MEAN ERGODICITY AND SPECTRUM OF THE CESÀRO OPERATOR ON WEIGHTED c_0 SPACES

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Abstract. A detailed investigation is made of the continuity, the compactness and the spectrum of the Cesàro operator C acting on the weighted Banach sequence space $c_0(w)$ for a bounded, strictly positive weight w. New features arise in the weighted setting (eg. existence of eigenvalues, compactness, mean ergodicity) which are not present in the classical setting of c_0.

1. Introduction

The discrete Cesàro operator C is defined on the linear sequence space $\mathbb{C}^{\mathbb{N}}$ by

$$\mathsf{C}x := \left(x_1, \frac{x_1+x_2}{2}, \dots, \frac{x_1+\dots+x_n}{n}, \dots\right), \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{1.1}$$

The operator C is said to act in a vector subspace $X \subseteq \mathbb{C}^{\mathbb{N}}$ if it maps X into itself. Of particular interest is the situation when X is a Banach space. Two of the fundamental questions in this case are: is $\mathsf{C}\colon X \to X$ continuous and, if so, what is the spectrum of $\mathsf{C}\colon X \to X$? Amongst the classical Banach spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ where precise answers are known we mention ℓ_p ($1 < p < \infty$), [7], [19], and c_0, [19], [25], and both c, ℓ_∞, [1], [19], as well as ces_p, $p \in \{0\} \cup (1,\infty)$, [9], the Bachelis spaces N^p, $2 \le p < \infty$, [10], and the spaces of bounded variation bv_0, [23], and bounded p-variation bv_p, $1 \le p < \infty$, [2]. In each case the operator norm of $\mathsf{C}\colon X \to X$ equals its spectral radius $r(\mathsf{C})$ and C has at most one eigenvalue, namely 1. Moreover, in all of these spaces C fails to be compact. This is not always so. It may happen, eg. for the weighted Banach spaces $X = \ell_p(w)$, $1 < p < \infty$, that $r(\mathsf{C}) < \|\mathsf{C}\|$ or that C has infinitely many eigenvalues. For certain weights w it can also happen that C is compact in $\ell_p(w)$, [4]. There is no claim that this list of spaces (and references) is complete.

One of the aims of this paper is to investigate the two questions mentioned above for C acting on the weighted Banach space $c_0(w)$. To be precise, let $w = (w(n))_{n=1}^\infty$ be a bounded, strictly positive sequence. Define

$$c_0(w) := \left\{x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \colon \lim_{n\to\infty} w(n)|x_n| = 0\right\},$$

equipped with the norm $\|x\|_{0,w} := \sup_{n\in\mathbb{N}} w(n)|x_n|$ for $x \in c_0(w)$. Then $c_0(w)$ is isometrically isomorphic to c_0 via the linear multiplication operator $\Phi_w\colon c_0(w) \to c_0$ given by

$$x = (x_n)_{n\in\mathbb{N}} \to \Phi_w(x) := (w(n)x_n)_{n\in\mathbb{N}}. \tag{1.2}$$

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Therefore, $c_0(w)$ is a Banach space. The dual space $(c_0(w))'$ of $c_0(w)$ is the Banach space $\ell_1(v)$, where $v(n) = w(n)^{-1}$ for $n \in \mathbb{N}$. In particular, the canonical unit vectors $e_k := (\delta_{kn})_{n\in\mathbb{N}}$, for $k \in \mathbb{N}$, form an unconditional basis in $c_0(w)$. If $\inf_{n\in\mathbb{N}} w(n) > 0$, then $c_0(w) = c_0$ with equivalent norms and we are in the standard situation. Accordingly, we are mainly interested in the case when $\inf_{n\in\mathbb{N}} w(n) = 0$.

The restriction to c_0 of the Cesàro operator $\mathsf{C}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ as given in (1.1) defines a linear operator on c_0 with operator norm equal to 1. Denote this operator by $\mathsf{C}^{(0)}$ so that $\|\mathsf{C}^{(0)}\| = 1$. In Section 2 we discuss various aspects of C when it is restricted to the larger space $c_0(w) \supseteq c_0$. Denote this restriction by $\mathsf{C}^{(0,w)}$ whenever it is continuous; see Corollary 2.3(i) for a characterization of the continuity of $\mathsf{C}^{(0,w)}$. A useful sufficient condition in this respect is that w *decreases* (cf. Corollary 2.3(i)). However, if we just have $\lim_{n\to\infty} w(n) = 0$ (rather than $w \downarrow 0$), then the continuity of C in $c_0(w)$ need not follow in general; see Remark 2.4(i). Section 2 is also devoted to characterizing the *compactness* of $\mathsf{C}^{(0,w)}$ (see Corollary 2.3(ii)). It turns out (cf. Proposition 2.7) that a sufficient condition for this is that

$$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} \in [0,1), \tag{1.3}$$

although this criterion is not necessary; see Remark 2.10. Nevertheless, condition (1.3) allows us to exhibit a large class of weights w for which $\mathsf{C}^{(0,w)}$ is compact. Moreover, whenever $\mathsf{C}^{(0,w)}$ is compact, then its spectrum consists precisely of the point 0 together with the eigenvalues $\{\frac{1}{m}\colon m \in \mathbb{N}\}$; see Proposition 3.9.

For a Fréchet space X let $\mathcal{L}(X)$ denote the space of all continuous linear operators of X into itself. Given $T \in \mathcal{L}(X)$, the *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{pt}(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If we need to stress the space X, then we also write $\sigma(T; X)$, $\sigma_{pt}(T; X)$ and $\rho(T; X)$.

The two main results in Section 3 present a precise description of both $\sigma(\mathsf{C}^{(0,w)})$ and $\sigma_{pt}(\mathsf{C}^{(0,w)})$; see Theorem 3.4 and Proposition 3.7. The estimates needed for this are possible because of the availability of an explicit formula for the inverse operator $(\mathsf{C} - \lambda I)^{-1}$ whenever $\lambda \neq \frac{1}{m}$, $m \in \mathbb{N}$; see Step 4 in the proof of Theorem 3.4. For the case when $\mathsf{C}^{(0,w)}$ is compact we refer to Proposition 3.9, where it is also shown that the sequence w is then *rapidly decreasing*. The converse is not valid (cf. Example 2.11 and Example 3.10(iii)). Several non-trivial examples of weights w are presented for which $\sigma(\mathsf{C}^{(0,w)})$ is explicitly identified. It is known that $\sigma(\mathsf{C}^{(0)}) = \{\lambda \in \mathbb{C}\colon |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}$, [19], [25]. There exist non-constant weights w such that $\mathsf{C}^{(0,w)}$ is not compact and $\sigma(\mathsf{C}^{(0,w)})$ is no longer a disc; see Examples 3.10(ii) and 3.12. In the event that w is *decreasing* it turns out (cf. Corollary 3.6) that always

$$\sigma(\mathsf{C}^{(0,w)}) \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}. \tag{1.4}$$

For the case when $\lim_{n\to\infty} \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} = 1$ the inclusion (1.4), actually an equality, is due to B.E. Rhoades and M. Yildirim, [26, Corollary 2]. In Example 2.11 a decreasing weight $w \downarrow 0$ is exhibited (i.e., (1.4) holds) for which the sequence

$\{\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)}\}_{n=1}^{\infty}$ fails to converge at all. Given any $k \in \mathbb{N}$, we point out that there exists a bounded, strictly positive sequence w such that $\mathsf{C}^{(0,w)}$ has precisely k eigenvalues; see Example 3.10(ii)

The study of mean ergodic operators T in Banach spaces was initiated in the 1930's-40's by N. Dunford, J. von Neumann, F. Riesz and others and has continued ever since. Depending on whether the averages $\{\frac{1}{n}\sum_{m=1}^{n} T^m\}_{n\in\mathbb{N}}$ of the iterates of T converge in the strong (resp. uniform) operator topology the operator T is called *mean ergodic* (resp. *uniformly mean ergodic*). The final section presents characterizations of when $\mathsf{C}^{(0,w)}$ is mean ergodic (cf. Proposition 4.3) and when it is uniformly mean ergodic (cf. Proposition 4.7). If $\mathsf{C}^{(0,w)}$ is compact and power bounded, then it is necessarily uniformly mean ergodic; see Proposition 4.8. According to Proposition 4.10 this is the case whenever (1.3) is satisfied. Examples of weights w are presented such that $\mathsf{C}^{(0,w)}$ is mean ergodic but not uniformly mean ergodic (cf. Example 4.12(i)) and also weights w such that $\mathsf{C}^{(0,w)}$ is uniformly mean ergodic but not compact (cf. Example 4.12(ii)). Crucial for the proofs of the above mentioned characterizations is a detailed knowledge of the range $(I - \mathsf{C}^{(0,w)})(c_0(w))$ of $I - \mathsf{C}^{(0,w)}$ and of its closure in $c_0(w)$; see Lemmas 4.1 and 4.5.

2. Continuity and compactness of C in $c_0(w)$

Concerning notation, the closed unit ball of a Banach space X is denoted by $\mathcal{B}[X]$ and the ideal of all compact operators on X by $\mathcal{K}(X)$.

Let $v = (v(n))_{n=1}^{\infty}$ and $w = (w(n))_{n=1}^{\infty}$ be two strictly positive sequences. Let $T_{v,w}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ denote the linear operator given by

$$T_{v,w}x := \left(\frac{w(n)}{n}\sum_{k=1}^{n}\frac{x_k}{v(k)}\right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{2.1}$$

Then $\Phi_w \mathsf{C} = T_{v,w}\Phi_v$. Therefore, the Cesàro operator C maps $c_0(v)$ continuously (resp., compactly) into $c_0(w)$ if and only if the operator $T_{v,w} \in \mathcal{L}(c_0)$ (resp., $T_{v,w} \in \mathcal{K}(c_0)$).

The following criterion is known (see, eg., [27, Theorem 4.51-C]).

Lemma 2.1. *Let $A = (a_{nm})_{n,m\in\mathbb{N}}$ be a matrix with entries from $\mathbb{C}$ and $T\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the linear operator defined by*

$$Tx := \left(\sum_{m=1}^{\infty} a_{nm}x_m\right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}}, \tag{2.2}$$

interpreted to mean that Tx exists in $\mathbb{C}^{\mathbb{N}}$ for every $x \in \mathbb{C}^{\mathbb{N}}$.

Then $T \in \mathcal{L}(c_0)$ if and only if the following two conditions are satisfied:

(i) $\lim_{n\to\infty} a_{nm} = 0$ *for each fixed $m \in \mathbb{N}$;*
(ii) $\sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}| < \infty$.

In this case, $\|T\| = \sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}|$.

An immediate application is the following result.

Proposition 2.2. *Let v and w be two bounded, strictly positive sequences.*

(i) *The Cesàro operator* C *maps* $c_0(v)$ *continuously into* $c_0(w)$ *if and only if*

$$\left\{\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{v(k)}\right\}_{n\in\mathbb{N}} \in \ell_\infty,$$

that is, if and only if there exists $K>0$ *such that*

$$\sum_{k=1}^{n}\frac{1}{v(k)} \leq K\frac{n}{w(n)}, \quad n\in\mathbb{N}. \tag{2.3}$$

In this case,

$$\|\mathsf{C}\| = M_{v,w} = \sup_{n\in\mathbb{N}}\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{v(k)},$$

with $M_{v,w}$ *denoting the smallest constant* K *satisfying* (2.3).

(ii) *The Cesàro operator* $\mathsf{C}\colon c_0(v)\to c_0(w)$ *is compact if and only if*

$$\left\{\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{v(k)}\right\}_{n\in\mathbb{N}} \in c_0. \tag{2.4}$$

Proof. (i) By the remarks immediately prior to Lemma 2.1 it suffices to show that the operator $T_{v,w}\in\mathcal{L}(c_0)$ if and only if $M_{v,w}<\infty$. This is a direct consequence of Lemma 2.1; see (2.1) and (2.2) with $T:=T_{v,w}$.

(ii) By the remarks immediately prior to Lemma 2.1 it suffices to show that the operator $T_{v,w}\in\mathcal{K}(c_0)$ if and only if (2.4) is satisfied.

Assume first that (2.4) holds. In particular, $T_{v,w}\in\mathcal{L}(c_0)$ as $M_{v,w}<\infty$; see part (i). According to (2.4) the element $z:=(z_n)_n$ given by $z_n:=\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{v(k)}$, for $n\in\mathbb{N}$, belongs to c_0. Moreover, for every $x\in\mathcal{B}[c_0]$, we have that

$$|(T_{v,w}x)_n| \leq \frac{w(n)}{n}\sum_{k=1}^{n}\frac{|x_k|}{v(k)} \leq \|x\|_0\, z_n \leq z_n, \quad n\in\mathbb{N}.$$

Accordingly, the bounded set $T_{v,w}(\mathcal{B}[c_0])$ is relatively compact in c_0 (see, [11, p.15], [13, p.339]) and hence, $T_{v,w}\in\mathcal{K}(c_0)$.

