}^{\infty} b^n D^n f$$

converges in A_p. This implies that $aI - D$ in invertible in A_p whenever $a \neq 0$.

The spectrum of J

Theorem

(i) If $p(r) = o(r)$ (resp. $p(r) = O(r)$) as $r \to \infty$, the spectrum of J in $A_p(\mathbb{C})$ (resp. in $A_p^0(\mathbb{C})$) is $\mathbb{C}$.

(ii) If $r = O(p(r))$ (resp. $r = o(p(r))$) as $r \to \infty$), the spectrum of J in $A_p(\mathbb{C})$ (resp. in $A_p^0(\mathbb{C})$) reduces to $\{0\}$.

Idea: (i) $Jf - \lambda f = 1$ has no solution for $\lambda \in \mathbb{C}$, since the exponentials do not belong to the space.

(ii) $Jf = 1$ has no solution and for $\lambda \neq 0$, $\sum_{n \geq 0} \left(\frac{J}{\lambda}\right)^n$ converges, and so, $(\lambda I - J)^{-1}$ exists.

References

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