Now assume that $T_{v,w}\in\mathcal{K}(c_0)$. Then $T_{v,w}(\mathcal{B}[c_0])$ is a relatively compact subset of c_0. Hence, for every $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$|(T_{v,w}x)_n| = \frac{w(n)}{n}\left|\sum_{k=1}^{n}\frac{x_k}{v(k)}\right| < \varepsilon, \quad x\in\mathcal{B}[c_0],\ n\geq n_0.$$

For each $n\in\mathbb{N}$, let $u^{(n)}=(u_k^{(n)})_k\in\mathcal{B}[c_0]$ be given by $u_k^{(n)}=1$ if $k\in\{1,\ldots,n\}$ and 0 otherwise. It follows that

$$\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{v(k)} = |(T_{v,w}u^{(n)})_n| < \varepsilon, \quad n\geq n_0.$$

Accordingly, $\lim_{n\to\infty}\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{v(k)}=0$ which is precisely (2.4). □

Corollary 2.3. *Let* w *be a bounded, strictly positive sequence. Then the following assertions hold.*

(i) *The Cesàro operator* $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ *if and only if*

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n\in\mathbb{N}} \in \ell_\infty. \tag{2.5}$$

Moreover, $\|\mathsf{C}^{(0,w)}\| \geq 1$.

If w *is decreasing, then* (2.5) *is satisfied and* $\|\mathsf{C}^{(0,w)}\| = 1$.

(ii) *The following conditions are equivalent.*

(a) $\mathsf{C}^{(0,w)}$ *is weakly compact.*

(b) $\mathsf{C}^{(0,w)}$ *is compact.*

(c) *The sequence*

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n\in\mathbb{N}} \in c_0. \tag{2.6}$$

Proof. (i) Applying Proposition 2.2(i) with $v = w$, it follows that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ with $\|\mathsf{C}^{(0,w)}\| = M_w := M_{w,w} < \infty$ if and only if (2.5) holds. On the other hand, $\mathsf{C}^{(0,w)}e_1 = (\frac{1}{n})_n$ and hence, $\|\mathsf{C}^{(0,w)}e_1\|_{0,w} = \sup_{n\in\mathbb{N}} \frac{w(n)}{n} \geq w(1) = \|e_1\|_{0,w}$. Accordingly, $\|\mathsf{C}^{(0,w)}\| \geq \frac{\|\mathsf{C}^{(0,w)}e_1\|_{0,w}}{\|e_1\|_{0,w}} \geq 1$.

If w is decreasing, then

$$\frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} = \frac{1}{n} \sum_{k=1}^{n} \frac{w(n)}{w(k)} \leq 1, \quad n \in \mathbb{N},$$

and so, $M_w \leq 1$. Accordingly, $\|\mathsf{C}^{(0,w)}\| = 1$ in this case.

(ii) (b)⇒(a) is clear.

(a)⇒(b). Observe, in the notation of (1.2) and (2.1), that $T_{w,w} \in \mathcal{L}(c_0)$ satisfies $T_{w,w} = \Phi_w \mathsf{C}^{(0,w)} \Phi_w^{-1}$ and hence, $T_{w,w}$ is weakly compact. It is well known that $T_{w,w}$ is then automatically compact, [16, Corollary 6, p.373], [21, Ex. 3.54(b), p.347], and hence, also $\mathsf{C}^{(0,w)} = \Phi_w^{-1} T_{w,w} \Phi_w$ is compact on $c_0(w)$.

(b)⇔(c) is a direct consequence of Proposition 2.2(ii) with $v := w$. □

For the Cesàro operator C acting in the weighted spaces $\ell_p(w)$, $1 < p < \infty$, various sufficient criteria for compactness are known but, unlike for $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$, no characterizations of compactness in terms of w, [4].

Remark 2.4. (i) Corollary 2.3 shows that if w *decreases* to 0, then $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. However, the condition $\lim_{n\to\infty} w(n) = 0$ by itself does not necessarily ensure the continuity of C acting in $c_0(w)$.

Let $w(2n+1) = \frac{1}{n+1}$ for $n \geq 0$ and $w(2n) = 2^{-n}$ for $n \geq 1$. Then

$$\sum_{k=1}^{2n} \frac{1}{w(k)} = (1 + 2 + \ldots + n) + (2 + 2^2 + \ldots + 2^n) = \frac{n(n+1)}{2} + 2^{n+1} - 2$$

whereas

$$\sum_{k=1}^{2n+1} \frac{1}{w(k)} = \frac{1}{w(2n+1)} + \sum_{k=1}^{2n} \frac{1}{w(k)} = \frac{n^2+3n+2}{2} + 2^{n+1} - 2.$$

In particular, $\frac{w(2n+1)}{2n+1}\sum_{k=1}^{2n+1}\frac{1}{w(k)} \geq \frac{2^{n+1}-2}{(2n+1)(n+1)} \to \infty$ for $n \to \infty$ and so the sequence $\{\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)}\}_{n\in\mathbb{N}} \notin \ell_\infty$, i.e., C does not act continuously in $c_0(w)$; see Corollary 2.3(i). A further example of this kind is given in Case (iii) of Remark 4.11(ii)(c).

(ii) Let $\alpha > 0$ and $w(n) := \frac{1}{n^\alpha}$ for all $n \in \mathbb{N}$. Since w is decreasing, Corollary 2.3(i) yields that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. But, for each $n \in \mathbb{N}$, we have that

$$\begin{aligned}\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} &= \frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}k^\alpha \geq \frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}\int_{k-1}^{k}x^\alpha\,dx \\ &= \frac{1}{n^{\alpha+1}}\int_0^n x^\alpha\,dx = \frac{1}{\alpha+1}.\end{aligned}$$

So, $\mathsf{C}^{(0,w)}$ cannot be compact; see Corollary 2.3(ii).

We recall the following fact; see [3, Proposition 4.1], [5, Propositions 4.3 and 4.4]. For convenience of notation we let $\Sigma := \{\frac{1}{m} \colon m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$. Recall that $\mathbb{C}^{\mathbb{N}}$ is a Fréchet space for the topology of coordinatewise convergence.

Lemma 2.5. (i) $\sigma(\mathsf{C};\mathbb{C}^{\mathbb{N}}) = \sigma_{pt}(\mathsf{C};\mathbb{C}^{\mathbb{N}}) = \Sigma$.

(ii) *Fix* $m \in \mathbb{N}$. *Let* $x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^{\mathbb{N}}$ *where* $x_n^{(m)} := 0$ *for* $n \in \{1, \ldots, m-1\}$, $x_m^{(m)} := 1$ *and* $x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!}$ *for* $n > m$. *Then the eigenspace*

$$\mathrm{Ker}\left(\frac{1}{m}I - \mathsf{C}\right) = \mathrm{span}\{x^{(m)}\} \subseteq \mathbb{C}^{\mathbb{N}}$$

is 1*-dimensional.*

Recall that a sequence $u = (u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is *rapidly decreasing* if $(n^m u_n)_{n\in\mathbb{N}} \in c_0$ for every $m \in \mathbb{N}$. The space of all rapidly decreasing, $\mathbb{C}$-valued sequences is denoted by s.

Proposition 2.6. *Let w be a bounded, strictly positive sequence. The following conditions are equivalent.*

(i) $(n^m w(n))_n \in c_0$ *for all* $m \in \mathbb{N}$.

(ii) $w \in s$.

If, in addition, $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$, *then* (i)-(ii) *are equivalent to*

(iii) $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(0,w)})$.

Proof. (i)⇔(ii) follows from the definition of the space s.

Assume now that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$.

(iii)⇒(i). Fix $m \in \mathbb{N}$. Then $\frac{1}{m+1} \in \sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \sigma_{pt}(\mathsf{C};\mathbb{C}^{\mathbb{N}})$ with each element $\alpha x^{(m+1)}$, for $\alpha \in \mathbb{C}$, an eigenvector in $c_0(w) \subseteq \mathbb{C}^{\mathbb{N}}$ corresponding to $\frac{1}{m+1}$; see Lemma 2.5, also for the definition of $x^{(m+1)}$. So, we must have $x^{(m+1)} \in c_0(w)$, i.e., $(x_n^{(m+1)}w(n))_n \in c_0$. Since $\frac{(n-1)!}{(n-m)!} \simeq n^{m-1}$ for $n \to \infty$, this happens only if $(n^m w(n))_n \in c_0$.

(i)⇒(iii). Fix $m \in \mathbb{N}$. Since $(n^{m-1}w(n))_n \in c_0$, the element $(x_n^{(m)}w(n))_n \in c_0$, i.e., $x^{(m)} \in c_0(w)$. As $x^{(m)}$ is an eigenvector corresponding to the eigenvalue $\frac{1}{m}$, it follows that $\frac{1}{m}$ also belongs to $\sigma_{pt}(\mathsf{C}^{(0,w)})$. □

Proposition 2.7. *Let w be bounded, strictly positive and satisfy* (1.3), *i.e.,*

$$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} \in [0,1).$$

Then $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$ *and*

$$\sigma_{pt}(\mathsf{C}^{(0,w)}) = \Sigma; \quad \sigma(\mathsf{C}^{(0,w)}) = \Sigma_0. \tag{2.7}$$

Proof. Via Corollary 2.3(ii) it suffices to prove that $\{\frac{w(n)}{n}\sum_{k=1}^n \frac{1}{w(k)}\}_{n\in\mathbb{N}} \in c_0$. By the assumption there exists $0<r<1$ such that

$$w(n+1) < rw(n), \quad n \geq n_0. \tag{2.8}$$

In particular, $\{w(n)\}_{n\geq n_0}$ is decreasing to 0. Fix $n > n_0$. It follows from (2.8) that

$$w(n) \leq r^{n-k} w(k), \quad n_0 \leq k \leq n, \tag{2.9}$$

and hence, that $\frac{1}{w(k)} \leq \frac{r^{n-k}}{w(n)}$ for $n_0 \leq k \leq n$. Accordingly,

$$\sum_{k=n_0}^n \frac{1}{w(k)} \leq \frac{1}{w(n)} \sum_{k=n_0}^n r^{n-k} = \frac{1-r^{n+1-n_0}}{(1-r)w(n)} \leq \frac{1}{(1-r)w(n)}.$$

Setting $M := \sum_{k=1}^{n_0-1} \frac{1}{w(k)}$ it follows that

$$\frac{w(n)}{n}\sum_{k=1}^n \frac{1}{w(k)} \leq \frac{w(n)}{n}\sum_{k=1}^{n_0-1} \frac{1}{w(k)} + \frac{w(n)}{n}\sum_{k=n_0}^n \frac{1}{w(k)} \leq \frac{M\|w\|_\infty}{n} + \frac{1}{(1-r)n}$$

for every $n \geq n_0$. This shows that (2.6) is valid, i.e., $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$.

According to (2.9) we have $w(n) \leq r^{n-n_0} w(n_0)$ for $n \geq n_0$, which implies that $w \in s$ (recall $0<r<1$). Hence, Proposition 2.6 yields that

$$\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \sigma(\mathsf{C}^{(0,w)})$$

and so $\Sigma_0 \subseteq \sigma(\mathsf{C}^{(0,w)})$. But, $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$. Keeping in mind Lemma 2.5(i) which implies that $\sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \sigma_{pt}(\mathsf{C};\mathbb{C}^{\mathbb{N}}) = \Sigma$, and that $\sigma(\mathsf{C}^{(0,w)}) = \{0\} \cup \sigma_{pt}(\mathsf{C}^{(0,w)})$, [21, Theorem 3.4.23], it follows that $\Sigma_0 = \sigma(\mathsf{C}^{(0,w)})$ and $\Sigma = \sigma_{pt}(\mathsf{C}^{(0,w)})$. This is precisely (2.7). □

Remark 2.8. A special case of Proposition 2.7 occurs if $w \downarrow 0$ (in which case $\frac{w(n+1)}{w(n)} \in (0,1]$ for all $n \in \mathbb{N}$) and the limit

$$\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = l \text{ exists in } [0,1).$$

An alternate proof of Proposition 2.7 is then possible via the Stolz-Cesàro criterion, [22, Theorem 1.22]; simply adapt the proof of Corollary 4.3 in [4]. Indeed, with $a_n := \sum_{k=1}^n \frac{1}{w(k)}$ and $b_n := \frac{1}{w(n)}$ for $n \in \mathbb{N}$, one shows as in [4] that $\sup_{n\in\mathbb{N}} \frac{a_n}{b_n} < \infty$ and hence,

$$\lim_{n\to\infty} \frac{w(n)}{n}\sum_{k=1}^n \frac{1}{w(k)} = \lim_{n\to\infty} \frac{1}{n}\frac{a_n}{b_n} = 0.$$

Examples 2.9. (i) According to Remark 2.8 we see that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$ whenever the weight w is one of the following sequences:

(1) $w(n) := a^{-\alpha_n}$, $n \in \mathbb{N}$, with $a > 1$, $\alpha_n \uparrow \infty$ and $\lim_{n\to\infty}(\alpha_n - \alpha_{n-1}) = \infty$.

(2) $w(n) := \frac{n^\alpha}{a^n}$ for $n \in \mathbb{N}$, where $a > 1$ and $\alpha \in \mathbb{R}$.

(3) $w(n) := \frac{a^n}{n!}$ for $n \in \mathbb{N}$, where $a \geq 1$.

(4) $w(n) := n^{-n}$ for $n \in \mathbb{N}$.

Each of the above weights w satisfies the condition: $\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = l \neq 1$; see, for instance, Remark 4.4(iii), (iv) of [4].

(ii) Define the sequence $\{a_n\}_{n\in\mathbb{N}}$ via $a_{2n-1} := \frac{1}{2}$ and $a_{2n} := \frac{1}{n+1}$ for $n \in \mathbb{N}$. Then define the bounded, strictly positive weight w via $w(1) = 1$ and $w(n+1) = a_n w(n)$ for $n \in \mathbb{N}$. Observe that $w \downarrow 0$ but, the sequence $\{\frac{w(n+1)}{w(n)}\}_{n\in\mathbb{N}}$ is *not* convergent (i.e., Remark 2.8 is not applicable). On the other hand, $\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} = \limsup_{n\to\infty} a_n = \frac{1}{2}$ and so Proposition 2.7 can be applied to conclude that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$.

Remark 2.10. (i) Let $w(n) := e^{-\sqrt{n}}$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = 1$ and so neither Proposition 2.7 nor Remark 2.8 can be applied. However, direct calculations yield, for every $n \in \mathbb{N}$, that

$$\sum_{k=1}^{n} \frac{1}{w(k)} = \sum_{k=1}^{n} e^{\sqrt{k}} \leq \int_1^{n+1} e^{\sqrt{x}}\, dx = \int_1^{\sqrt{n+1}} 2te^t\, dt = 2(\sqrt{n+1} - 1)e^{\sqrt{n+1}}$$

and hence, that

$$\frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \leq \frac{2(\sqrt{n+1} - 1)e^{\sqrt{n+1}}}{ne^{\sqrt{n}}}.$$

Since $\lim_{n\to\infty} \frac{2(\sqrt{n+1}-1)e^{\sqrt{n+1}}}{ne^{\sqrt{n}}} = 0$, it follows that also $\lim_{n\to\infty} \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} = 0$. According to Proposition 2.2(ii), we can conclude that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$.

(ii) Let $\beta > 1$ and $w(n) := e^{-(\log n)^\beta}$ for $n \in \mathbb{N}$. Since, for each $n \geq 1$, we have $\frac{w(n+1)}{w(n)} = e^{-f'(\xi_n)}$ for some $\xi_n \in (n, n+1)$ with $f(x) := (\log x)^\beta$, $x \in (1,\infty)$, satisfying $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{\beta(\log x)^{\beta-1}}{x} = 0$, it follows that $\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = 1$ and so again Proposition 2.7 cannot be applied.

For each $n \in \mathbb{N}$, observe that $\alpha_n := (\log n)^\beta$ satisfies $w(n) = e^{-\alpha_n}$ for all $n \in \mathbb{N}$. Then

$$\frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} = \frac{e^{\alpha_1} + \ldots + e^{\alpha_n}}{ne^{\alpha_n}}, \quad n \in \mathbb{N}.$$

Set $a_n := \sum_{k=1}^{n} e^{\alpha_k}$ and $b_n := ne^{\alpha_n}$ for $n \in \mathbb{N}$. Then $b_n \uparrow \infty$. In particular, for every $n \in \mathbb{N}$, we have that

$$b_n - b_{n-1} = (1 + \beta(\log t_n)^{\beta-1})e^{(\log t_n)^\beta}$$

for some $t_n \in (n, n+1)$; apply the mean value theorem to $h(x) := xe^{(\log x)^\beta}$, $x \in (1,\infty)$, and observe that $b_n - b_{n-1} = h(n) - h(n-1)$. Hence,

$$b_n - b_{n-1} \geq (1 + \beta(\log(n-1))^{\beta-1})e^{(\log(n-1))^\beta} = (1 + \beta(\log(n-1))^{\beta-1})e^{\alpha_{n-1}}.$$

Since $a_n - a_{n-1} = e^{\alpha_n}$ it follows for every $n \in \mathbb{N}$ that

$$0 \leq \frac{a_n - a_{n-1}}{b_n - b_{n-1}} \leq \frac{e^{\alpha_n}}{(1+\beta(\log(n-1))^{\beta-1})e^{\alpha_{n-1}}} = \frac{e^{\alpha_n - \alpha_{n-1}}}{1+\beta(\log(n-1))^{\beta-1}}.$$

But, $e^{\alpha_n-\alpha_{n-1}} = e^{f(n)-f(n-1)} = e^{f'(\xi_n)}$ with f as above and so $e^{\alpha_n-\alpha_{n-1}} \to 1$ for $n \to \infty$. Hence, $\lim_{n\to\infty} \frac{e^{\alpha_n-\alpha_{n-1}}}{1+\beta(\log(n-1))^{\beta-1}} = 0$ which implies that also $\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = 0$. According to the Stolz-Cesàro criterion, [22, Theorem 1.22], we deduce that

$$\lim_{n\to\infty} \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} = 0$$

and so, by Corollary 2.3(ii) we can conclude that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$.

Example 2.11. There exist weights $w \in s$ such that $\mathsf{C}^{(0,w)} \notin \mathcal{K}(c_0(w))$. Indeed, define w via $w(1) := 1$ and $w(n) := \frac{1}{j^j}$ if $n \in \{2^{j-1}+1, \ldots, 2^j\}$ for $j \in \mathbb{N}$. Then w is a positive weight with $w \downarrow 0$ and $w \in s$. To see this, fix $m \in \mathbb{N}$. Observe, for each $j \in \mathbb{N}$ and $n \in \{2^{j-1}+1, \ldots, 2^j\}$, that

$$0 \leq n^m w(n) \leq \frac{2^{mj}}{j^j}.$$

Since $\lim_{j\to\infty} \frac{2^{mj}}{j^j} = 0$, it follows that also $\lim_{n\to\infty} n^m w(n) = 0$. So, as m is arbitrary, Proposition 2.6 implies that $w \in s$.

To see that $\mathsf{C}^{(0,w)} \notin \mathcal{K}(c_0(w))$ it suffices, via Corollary 2.3(ii), to show that (2.6) is not satisfied. Observe, for every $n = 2^j$ with $j \in \mathbb{N}$, that

$$\frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} = \frac{1}{(2j)^j} \left(1 + \sum_{k=1}^{j} \sum_{i=2^{k-1}+1}^{2^k} k^k \right) = \frac{1}{(2j)^j} \left(1 + \sum_{k=1}^{j} 2^{k-1} k^k \right)$$

which does not converge to 0 for $j \to \infty$. To see this, set $a_j := \sum_{k=1}^{j} 2^{k-1} k^k$ and $b_j := (2j)^j$ for $j \in \mathbb{N}$. Then $b_j \uparrow \infty$. Moreover, we have that

$$\frac{a_j - a_{j-1}}{b_j - b_{j-1}} = \frac{2^{j-1} j^j}{(2j)^j - (2(j-1))^{j-1}} = \frac{1}{2 - \frac{1}{j}\left(1 - \frac{1}{j}\right)^{j-1}} \to \frac{1}{2},$$

for $j \to \infty$, According to the Stolz-Cesàro criterion it follows that also $\lim_{j\to\infty} \frac{a_j}{b_j} = \frac{1}{2}$ and hence, that $\lim_{j\to\infty} \frac{w(2^j)}{2^j} \sum_{k=1}^{2^j} \frac{1}{w(k)} = \frac{1}{2}$. So, (2.6) is not satisfied.

Actually, the sequence $\{\frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)}\}_{n\in\mathbb{N}}$ does not converge at all. Indeed, for every $n = 2^{j-1}+1$ with $j \in \mathbb{N}$, we observe that

$$\begin{aligned} \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} &= \frac{1}{(2^{j-1}+1)j^j} \left(1 + \sum_{k=1}^{j-1} \sum_{k=1}^{j} 2^{k-1} k^k + j^j \right) \\ &= \frac{1}{(2^{j-1}+1)j^j} \left(1 + \sum_{k=1}^{j-1} 2^{k-1} k^k + j^j \right) \\ &\simeq \frac{1}{(2^{j-1}+1)j^j} + \frac{2^{j-2}(j-1)^{j-1}}{(2^{j-1}+1)j^j} + \frac{1}{2^{j-1}+1}, \end{aligned}$$

where $\frac{2^{j-2}(j-1)^{j-1}}{(2^{j-1}+1)j^j} \simeq \frac{1}{2e^j}$ for large j, and so it converges to 0 for $j \to \infty$.

For a further example exhibiting these features we refer to Case (ii) of Remark 4.11(ii)(c).

3. Spectrum of $\mathsf{C}^{(0,w)}$

The aim of this section is to provide a detailed knowledge of the spectrum of $\mathsf{C}^{(0,w)}$. Unlike for the Cesàro operator $\mathsf{C}^{(0)} \in \mathcal{L}(c_0)$, it can now happen that many eigenvalues appear; see Theorem 3.4. Of course, if $\lim_{n\to\infty} w(n) = 0$, then the constant vector $\mathbf{1} := (1)_n$ lies in $c_0(w)$ and satisfies $\mathsf{C}^{(0,w)}\mathbf{1} = \mathbf{1}$, that is, $1 \in \sigma_{pt}(\mathsf{C}^{(0,w)})$. Given a bounded, strictly positive sequence w, let $S_w := \{s \in \mathbb{R}\colon \sum_{n=1}^\infty \frac{1}{n^s w(n)} < \infty\}$. In case $S_w \neq \emptyset$ we define $s_0 := \inf S_w$. Moreover, let $R_w := \{t \in \mathbb{R}\colon \lim_{n\to\infty} n^t w(n) = 0\}$. In case $R_w \neq \mathbb{R}$ we define $t_0 := \sup R_w$. If $R_w = \mathbb{R}$ we set $t_0 = \infty$.

Let $w(n) = 2^{-n}$ for $n \in \mathbb{N}$. Then $S_w = \emptyset$, i.e., it can happen that S_w is empty. However, in the event that $S_w \neq \emptyset$, then $s_0 \geq 1$. Indeed, for any fixed $s \in \mathbb{R}$ we have

$$\sum_{n=1}^\infty \frac{1}{n^s w(n)} \geq \|w\|_\infty^{-1} \sum_{n=1}^\infty \frac{1}{n^s}. \tag{3.1}$$

So, whenever $s \in S_w$ it follows that $\sum_{n=1}^\infty \frac{1}{n^s} < \infty$, that is, $s > 1$. Hence, $S_w \subseteq (1,\infty)$ which implies that $s_0 \geq 1$. Moreover, for any $r > s \in S_w$ we have

$$\sum_{n=1}^\infty \frac{1}{n^r w(n)} < \sum_{n=1}^\infty \frac{1}{n^s w(n)}$$

and so also $r \in S_w$. Accordingly, whenever $S_w \neq \emptyset$, then it is an infinite interval, i.e., $S_w = [s_0,\infty)$ or $S_w = (s_0,\infty)$ with $s_0 \geq 1$. It is a consequence of (3.1) that $1 \notin S_w$, for all positive, bounded sequences w.

In the event that $a_w := \inf_{n\in\mathbb{N}} w(n) > 0$ it follows that necessarily $s_0 = 1$. Indeed, in this case $w(n)^{-1} \leq a_w^{-1}$, $n \in \mathbb{N}$, which implies that $\frac{1}{n^s w(n)} \leq \frac{1}{n^s a_w}$, for all $n \in \mathbb{N}$ and $s \in \mathbb{R}$. Hence, $(1,\infty) \subseteq S_w$ and so $s_0 \leq 1$. Since we are assuming that $S_w \neq \emptyset$, we already know that $s_0 \geq 1$. Accordingly, $s_0 = 1$ whenever $a_w > 0$.

Fix $\alpha > 0$. For $w(n) = 1/n^\alpha$ and any $s \in \mathbb{R}$ it follows that $\sum_{n=1}^\infty \frac{1}{n^s w(n)} = \sum_{n=1}^\infty \frac{1}{n^{s-\alpha}} < \infty$ precisely when $s > (1+\alpha)$ and so $s_0 = 1+\alpha$. Hence, given any $\beta > 1$, there exists a positive, decreasing weight $w \downarrow 0$ such that $S_w = (\beta,\infty)$, that is, $s_0 = \beta$.

Concerning the set R_w, a similar discussion applies. For $w(n) = 2^{-n}$ it turns out that $R_w = \mathbb{R}$ with $t_0 = \infty$. However, if $R_w \neq \mathbb{R}$, then t_0 is finite with $t_0 \geq 0$ and $R_w = (-\infty, t_0)$ or $R_w = (-\infty, t_0]$. Moreover, $R_w = \emptyset$ is not possible as $n^t w(n) \leq \|w\|_\infty n^t$ with $n^t \to 0$ whenever $t < 0$. If $a_w > 0$, then necessarily $t_0 = 0$ but, $0 \notin R_w$ as $n^t w(n) \geq a_w n^t$ for all $t \in \mathbb{R}$. In the event that w is also decreasing, we point out that $w \downarrow 0$ if and only if $0 \in R_w$.

Proposition 3.1. *Let w be a bounded, strictly positive sequence.*

(i) *If $S_w \neq \emptyset$, then $t_0 \leq s_0$. In particular, $R_w \neq \mathbb{R}$.*
(ii) *If $R_w \neq \mathbb{R}$, then $S_w \subseteq [1+t_0,\infty)$.*
(iii) *If $w \in s$, then $S_w = \emptyset$.*

Proof. (i) Suppose that $S_w \neq \emptyset$. Fix any $s > s_0$. Since $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)} < \infty$, there exists $N \in \mathbb{N}$ such that $\frac{1}{n^s w(n)} \leq 1$ for $n \geq N$ and hence, $n^s w(n) \geq 1$ for $n \geq N$. So, the sequence $(n^s w(n))_{n\in\mathbb{N}}$ cannot converge to 0 which yields that $t_0 \leq s$. Accordingly, $t_0 \leq s_0$. In particular, $R_w \neq \mathbb{R}$.

(ii) Fix any $t < t_0$, in which case $\lim_{n\to\infty} n^t w(n) = 0$. Hence, there exists $K \in \mathbb{N}$ such that $n^t \leq \frac{1}{w(n)}$ for $n \geq K$. So, for any $s \in \mathbb{R}$ we have (as $\frac{1}{n^s} > 0$ for each $n \in \mathbb{N}$) that

$$\frac{1}{n^{s-t}} = \frac{n^t}{n^s} \leq \frac{1}{n^s w(n)}, \quad n \geq K.$$

Choose now any $s \leq 1 + t$. The previous inequality implies that $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)}$ diverges. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)}$ diverges whenever $s \leq 1 + t$, for some $t < t_0$, that is, whenever $s \in (-\infty, 1 + t_0)$. So, $S_w \subseteq [1 + t_0, \infty)$.

(iii) Suppose first that $r \leq 0$. Then $\frac{1}{n^r} \geq 1$ for all $n \in \mathbb{N}$. Since $w \in c_0$ there exists $K \in \mathbb{N}$ such that also $\frac{1}{w(n)} \geq 1$ for all $n \geq K$. Hence, $r \notin S_w$ because

$$\sum_{n=1}^{\infty} \frac{1}{n^r w(n)} \geq \sum_{n=K}^{\infty} \frac{1}{n^r w(n)} = \infty.$$

Suppose that $r > 0$. Let $m := 1 + [r]$. Since $\sum_{n=1}^{\infty} n^m w(n) < \infty$, there exists $M \in \mathbb{N}$ such that $\frac{1}{w(n)} \geq n^m$ for all $n \geq M$. Again $r \notin S_w$ because

$$\sum_{n=1}^{\infty} \frac{1}{n^r w(n)} \geq \sum_{n=M}^{\infty} \frac{n^m}{n^r} = \infty.$$

□

Remark 3.2. The converse of Proposition 3.1(iii) is not valid. Define an increasing sequence inductively in $\mathbb{N}$ by $n(1) := 1$ and $n(k+1) := 2(k+1)(n(k))^k$ for $k \in \mathbb{N}$, in which case $n(k+1) > 1 + k(n(k))^k$ for $k \in \mathbb{N}$. Now define a strictly positive weight $w \downarrow 0$ as follows. For $k \geq 1$ let

$$w(j) := \frac{1}{k(n(k))^k}, \quad n(k) \leq j < n(k+1).$$

Fix any $t \in \mathbb{R}$. Then for each positive integer $k > t$ we have

$$\frac{1}{(n(k))^t w(n(k))} = k(n(k))^{k-t} \geq k$$

and so $\sum_{n=1}^{\infty} \frac{1}{n^t w(n)}$ diverges. Hence, $S_w = \emptyset$.

To verify that $w \notin s$ it suffices to show that $(nw(n))_n \notin c_0$; see Proposition 2.6. But, this follows from the inequalities

$$(n(k+1)-1)w(n(k+1)-1) = \frac{n(k+1)-1}{k(n(k))^k} > \frac{k(n(k))^k}{k(n(k))^k} = 1, \quad k \in \mathbb{N}.$$

Since $w \notin s$, the operator $\mathsf{C}^{(0,w)} \notin \mathcal{K}(c_0(w))$; see Proposition 3.9 below.

The following result, [4, Lemma 3.2], [25, Lemma 7], will be needed later.

Lemma 3.3. (i) *Let $\lambda \in \mathbb{C} \setminus \Sigma_0$ and set $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$. Then there exist constants $d > 0$ and $D > 0$ (depending on α) such that*

$$\frac{d}{n^\alpha} \le \prod_{k=1}^{n} \left| 1 - \frac{1}{k\lambda} \right| \le \frac{D}{n^\alpha}, \quad n \in \mathbb{N}. \tag{3.2}$$

(ii) *For each $m \in \mathbb{N}$ we have that*

$$\frac{(n-1)!}{(n-m)!} \simeq n^{m-1}, \quad \textit{for all large } n \in \mathbb{N}. \tag{3.3}$$

If $S_w \neq \emptyset$, then $s_0 \ge 1$ and so $\frac{1}{2s_0} \le \frac{1}{2}$, an observation which is relevant for the following few results. We now come to the main feature of this section. Observe that parts (1) and (2) provide a *characterization* of $\sigma(\mathsf{C}^{(0,w)})$.

Theorem 3.4. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$.*

(1) *The following inclusion holds:*

$$\Sigma_0 \subseteq \sigma(\mathsf{C}^{(0,w)}). \tag{3.4}$$

(2) *Let $\lambda \notin \Sigma_0$. Then $\lambda \in \rho(\mathsf{C}^{(0,w)})$ if and only if both of the conditions*

(i) $\lim_{n\to\infty} \frac{w(n)}{n^{1-\alpha}} = 0$, *and*

(ii) $\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{m=1}^{n-1} \frac{w(n)n^\alpha}{w(m)m^\alpha} < \infty$,

are satisfied, where $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$.

(3) *Suppose that $R_w \neq \mathbb{R}$, i.e., $t_0 < \infty$. Then we have the inclusions*

$$\left\{ \frac{1}{m} \colon m \in \mathbb{N},\ 1 \le m < t_0 + 1 \right\} \subseteq \sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \left\{ \frac{1}{m} \colon m \in \mathbb{N},\ 1 \le m \le t_0 + 1 \right\}. \tag{3.5}$$

In particular, $\sigma_{pt}(\mathsf{C}^{(0,w)})$ is a proper subset of Σ.

If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(\mathsf{C}^{(0,w)}) = \Sigma. \tag{3.6}$$

Proof. The proof has certain similarities with that of Theorem 3.3 in [4], together with new features due to the different setting.

(1) The dual operator $A := (\mathsf{C}^{(0,w)})' \in \mathcal{L}(\ell_1(w^{-1}))$ satisfies $\|A\| = \|\mathsf{C}^{(0,w)}\|$ and is given by

$$Ay = \left(\sum_{k=n}^{\infty} \frac{y_k}{k} \right)_{n\in\mathbb{N}}, \quad y = (y_n)_{n\in\mathbb{N}} \in \ell_1(w^{-1}). \tag{3.7}$$

Step 1. $0 \notin \sigma_{pt}(A)$.

Observe that $Ay = 0$, for some $y \in \ell_1(w^{-1})$, implies that $z_n := \sum_{k=n}^{\infty} \frac{y_k}{k} = 0$ for all $n \in \mathbb{N}$. Hence, $y_n = n(z_n - z_{n+1}) = 0$, for $n \in \mathbb{N}$, and so A is injective.

Step 2. $\Sigma \subseteq \sigma_{pt}(A)$.

Suppose first that $\lambda \in \mathbb{C} \setminus \{0\}$. Then $Ay = \lambda y$ for some non-zero $y \in \ell_1(w^{-1})$ if and only if $\lambda y_n = \sum_{k=n}^{\infty} \frac{y_k}{k}$ for all $n \in \mathbb{N}$. This yields, for every $n \in \mathbb{N}$, that $\lambda(y_n - y_{n+1}) = \frac{y_n}{n}$ and so $y_{n+1} = \left(1 - \frac{1}{\lambda n}\right) y_n$. It follows, with $y_1 \neq 0$, that necessarily

$$y_{n+1} = y_1 \prod_{k=1}^{n} \left(1 - \frac{1}{\lambda k}\right), \quad n \in \mathbb{N}. \tag{3.8}$$

In particular, each eigenvalue of A is simple.

Now let $\lambda \in \Sigma$, i.e., $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$. According to (3.8), the non-zero vector $y = (y_n)_{n\in\mathbb{N}}$ defined via $y_1 \in \mathbb{C} \setminus \{0\}$ arbitrary, $y_n := y_1 \prod_{k=1}^{n-1} \left(1 - \frac{1}{\lambda k}\right)$ for $1 < n \le m$ and $y_n := 0$ for $n > m$, clearly belongs to $\ell_1(w^{-1})$ and satisfies $Ay = \lambda y$. Hence, $\lambda \in \sigma_{pt}(A)$.

Step 3. $\Sigma_0 \subseteq \sigma(\mathsf{C}^{(0,w)})$.

For every $T \in \mathcal{L}(X)$ with X a Banach space, it is known that $\sigma_{pt}(T') \subseteq \sigma(T)$, [13, p.581], with $\sigma(T)$ a closed subset of $\mathbb{C}$. According to Step 2, we then have that $\Sigma_0 \subseteq \sigma(\mathsf{C}^{(0,w)})$ which concludes the proof of part (1).

(2) **Step 4.** We proceed as in Step 5 of the proof of Theorem 3.3 in [4]; see also [8].

We recall the formula for the inverse operator $(\mathsf{C} - \lambda I)^{-1} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ whenever $\lambda \notin \Sigma_0$, [25, p.266]. For $n \in \mathbb{N}$ the n-th row of the matrix for $(\mathsf{C} - \lambda I)^{-1}$ has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^{n} \left(1 - \frac{1}{\lambda k}\right)}, \quad 1 \le m < n,$$

$$\frac{n}{1 - n\lambda} = \frac{1}{\frac{1}{n} - \lambda}, \quad m = n,$$

and all the other entries in row n are equal to 0. So, we can write

$$(\mathsf{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda, \tag{3.9}$$

where the diagonal operator $D_\lambda = (d_{nm})_{n,m\in\mathbb{N}}$ is given by $d_{nn} := \frac{1}{\frac{1}{n} - \lambda}$ and $d_{nm} := 0$ if $n \ne m$. The operator $E_\lambda = (e_{nm})_{n,m\in\mathbb{N}}$ is then the lower triangular matrix with $e_{1m} = 0$ for all $m \in \mathbb{N}$, and for every $n \ge 2$ with $e_{nm} := \frac{1}{n \prod_{k=m}^{n} \left(1 - \frac{1}{\lambda k}\right)}$ if $1 \le m < n$ and $e_{nm} := 0$ if $m \ge n$.

Fix $\lambda \notin \Sigma_0$. Then $d(\lambda) := \operatorname{dist}(\lambda, \Sigma_0) > 0$ and $|d_{nn}| \le \frac{1}{d(\lambda)}$ for $n \in \mathbb{N}$. Hence, for every $x \in c_0(w)$, we have

$$\|D_\lambda(x)\|_{0,w} = \sup_{n\in\mathbb{N}} |d_{nn} x_n| w(n) \le \frac{1}{d(\lambda)} \sup_{n\in\mathbb{N}} |x_n| w(n) = \frac{1}{d(\lambda)} \|x\|_{0,w}.$$

This means that $D_\lambda \in \mathcal{L}(c_0(w))$. So, by (3.9) it remains to show that $E_\lambda \in \mathcal{L}(c_0(w))$ if and only if conditions (i) and (ii) are satisfied with $\alpha := \operatorname{Re}\left(\frac{1}{\lambda}\right)$.

To this end, we note that $E_\lambda \in \mathcal{L}(c_0(w))$ if and only if the operator $\tilde{E}_\lambda := \Phi_w E_\lambda \Phi_w^{-1}$ (cf. (1.2)), i.e., the restriction to c_0 of

$$(\tilde{E}_\lambda(x))_n = w(n) \sum_{m=1}^{n-1} \frac{e_{nm}}{w(m)} x_m, \quad x \in \mathbb{C}^{\mathbb{N}},\ n \in \mathbb{N},$$

with $(\tilde{E}_\lambda(x))_1 := 0$, defines a continuous linear operator on c_0. Observe that $\tilde{E}_\lambda = (\tilde{e}_{nm})_{n,m\in\mathbb{N}}$ is the lower triangular matrix given by $\tilde{e}_{1m} = 0$ for $m \in \mathbb{N}$ and $\tilde{e}_{nm} = \frac{w(n)}{w(m)} e_{nm}$ for $n \ge 2$ and $m \in \mathbb{N}$. So, we need to verify that $\tilde{E}_\lambda \in \mathcal{L}(c_0)$ if and only if conditions (i) and (ii) are satisfied with $\alpha := \operatorname{Re}\left(\frac{1}{\lambda}\right)$. Since $\lambda \in \mathbb{C} \setminus \Sigma_0$, we first observe that it follows from (3.2) that

$$\begin{aligned} &\frac{D^{-1}}{n^{1-\alpha}} \le |e_{n1}| \le \frac{d^{-1}}{n^{1-\alpha}}, \quad n \ge 2, \\ &\frac{d' D^{-1}}{n^{1-\alpha} m^{\alpha}} \le |e_{nm}| \le \frac{d^{-1} D'}{n^{1-\alpha} m^{\alpha}}, \quad 2 \le m < n, \end{aligned} \tag{3.10}$$

for some constants $d' > 0$ and $D' > 0$ depending on λ.

Suppose then that $\tilde{E}_\lambda \in \mathcal{L}(c_0)$. By Lemma 2.1(i) applied to $T = \tilde{E}_\lambda$ it follows that $\lim_{n\to\infty} \frac{w(n)|e_{nm}|}{w(m)} = 0$ for all $m \in \mathbb{N}$, i.e., $\lim_{n\to\infty} w(n)|e_{nm}| = 0$ for all $m \in \mathbb{N}$. Hence, by (3.10) it follows that $\lim_{n\to\infty} \frac{w(n)}{n^{1-\alpha} m^{\alpha}} = 0$ for each fixed $m \in \mathbb{N}$. So, condition (i) is satisfied. From Lemma 2.1(ii) it follows that $\sup_{n\in\mathbb{N}} \sum_{m=1}^{n-1} \frac{w(n)|e_{nm}|}{w(m)} < \infty$. Thus, again by (3.10) it follows that $\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{m=1}^{n-1} \frac{w(n) n^{\alpha}}{w(m) m^{\alpha}} < \infty$, i.e., condition (ii) is satisfied.

Conversely, suppose that conditions (i) and (ii) are satisfied. Clearly, condition (i) implies that $\lim_{n\to\infty} \frac{w(n)}{w(m) n^{1-\alpha} m^{\alpha}} = 0$ for each fixed $m \in \mathbb{N}$. So, by (3.10) it follows that $\lim_{n\to\infty} \frac{w(n) e_{nm}}{w(m)} = 0$, i.e., $\lim_{n\to\infty} \tilde{e}_{nm} = 0$ for all $m \in \mathbb{N}$. On the other hand, by condition (ii) we have that $\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{m=1}^{n-1} \frac{w(n) n^{\alpha}}{w(m) m^{\alpha}} < \infty$. In view of (3.10) this implies that $\sup_{n\in\mathbb{N}} \sum_{m=1}^{n-1} \frac{w(n)|e_{nm}|}{w(m)} < \infty$, i.e., $\sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} \tilde{e}_{nm} < \infty$. Therefore, by Lemma 2.1 we can conclude that $\tilde{E}_\lambda \in \mathcal{L}(c_0)$. This completes the proof showing that $\tilde{E}_\lambda \in \mathcal{L}(c_0)$ if and only if conditions (i) and (ii) are satisfied.

The proof of part (2) is thereby complete.

(3) Suppose first that $R_w \neq \mathbb{R}$.

Step 5. *Both of the inclusions in* (3.5) *are valid.*

The Cesàro operator $\mathsf{C}^{(0,w)}$ is clearly injective. So, $0 \notin \sigma_{pt}(\mathsf{C}^{(0,w)})$. Let $\lambda \in \mathbb{C} \setminus \{0\}$. Consider the equation $(\lambda I - \mathsf{C})x = 0$ with $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \setminus \{0\}$. As for the proof of Step 7 in Theorem 3.3 in [4], with $m \in \mathbb{N}$ being the smallest positive integer such that $x_m \neq 0$, it follows that $\lambda = \frac{1}{m}$ and we deduce that

$$x_n = x_{m+(n-m)} = \frac{(n-1)!}{(m-1)!(n-m)!} x_m, \quad n > m. \tag{3.11}$$

According to (3.3) we have $\frac{(n-1)!}{(m-1)!(n-m)!} \simeq \frac{1}{(m-1)!} \cdot n^{m-1}$, for each $m \in \mathbb{N}$. So, $x \in c_0(w)$ if and only if $\lim_{n\to\infty} n^{(m-1)} w(n) = 0$. But, $\lim_{n\to\infty} n^{(m-1)} w(n) = 0$ precisely when $(m-1) \in R_w$. In this case, $(m-1) \le t_0$, i.e., $m \le t_0 + 1$. So, $\sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \{\frac{1}{m} \colon m \in \mathbb{N},\ 1 \le m \le t_0 + 1\}$.

Conversely, if $m < t_0+1$ for some $m \in \mathbb{N}$, i.e., $(m-1) < t_0$, then $(m-1) \in R_w$ as $t_0 = \sup R_w$. Then $x \in \mathbb{C}^{\mathbb{N}}$ defined according to (3.11), with $x_1 = \ldots = x_{m-1} = 0$ and for any arbitrary $x_m \neq 0$, belongs to $c_0(w)$. Therefore, $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(0,w)})$.

Step 6. *Assume now that* $R_w = \mathbb{R}$. *Then* (3.6) *is valid.*

As argued in Step 5, the point $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(0,w)})$ if and only if $(m-1) \in R_w$. But, for $R_w = \mathbb{R}$, this is satisfied for every $m \in \mathbb{N}$ and so $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(0,w)})$. On the other hand, it is also shown in the proof of Step 5 that *every* eigenvalue λ of $\mathsf{C} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ must have the form $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$. Since every eigenvalue of $\mathsf{C}^{(0,w)}$ is also an eigenvalue of C (as $c_0(w) \subseteq \mathbb{C}^{\mathbb{N}}$), it follows that $\sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \Sigma$. □

Remark 3.5. The following observation (which is routine to establish) is used in the sequel. Let $r > 0$ be fixed. Then $\lambda \in \mathbb{C} \setminus \{0\}$ satisfies

$$\left|\lambda - \frac{1}{2r}\right| > \frac{1}{2r} \quad \text{if and only if} \quad \operatorname{Re}\left(\frac{1}{\lambda}\right) < r. \tag{3.12}$$

Similarly, it is the case that $\left|\lambda - \frac{1}{2r}\right| = \frac{1}{2r}$ if and only if $\operatorname{Re}\left(\frac{1}{\lambda}\right) = r$ and that $\left|\lambda - \frac{1}{2r}\right| < \frac{1}{2r}$ if and only if $\operatorname{Re}\left(\frac{1}{\lambda}\right) > r$.

Corollary 3.6. *Let w be a strictly positive, decreasing sequence. Then*

$$\sigma(\mathsf{C}^{(0,w)}) \subseteq \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}. \tag{3.13}$$

Proof. To establish this it suffices to show that, for every $\lambda \in \mathbb{C}$ with $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$, we have that $\lambda \in \rho(\mathsf{C}^{(0,w)})$, i.e., that conditions (i) and (ii) in part (2) of Theorem 3.4 are satisfied. So, fix $\lambda \in \mathbb{C}$ with $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. Then necessarily $\lambda \notin \Sigma_0$ and $\alpha := \operatorname{Re}\left(\frac{1}{\lambda}\right) < 1$; see (3.12).

Clearly, $\lim_{n\to\infty} \frac{w(n)}{n^{1-\alpha}} = 0$ as w is bounded and $\alpha < 1$. Hence, condition (i) of part (2) of Theorem 3.4 holds.

On the other hand, for a fixed $n \in \mathbb{N}$, we have $\frac{w(n)}{w(m)} \leq 1$ for all $1 \leq m < n$ (as w is decreasing) and so

$$\frac{1}{n}\sum_{m=1}^{n-1} \frac{w(n)n^\alpha}{w(m)m^\alpha} \leq \frac{1}{n^{1-\alpha}}\sum_{m=1}^{n-1}\frac{1}{m^\alpha}.$$

Accordingly, $\sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{m=1}^{n-1} \frac{w(n)n^\alpha}{w(m)m^\alpha} < \infty$ whenever $\sup_{n\in\mathbb{N}} \frac{1}{n^{1-\alpha}}\sum_{m=1}^{n-1}\frac{1}{m^\alpha} < \infty$. So, condition (ii) of part (2) of Theorem 3.4 will be satisfied *provided* we can verify that $\sup_{n\in\mathbb{N}} \frac{1}{n^{1-\alpha}}\sum_{m=1}^{n-1}\frac{1}{m^\alpha} < \infty$. To establish this, we need to distinguish the three cases; a) $\alpha = 0$; b) $\alpha < 0$ and c) $0 < \alpha < 1$.

Case a). We have, for every $n \in \mathbb{N}$, that

$$\frac{1}{n^{1-\alpha}}\sum_{m=1}^{n-1}\frac{1}{m^\alpha} = \frac{1}{n}\sum_{m=1}^{n-1} 1 = \frac{n-1}{n} \leq 1$$

and hence, that $\sup_{n\in\mathbb{N}} \frac{1}{n^{1-\alpha}}\sum_{m=1}^{n-1}\frac{1}{m^\alpha} \leq 1$.

Case b). Fix $n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{m=1}^{n-1}\frac{1}{m^\alpha} &\leq \sum_{m=1}^{n-1}\int_m^{m+1} x^{-\alpha}\,dx = \int_1^n x^{-\alpha}\,dx \\ &= \frac{1}{1-\alpha}\left(n^{1-\alpha} - 1\right) \leq \frac{n^{1-\alpha}}{1-\alpha}.\end{aligned}$$

Accordingly,

$$\frac{1}{n^{1-\alpha}}\sum_{m=1}^{n-1}\frac{1}{m^\alpha} \leq \frac{1}{n^{1-\alpha}}\frac{n^{1-\alpha}}{1-\alpha} = \frac{1}{1-\alpha}.$$

It follows that $\sup_{n\in\mathbb{N}} \frac{1}{n^{1-\alpha}}\sum_{m=1}^{n-1}\frac{1}{m^\alpha} \leq \frac{1}{1-\alpha}$.

Case c). Fix $n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{m=1}^{n-1} \frac{1}{m^\alpha} &\leq 1 + \sum_{m=2}^{n-1} \int_{m-1}^{m} x^{-\alpha}\, dx = 1 + \int_1^{n-1} x^{-\alpha}\, dx \\ &= 1 + \frac{(n-1)^{1-\alpha} - 1}{1-\alpha} \leq \frac{(n-1)^{1-\alpha}}{1-\alpha}.\end{aligned}$$

Accordingly,

$$\frac{1}{n^{1-\alpha}} \sum_{m=1}^{n-1} \frac{1}{m^\alpha} \leq \frac{1}{n^{1-\alpha}} \frac{(n-1)^{1-\alpha}}{1-\alpha} < \frac{1}{1-\alpha}.$$

It follows that $\sup_{n\in\mathbb{N}} \frac{1}{n^{1-\alpha}} \sum_{m=1}^{n-1} \frac{1}{m^\alpha} \leq \frac{1}{1-\alpha}$.

These 3 cases verify that condition (ii) in part (2) of Theorem 3.4 is fulfilled. □

Proposition 3.7. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$.*

Suppose that $S_w \neq \emptyset$. Then, for the dual operator $(\mathsf{C}^{(0,w)})' \in \mathcal{L}(\ell_1(w^{-1}))$ of $\mathsf{C}^{(0,w)}$, it is the case that

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_0}\right| < \frac{1}{2s_0}\right\} \cup \Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(0,w)})') \tag{3.14}$$

and

$$\sigma_{pt}((\mathsf{C}^{(0,w)})') \setminus \Sigma_0 \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\}. \tag{3.15}$$

For the Cesàro operator $\mathsf{C}^{(0,w)}$ itself we have

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\} \cup \Sigma \subseteq \sigma(\mathsf{C}^{(0,w)}). \tag{3.16}$$

Proof. Again we proceed by a series of steps. As before, let $A = (\mathsf{C}^{(0,w)})'$.

Step 1. $\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_0}\right| < \frac{1}{2s_0}\right\} \subseteq \sigma_{pt}(A)$.

That $\Sigma \subseteq \sigma_{pt}(A)$ was established in Step 2 of the proof of Theorem 3.4.

So, let $\lambda \in \mathbb{C} \setminus \Sigma$ satisfy $\left|\lambda - \frac{1}{2s_0}\right| < \frac{1}{2s_0}$ (equivalently, via Remark 3.5, $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right) > s_0$). For such a λ the vector $y = (y_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ defined by (3.8), with $y_1 \neq 0$, actually belongs to $\ell_1(w^{-1})$. Indeed, via Lemma 3.3(i) there exists $c = c(\lambda) > 0$ such that

$$\prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right| \leq c n^{-\mathrm{Re}(1/\lambda)}, \quad n \in \mathbb{N}.$$

It then follows from (3.8) that

$$\begin{aligned}\sum_{n=1}^{\infty} |y_n| w(n)^{-1} &= |y_1| w(1)^{-1} + |y_1| \sum_{n=2}^{\infty} \prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right| w(n)^{-1} \\ &\leq |y_1| w(1)^{-1} + c|y_1| \sum_{n=2}^{\infty} n^{-\mathrm{Re}(1/\lambda)} w(n)^{-1},\end{aligned}$$

where the series $\sum_{n=2}^{\infty} n^{-\mathrm{Re}(1/\lambda)} w(n)^{-1}$ converges because $\mathrm{Re}(1/\lambda) \in S_w$, that is, $y \in \ell_1(w^{-1})$. Since $Ay = \lambda y$, it follows that $\lambda \in \sigma_{pt}(A)$.

Step 2. $\sigma_{pt}(A) \setminus \Sigma_0 \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\}$.

Fix $\lambda \in \sigma_{pt}(A) \setminus \Sigma_0$. According to (3.2) there exists $\beta = \beta(\lambda) > 0$ such that

$$\prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right| \geq \beta \cdot n^{-\mathrm{Re}(1/\lambda)}, \quad n \in \mathbb{N}. \tag{3.17}$$

But, as argued in Step 1 (for any $y_1 \in \mathbb{C} \setminus \{0\}$) the eigenvector $y = (y_n)_{n\in\mathbb{N}}$ corresponding to the eigenvalue λ of A, which necessarily belongs to $\ell_1(w^{-1})$, i.e., $\sum_{n=1}^{\infty} |y_n| w(n)^{-1} < \infty$, is given by (3.8). Then (3.17) implies that also $\sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(1/\lambda)} w(n)} < \infty$, i.e., $\mathrm{Re}\left(\frac{1}{\lambda}\right) \in S_w$ and so $\mathrm{Re}\left(\frac{1}{\lambda}\right) \geq s_0$, that is (via Remark 3.5), $\lambda \in \left\{\mu \in \mathbb{C}\colon \left|\mu - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\}$.

It is clear that Steps 1-2 establish the two containments in (3.14) and (3.15).

For every $T \in \mathcal{L}(X)$ with X a Banach space, it is known that $\sigma_{pt}(T') \subseteq \sigma(T)$, [13, p.581], with $\sigma(T)$ a closed subset of $\mathbb{C}$. Accordingly, (3.16) follows from (3.14). □

Remark 3.8. For a Banach space operator $T \in \mathcal{L}(X)$ if $\lambda \in \sigma(T) \setminus \sigma_{pt}(T)$, then $T - \lambda I$ is not surjective. Such a point λ belongs to the *continuous spectrum* $\sigma_c(T)$ of T if the range $(T - \lambda I)(X)$ is dense in X. Otherwise λ belongs to the *residual spectrum* $\sigma_r(T)$ of T. Accordingly, there is the disjoint decomposition $\sigma(T) = \sigma_{pt}(T) \cup \sigma_r(T) \cup \sigma_c(T)$, [13, p.580]. For the dual operator $T' \in \mathcal{L}(X')$ we have

$$\sigma_r(T) \subseteq \sigma_{pt}(T') \subseteq \sigma_r(T) \cup \sigma_{pt}(T),$$

[13, p.581]. In particular,

$$\sigma_{pt}(T') \setminus \sigma_{pt}(T) = \sigma_r(T). \tag{3.18}$$

For the classical Cesàro operator $\mathsf{C}^{(0)} \in \mathcal{L}(c_0)$ it is known that $\sigma_{pt}(\mathsf{C}^{(0)}) = \emptyset$, $\sigma_c(\mathsf{C}^{(0)}) = \{\lambda \in \mathbb{C}\colon |\lambda - \frac{1}{2}| = \frac{1}{2}\} \setminus \{1\}$ and $\sigma_r(\mathsf{C}^{(0)}) = \{1\} \cup \{\lambda \in \mathbb{C}\colon |\lambda - \frac{1}{2}| < \frac{1}{2}\}$, [1, Theorem 2.7].

Suppose now that w is a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. Theorem 3.4 and Proposition 3.7 provide relevant information about the nature of the fine spectrum of $\mathsf{C}^{(0,w)}$. First, it is always the case that

$$0 \in \sigma_c(\mathsf{C}^{(0,w)}). \tag{3.19}$$

Indeed, Step 2 in the proof of Theorem 3.4 implies that $0 \in \sigma((\mathsf{C}^{(0,w)})')$ and hence, $\mathsf{C}^{(0,w)}$ cannot be surjective, [15, Theorem II.3.13]. Since $(\mathsf{C}^{(0,w)})'$ is injective (cf. Step 1 in the proof of Theorem 3.4), it is known that $\mathsf{C}^{(0,w)}$ has dense range in $c_0(w)$, [15, Theorem II.3.7]. Then (3.19) is clear because $\mathsf{C}^{(0,w)}$ is injective, i.e., $0 \notin \sigma_{pt}(\mathsf{C}^{(0,w)})$, which follows immediately from (1.1).

In the event that $t_0 \in [0, \infty)$ is finite, Theorem 3.4(3) implies that

$$\left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \leq m < t_0 + 1\right\} \subseteq \sigma_{pt}(\mathsf{C}^{(0,w)}).$$

Moreover, (3.5) and Step 2 in the proof of Theorem 3.4 show that

$$\left\{\frac{1}{m}\colon m \in \mathbb{N},\ m > t_0 + 1\right\} \subseteq \sigma_{pt}((\mathsf{C}^{(0,w)})') \setminus \sigma_{pt}(\mathsf{C}^{(0,w)})$$

and hence, via (3.18), it follows that

$$\left\{\frac{1}{m} : m \in \mathbb{N},\ m > t_0 + 1\right\} \subseteq \sigma_r(\mathsf{C}^{(0,w)}). \tag{3.20}$$

Suppose now that $S_w \neq \emptyset$, in which case t_0 is necessarily finite as $t_0 \leq s_0$ (cf. Proposition 3.1(i)). According to (3.14), (3.18) and (3.20) it follows that

$$\left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_0}\right| < \frac{1}{2s_0}\right\} \cup \left\{\frac{1}{m} : m \in \mathbb{N},\ m > t_0 + 1\right\} \subseteq \sigma_r(\mathsf{C}^{(0,w)}). \tag{3.21}$$

Moreover, the inclusion $\sigma_r(\mathsf{C}^{(0,w)}) \setminus \Sigma_0 \subseteq \sigma_{pt}((\mathsf{C}^{(0,w)})') \setminus \Sigma_0$, which follows from (3.18) and $\sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \Sigma$, together with (3.15) and (3.20) imply that

$$\sigma_r(\mathsf{C}^{(0,w)}) \setminus \Sigma_0 \subseteq \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\} \setminus \{0\}. \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$\sigma_c(\mathsf{C}^{(0,w)}) \cap \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\} \subseteq \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_0}\right| = \frac{1}{2s_0}\right\}.$$

In the event that (3.16) is an equality we can conclude that

$$\sigma_c(\mathsf{C}^{(0,w)}) \subseteq \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_0}\right| = \frac{1}{2s_0}\right\}.$$

This includes the weights w of Examples 3.10(i), (ii) and Example 3.12 below and, of course, the case when $\inf_{n\in\mathbb{N}} w(n) > 0$ (for which $s_0 = 1$ and $c_0(w)$ is isomorphic to c_0 with $\mathsf{C}^{(0,w)} = \mathsf{C}^{(0)}$).

The following result extends Proposition 2.6.

Proposition 3.9. *Let w be a bounded, strictly positive sequence. The following assertions are equivalent.*

(i) $w \in s$.
(ii) $R_w = \mathbb{R}$.

If, in addition, $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$, then (i) *and* (ii) *are equivalent to*

(iii) $\sigma_{pt}(\mathsf{C}^{(0,w)}) = \Sigma$.

Moreover, the inclusion $\Sigma_0 \subseteq \sigma(\mathsf{C}^{(0,w)})$ always holds.

In the event that $S_w \neq \emptyset$, the operator $\mathsf{C}^{(0,w)}$ is not compact.

Whenever $\mathsf{C}^{(0,w)}$ satisfies (2.7)*, then necessarily $S_w = \emptyset$.*

If $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$, then (2.7) *holds, that is,*

$$\sigma_{pt}(\mathsf{C}^{(0,w)}) = \Sigma \quad \textit{and} \quad \sigma(\mathsf{C}^{(0,w)}) = \Sigma_0.$$

Moreover, $w \in s$ and $S_w = \emptyset$.

Proof. (i)⇒(ii). Since $w \in \ell_\infty$, it is clear that $(-\infty, 0) \subseteq R_w$. For $t > 0$ fixed, set $m := [t] + 1$. Proposition 2.6 implies that $\lim_{n\to\infty} n^m w(n) = 0$ and hence, also $\lim_{n\to\infty} n^t w(n) = 0$. Accordingly, $t \in R_w$.

(ii)⇒(i). Obvious in view of Proposition 2.6.

Assume now that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$.

(ii)⇔(iii). Given $m \in \mathbb{N}$ it was shown in Steps 5 and 6 in the proof of Theorem 3.4 that $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(0,w)})$ if and only if $(m-1) \in R_w$. Accordingly, $\sigma_{pt}(\mathsf{C}^{(0,w)}) = \Sigma$

if and only if $m \in R_w$ for every $m \in \mathbb{N}$, i.e., if and only if $w \in s$. By (i)$\Leftrightarrow$(ii) this is the case if and only if $R_w = \mathbb{R}$.

That always $\Sigma_0 \subseteq \sigma(\mathsf{C}^{(0,w)})$ is part (1) of Theorem 3.4.

It is immediate from Proposition 3.7 (cf. (3.16)) that $\mathsf{C}^{(0,w)} \notin \mathcal{K}(c_0(w))$ whenever $S_w \neq \emptyset$.

Let $\mathsf{C}^{(0,w)}$ satisfy (2.7). If $S_w \neq \emptyset$, then (3.16) provides a contradiction. Accordingly, $S_w = \emptyset$.

Finally, suppose that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$. Then $\sigma(\mathsf{C}^{(0,w)}) = \{0\} \cup \sigma_{pt}(\mathsf{C}^{(0,w)})$, [21, Theorem 3.4.23], which combined with the inclusions $\Sigma \subseteq \sigma(\mathsf{C}^{(0,w)})$ and $\sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \sigma_{pt}(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \Sigma$ (cf. Lemma 2.5(i)) yields (2.7). Since $\sigma_{pt}(\mathsf{C}^{(0,w)}) = \Sigma$, Proposition 2.6 implies that $w \in s$. That $S_w = \emptyset$ follows from (2.7) has already been verified. □

We now present some relevant examples.

Examples 3.10. (i) Suppose that $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \geq 0$. Then $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)} < \infty$ if and only if $s > 1$ and hence, $s_0 = 1$. Moreover, $\lim_{n\to\infty} n^t w(n) = 0$ if and only if $t < 0$ or $t \leq 0$ in case $\gamma > 0$. Hence, $t_0 = 0$. According to Corollary 3.6 and Proposition 3.7 we have

$$\sigma(\mathsf{C}^{(0,w)}) = \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\},$$

whereas (3.5), for $t_0 = 0$, implies that $\sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \{1\}$. Since the constant sequence $\mathbf{1} \in c_0(w)$ if and only if $\gamma > 0$ and because $\mathsf{C}\mathbf{1} = \mathbf{1}$, it follows that

$$\sigma_{pt}(\mathsf{C}^{(0,w)}) = \emptyset \text{ if } \gamma = 0; \quad \sigma_{pt}(\mathsf{C}^{(0,w)}) = \{1\} \text{ if } \gamma > 0.$$

In particular, equality may occur in (3.13).

For the case when $\gamma = 0$ (so that $w(n) = 1$ for $n \in \mathbb{N}$), we recover the known result that $\sigma(\mathsf{C}^{(0)}) = \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$; see [7], [19].

(ii) Suppose that $w(n) = \frac{1}{n^\beta (\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\beta \geq 0$ and $\gamma \geq 0$. The claim is that

$$\left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2(\beta+1)}\right| \leq \frac{1}{2(\beta+1)}\right\} \cup \Sigma = \sigma(\mathsf{C}^{(0,w)}). \tag{3.23}$$

It is routine to check that $s_0 = 1 + \beta$. In particular, $S_w \neq \emptyset$. By (3.16) and Theorem 3.4(2), to verify (3.23) it suffices to prove that every $\lambda \in \mathbb{C} \setminus \Sigma_0$ satisfying $\left|\lambda - \frac{1}{2(\beta+1)}\right| > \frac{1}{2(\beta+1)}$ belongs to $\rho(\mathsf{C}^{(0,w)})$, i.e., that conditions (i) and (ii) in Theorem 3.4(2) are satisfied. So, fix such a λ and recall that $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right) < (\beta+1)$; see (3.12). Accordingly,

$$\lim_{n\to\infty} \frac{w(n)}{n^{1-\alpha}} = \lim_{n\to\infty} \frac{1}{n^{1-\alpha+\beta} \log^\gamma(n+1)} = 0$$

as $1-\alpha+\beta>0$. Moreover, for every $n\in\mathbb{N}$, we have

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^{n-1}\frac{w(n)n^\alpha}{w(m)m^\alpha} &= \frac{1}{n^{1-\alpha+\beta}\log^\gamma(n+1)}\sum_{m=1}^{n-1}\frac{\log^\gamma(m+1)}{m^{\alpha-\beta}}\\ &\leq \frac{1}{n^{1-\alpha+\beta}}\frac{\log^\gamma(n)}{\log^\gamma(n+1)}\sum_{m=1}^{n-1}\frac{1}{m^{\alpha-\beta}}\\ &\leq \frac{1}{n^{1-\alpha+\beta}}\sum_{m=1}^{n-1}\frac{1}{m^{\alpha-\beta}}\leq\frac{1}{1-\alpha+\beta}\end{aligned}$$

as $\alpha-\beta<1$ (see cases b) and c) in the proof of Corollary 3.6) and so

$$\sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{m=1}^{n-1}\frac{w(n)n^\alpha}{w(m)m^\alpha}<\infty.$$

This establishes the identity (3.23).

Now, observe that $\lim_{n\to\infty}n^tw(n)=0$ if and only if $t<\beta$ or $t\leq\beta$ in case $\gamma>0$. Hence, $t_0=\beta$. So, if $\beta\notin\mathbb{N}$, then by Theorem 3.4(3)

$$\left\{\frac{1}{m}\colon m\in\mathbb{N},\ 1\leq m<\beta+1\right\}=\sigma_{pt}(\mathsf{C}^{(0,w)}). \tag{3.24}$$

Suppose that $\beta\in\mathbb{N}$. If $\gamma=0$, then the identity (3.24) also holds because the eigenvector $x:=(x_n)_n$ of C corresponding to $\frac{1}{\beta+1}$ satisfies $x_n\simeq n^\beta$ for $n\to\infty$ (see (3.3) and (3.11)) and so $x\notin c_0(w)$. However, if $\gamma>0$, then $x\in c_0(w)$ and hence,

$$\left\{\frac{1}{m}\colon m\in\mathbb{N},\ 1\leq m\leq\beta+1\right\}=\sigma_{pt}(\mathsf{C}^{(0,w)}).$$

(iii) We return to Example 2.11 where $w\in s$, given by $w(1):=1$ and $w(n):=\frac{1}{j^j}$ if $n\in\{2^{j-1}+1,\dots,2^j\}$ for $j\in\mathbb{N}$, satisfies $w\downarrow 0$. It was shown there that $\mathsf{C}^{(0,w)}$ is *not* compact. Nevertheless, it is still the case that

$$\sigma_{pt}(\mathsf{C}^{(0,w)})=\Sigma;\quad \sigma(\mathsf{C}^{(0,w)})=\Sigma_0, \tag{3.25}$$

which should be compared with Propositions 2.7 and 3.9. Indeed, Proposition 3.9 yields that $\sigma_{pt}(\mathsf{C}^{(0,w)})=\Sigma$. According to Corollary 3.6 we have

$$\sigma(\mathsf{C}^{(0,w)})\subseteq\left\{\lambda\in\mathbb{C}\colon \left|\lambda-\frac{1}{2}\right|\leq\frac{1}{2}\right\}.$$

Hence, to establish (3.25) it remains to show that $\lambda\in\rho(\mathsf{C}^{(0,w)})$ whenever $\lambda\in\mathbb{C}\setminus\Sigma_0$ satisfies $|\lambda-\frac{1}{2}|\leq\frac{1}{2}$. Set $\alpha:=\mathrm{Re}\left(\frac{1}{\lambda}\right)$ in which case $\alpha\geq 1$; see Remark 3.5. To verify that $\lambda\in\rho(\mathsf{C}^{(0,w)})$ we establish that conditions (i) and (ii) in part (2) of Theorem 3.4 are satisfied.

Condition (i) is clear because $w\in s$.

Since $\frac{1}{n}\sum_{m=1}^{n-1}\frac{w(n)n^\alpha}{w(m)m^\alpha}\leq\frac{1}{n}\sum_{m=1}^{n}\frac{w(n)n^\alpha}{w(m)m^\alpha}$ for $n\in\mathbb{N}$, to verify (ii) we only need to estimate $\frac{1}{n}\sum_{m=1}^{n}\frac{w(n)n^\alpha}{w(m)m^\alpha}$. So, fix $j\in\mathbb{N}$ and then choose any $n\in(2^{j-1},2^j]$.

In particular, $n \leq 2^j$ and so

$$\begin{aligned}
\frac{1}{n}\sum_{m=1}^{n}\frac{w(n)n^\alpha}{w(m)m^\alpha} &= n^{\alpha-1}w(n)\sum_{m=1}^{n}\frac{1}{w(m)m^\alpha} \leq \frac{2^{(\alpha-1)j}}{j^j}\sum_{m=1}^{2^j}\frac{1}{w(m)m^\alpha} \\
&= \frac{2^{(\alpha-1)j}}{j^j}\left(1+\sum_{k=1}^{j}\sum_{i=2^{k-1}+1}^{2^k}\frac{k^k}{i^\alpha}\right) \leq \frac{2^{(\alpha-1)j}}{j^j}\left(1+\sum_{k=1}^{j}\frac{2^{k-1}k^k}{2^{(k-1)\alpha}}\right) \\
&= \frac{2^{(\alpha-1)j}}{j^j}\left(1+\sum_{k=1}^{j}\frac{k^k}{2^{(k-1)(\alpha-1)}}\right) =: D_j,
\end{aligned}$$

where we have used the fact that $\frac{k^k}{i^\alpha} \leq \frac{k^k}{(2^{k-1})^\alpha}$ for all $2^{k-1} < i \leq 2^k$. Define $b_j := \frac{j^j}{2^{(\alpha-1)j}}$ and $a_j := b_j D_j$ for $j \in \mathbb{N}$ and observe that

$$\lim_{j\to\infty}\frac{a_j - a_{j-1}}{b_j - b_{j-1}} = \lim_{j\to\infty}\left(\frac{1}{2^{\alpha-1}} - \frac{1}{j}\left(1-\frac{1}{j}\right)^{j-1}\right)^{-1} = 2^{\alpha-1}.$$

Since $b_j \uparrow \infty$ for $j \to \infty$, according to the Stolz-Cesàro criterion it follows that also $\lim_{j\to\infty}\frac{a_j}{b_j} = 2^{\alpha-1}$, i.e., $D_j \to 2^{\alpha-1}$ for $j \to \infty$. In particular,

$$\sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{m=1}^{n-1}\frac{w(n)n^\alpha}{w(m)m^\alpha} \leq \sup_{j\in\mathbb{N}} D_j < \infty.$$

So, condition (ii) is indeed satisfied.

According to (3.25) the operator $\mathsf{C}^{(0,w)}$ satisfies (2.7) and so Proposition 3.9 yields that $S_w = \emptyset$.

The following comparison type result is simple but, can be useful in practise.

Proposition 3.11. *Let v be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,v)} \in \mathcal{L}(c_0(v))$ and w be a weight satisfying*

$$Av(n) \leq w(n) \leq Bv(n), \quad n \in \mathbb{N}, \tag{3.26}$$

for positive constants A and B. Then $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ and

$$\sigma(\mathsf{C}^{(0,w)}) = \sigma(\mathsf{C}^{(0,v)}). \tag{3.27}$$

Proof. It is routine to verify that $\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} \leq \frac{A}{B}\frac{v(n)}{n}\sum_{k=1}^{n}\frac{1}{v(k)}$ for each $n \in \mathbb{N}$ and so, via Corollary 2.3(i), we see that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$.

It is clear from (3.26) that $c_0(w) = c_0(v)$ as vector spaces and that the two norms $\|\cdot\|_{0,v}$ and $|||x||| := \sup_{n\in\mathbb{N}} w(n)|x_n|$ are equivalent on $c_0(v)$. So, $\mathsf{C}^{(0,w)}$ is precisely the operator $\mathsf{C}\colon c_0(v) \to c_0(v)$ for the equivalent norm $|||\cdot|||$ in $c_0(v)$. The identity (3.27) is then clear. □

An immediate application of the previous result is as follows.

Example 3.12. Consider the strictly positive weight w given by $w(1) := 1$ with $w(2n) := \frac{1}{2n-1}$ and $w(2n+1) := \frac{1}{n+1}$ for $n \in \mathbb{N}$. It is routine to verify that $s_0 = 2$ and $t_0 = 1$. Observe that

$$A_{2n} := \frac{w(2n)}{2n}\sum_{k=1}^{2n}\frac{1}{w(k)} = \frac{1}{2n(2n-1)}\left(\sum_{k=1}^{n}(2k-1) + \sum_{k=1}^{n}k\right) = \frac{3n+1}{4(2n-1)},\ n \in \mathbb{N},$$

and so, $A_{2n} \leq 1$ with $\lim_{n\to\infty} A_{2n} = \frac{3}{8}$. On the other hand,

$$\begin{aligned} A_{2n+1} &:= \frac{w(2n+1)}{2n+1}\sum_{k=1}^{2n+1}\frac{1}{w(k)} = \frac{1}{(2n+1)(n+1)}\left(\frac{1}{w(2n+1)}+\sum_{k=1}^{2n}\frac{1}{w(k)}\right) \\ &= \frac{3n^2+3n+2}{2(2n+1)(n+1)}, \quad n\in\mathbb{N}, \end{aligned}$$

and so, $A_{2n+1} \leq 1$ with $\lim_{n\to\infty} A_{2n+1} = \frac{3}{4}$. It follows from Proposition 2.2(i) (with $v := w$) that $\|\mathsf{C}^{(0,w)}\| = 1$, even though w is *not* a decreasing weight; compare with Corollary 2.3(i). We point out that $\{w(n)\}_{n\geq n_0}$ fails to be decreasing for *every* $n_0 \in \mathbb{N}$.

Observe, for each $n \in \mathbb{N}$, that

$$2n \geq 2n-1 = \frac{1}{w(2n)} \geq n = \frac{2n}{2}$$

and that

$$2n+1 \geq n+1 = \frac{1}{w(2n+1)} \geq \frac{2(n+1)}{2} \geq \frac{2n+1}{2}.$$

It follows that $\frac{1}{n} \leq w(n) \leq \frac{2}{n}$ for all $n \in \mathbb{N}$. Applying Proposition 3.11 above (with $v(n) = \frac{1}{n}$ for $n \in \mathbb{N}$) and Example 3.10(ii) with $\gamma = 0$ and $\beta = 1$, we can conclude that

$$\sigma(\mathsf{C}^{(0,w)}) = \left\{\lambda\in\mathbb{C}\colon \left|\lambda-\frac{1}{4}\right| \leq \frac{1}{4}\right\} \cup \{1\}.$$

Observe that (3.16) is an equality in this case.

At this stage it is relevant to compare our results with previous ones on factorable matrices acting on c_0 due to Rhoades and Yildirim, [26].

Let w be any bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. The Cesàro operator C is similar (via an isometry) to a continuous linear operator T_w acting on c_0 which is defined by the *factorable matrix* $A(w) = (a_{nk})_{n,k\in\mathbb{N}}$ with entries $a_{nk} = a_n b_k = \frac{w(n)}{n}\cdot w(k)^{-1}$ for $1 \leq k \leq n$ and $a_{nk} = 0$ for $k > n$ (see (2.1)). In particular, $\sigma(\mathsf{C}^{(0,w)}) = \sigma(T_w; c_0)$. Observe that the matrix $A(w)$ satisfies the following two conditions:

(i) $\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}|a_{nk}| = \sup_{n\in\mathbb{N}}\frac{w(n)}{n}\sum_{k=1}^{n}w(k)^{-1} < \infty$, via Corollary 2.3(i), and

(ii) $f_k := \lim_{n\to\infty} a_{nk} = w(k)^{-1}\lim_{n\to\infty}\frac{w(n)}{n} = 0$, $k\in\mathbb{N}$, since $w \in \ell_\infty$.

If, in addition, the matrix $A(w)$ also satisfies the condition

(iii) $u := \lim_{n\to\infty}\sum_{k=1}^{\infty}a_{nk} = \lim_{n\to\infty}\frac{w(n)}{n}\sum_{k=1}^{n}w(k)^{-1}$ exists,

then the linear operator corresponding to $A(w)$ is a selfmap of c, the space of all convergent sequences, that is, $A(w)$ is *conservative*, [26, p.265].

In the case when $A(w)$ satisfies condition (iii) with $u \neq \sum_{k=1}^{\infty} f_k$ $(= 0)$, the matrix $A(w)$ is called *coregular*, [26]. In particular, $\mathsf{C}^{(0,w)} \notin \mathcal{K}(c_0(w))$; see Corollary 2.3(ii). For $A(w)$ observe that $\delta := \lim_{n\to\infty} a_n b_n = 0$. By [26, Corollary 1] applied to $A(w)$ we can conclude that

$$\sigma(\mathsf{C}^{(0,w)}) = \left\{\lambda\in\mathbb{C}\colon \left|\lambda-\frac{u}{2}\right| \leq \frac{u}{2}\right\} \cup \Sigma_0, \tag{3.28}$$

since $S := \overline{\{a_n b_n\colon n\in\mathbb{N}\}} = \Sigma_0$.

If the matrix $A(w)$ happens to satisfy condition (iii) with $u = 1$, then $A(w)$ is called *regular*, [26]. In this case [26, Corollary 2] yields that

$$\sigma(\mathsf{C}^{(0,w)}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}. \tag{3.29}$$

It is important to point out that there exist bounded, strictly positive sequences w such that the corresponding matrix $A(w)$ is not coregular, i.e., condition (iii) does not hold. For example, this is so for the decreasing sequence $w \in s$ in Example 2.11, where it is explicitly shown that the limit in condition (iii) fails to exist. For this weight w the spectrum of $\mathsf{C}^{(0,w)}$ is determined in Example 3.10(iii). Also for the weight w in Example 3.12 above the limit in condition (iii) fails to exist; the spectrum of $\mathsf{C}^{(0,w)}$ is also determined there. A further example w for which u fails to exist occurs in Case (ii) of Remark 4.11(c).

As seen in Example 3.12, $s_0 = 2$ exists but not the limit u in condition (iii). However, whenever $u \neq 0$ *does* exist for a bounded, strictly positive sequence w and $S_w \neq \emptyset$, then it follows from (3.16) and (3.28) that necessarily $s_0 \geq \frac{1}{u}$.

Remark 3.13. The formulae (3.28) and (3.29) provide an alternate approach to determine the spectra in Examples 3.10(i), (ii), where $w(n) := \frac{1}{n^\beta \log^\gamma(n+1)}$ for $n \in \mathbb{N}$, with $\beta \geq 0$ and $\gamma \geq 0$. Indeed, for

$$B_n := \frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} = \frac{1}{n^{\beta+1}\log^\gamma(n+1)}\sum_{k=1}^{n} k^\beta \log^\gamma(k+1), \quad n \in \mathbb{N},$$

it suffices to verify that $u := \lim_{n\to\infty} B_n = \frac{1}{1+\beta}$. In this case $u = \frac{1}{s_0}$.

If $\beta = \gamma = 0$, then $w(n) = 1$ for $n \in \mathbb{N}$, in which case clearly $B_n \to \frac{1}{1+\beta}$ for $n \to \infty$. So, we may assume that at least one of β, γ is strictly positive.

Observe, for each $k \in \mathbb{N}$, that

$$k^\beta \log^\gamma(k+1) \leq \int_k^{k+1} x^\beta \log^\gamma(x+1)\,dx \leq (k+1)^\beta \log^\gamma(k+2).$$

It follows, for each $n \in \mathbb{N}$, that

$$\sum_{k=1}^{n} k^\beta \log^\gamma(k+1) \leq \int_1^{n+1} x^\beta \log^\gamma(x+1)\,dx \leq \sum_{k=1}^{n}(k+1)^\beta \log^\gamma(k+2). \tag{3.30}$$

Since $\sum_{k=1}^{n}(k+1)^\beta \log^\gamma(k+2) = \left(\sum_{h=1}^{n+1} h^\beta \log^\gamma(h+1)\right) - \log^\gamma(2)$, it is clear from (3.30) that $\lim_{n\to\infty} B_n = \frac{1}{1+\beta}$ will follow if we can establish that

$$\lim_{n\to\infty} D_n := \lim_{n\to\infty}\frac{1}{n^{\beta+1}\log^\gamma(n+1)}\int_1^{n+1} x^\beta \log^\gamma(x+1)\,dx = \frac{1}{1+\beta}. \tag{3.31}$$

If $\gamma = 0$, then clearly $D_n = \frac{1}{n^{\beta+1}}\int_1^{n+1} x^\beta\,dx \to \frac{1}{1+\beta}$ for $n \to \infty$. So, we may assume that $\gamma > 0$. An integration by parts yields that

$$\int_1^{n+1} x^\beta \log^\gamma(x+1)\,dx = A_1(n) - A_2(n), \quad n \in \mathbb{N},$$

where

$$A_1(n) := \frac{(n+1)^{\beta+1}}{1+\beta}\log^\gamma(n+2) - \frac{\log^\gamma(2)}{1+\beta};\ A_2(n) := \int_1^{n+1}\frac{x^{\beta+1}}{1+\beta}\gamma\log^{\gamma-1}(x+1)\frac{dx}{x+1}.$$

Since $\gamma > 0$, it is clear that $\frac{A_1(n)}{n^{\beta+1}log^{\gamma}(n+1)} \to \frac{1}{1+\beta}$ for $n \to \infty$. So, to establish (3.31) reduces to showing that $\frac{A_2(n)}{n^{\beta+1}log^{\gamma}(n+1)} \to 0$ for $n \to \infty$, for which we consider two cases.

Case (a). $\gamma \geq 1$.

As $\frac{1}{1+x} \leq \frac{1}{x}$ and $x \mapsto x^\beta \log^{\gamma-1}(1+x)$ is increasing on $(0,\infty)$ we have, for each $n \in \mathbb{N}$, that

$$0 \leq A_2(n) \leq \frac{\gamma}{1+\beta} \int_1^{n+1} x^\beta \log^{\gamma-1}(x+1)\, dx \leq \frac{\gamma}{1+\beta}(n+1)^{\beta+1} \log^{\gamma-1}(n+2)$$

and so $0 \leq \frac{A_2(n)}{n^{\beta+1}\log^{\gamma}(n+1)} \leq \frac{\gamma}{1+\beta}\left(\frac{n+1}{n}\right)^{\beta+1} \frac{\log^{\gamma-1}(n+2)}{\log^{\gamma}(n+1)}$ reveals that $\frac{A_2(n)}{n^{\beta+1}\log^{\gamma}(n+1)} \to 0$ for $n \to \infty$.

Case (b). $0 < \gamma \leq 1$.

In this case for each $n \in \mathbb{N}$ we have

$$0 \leq A_2(n) \leq \frac{\gamma}{1+\beta}(n+1)^{\beta+1} \log^{\gamma-1}(3)$$

and hence, $0 \leq \frac{A_2(n)}{n^{\beta+1}\log^{\gamma}(n+1)} \leq \frac{\gamma}{1+\beta}\left(\frac{n+1}{n}\right)^{\beta+1} \frac{\log^{\gamma-1}(3)}{\log^{\gamma}(n+1)}$. Again it follows that $\frac{A_2(n)}{n^{\beta+1}\log^{\gamma}(n+1)} \to 0$ for $n \to \infty$.

This completes the proof of (3.31), that is, $u = \lim_{n\to\infty} B_n = \frac{1}{1+\beta}$.

4. Mean ergodic properties of $\mathsf{C}^{(0,w)}$

For X a Banach space, recall that an operator $T \in \mathcal{L}(X)$ is *mean ergodic* if its sequence of Cesàro averages

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}, \tag{4.1}$$

converges to some operator $P \in \mathcal{L}(X)$ in the strong operator topology τ_s, i.e., $\lim_{n\to\infty} T_{[n]}x = Px$ for each $x \in X$, [13, Ch.VIII]. According to [13, VIII Corollary 5.2] one then has the direct decomposition

$$X = \operatorname{Ker}(I-T) \oplus \overline{(I-T)(X)}. \tag{4.2}$$

To characterize the mean ergodicity of $\mathsf{C}^{(0,w)}$ we require the following fact.

Lemma 4.1. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$.*

(i) *The range of the operator $(I - \mathsf{C}^{(0,w)})$ satisfies*

$$\overline{(I - \mathsf{C}^{(0,w)})(c_0(w))} = \{x \in c_0(w) \colon x_1 = 0\} = \overline{\operatorname{span}\{e_r \colon r \geq 2\}}. \tag{4.3}$$

Moreover, the range of $(I - \mathsf{C}^{(0,w)})$ is itself closed in $c_0(w)$ if and only if

$$(I - \mathsf{C}^{(0,w)})(c_0(w)) = \{x \in c_0(w) \colon x_1 = 0\}. \tag{4.4}$$

(ii) *It is the case that* $\operatorname{Ker}(I - \mathsf{C}^{(0,w)}) = \operatorname{span}\{\mathbf{1}\}$ *if and only if* $\mathbf{1} \in c_0(w)$. *Otherwise,* $\operatorname{Ker}(I - \mathsf{C}^{(0,w)}) = \{0\}$.

Proof. (i) Since $\{e_r\}_{r\in\mathbb{N}}$ is a basis of $c_0(w)$ it is routine to check that

$$\{x \in c_0(w)\colon x_1 = 0\} = \overline{\mathrm{span}\{e_r\colon r \geq 2\}}. \tag{4.5}$$

The identities $e_{r+1} = (I - \mathsf{C}^{(0,w)})y_r$ for $r \in \mathbb{N}$ with

$$y_r := e_{r+1} - \frac{1}{r}\sum_{k=1}^{r} e_k, \quad r \in \mathbb{N},$$

which can be verified via direct calculation, together with the fact that $\{y_r\}_{r\in\mathbb{N}} \subseteq c_0(w)$, show that $\{e_r\}_{r\geq 2} \subseteq (I - \mathsf{C}^{(0,w)})(c_0(w))$. Accordingly,

$$\overline{\mathrm{span}\{e_r\colon r \geq 2\}} \subseteq \overline{(I - \mathsf{C}^{(0,w)})(c_0(w))}. \tag{4.6}$$

Clearly, $(I - \mathsf{C}^{(0,w)})(c_0(w)) \subseteq \{x \in c_0(w)\colon x_1 = 0\}$ and so, via (4.5), also $\overline{(I - \mathsf{C}^{(0,w)})(c_0(w))} \subseteq \overline{\mathrm{span}\{e_r\colon r \geq 2\}}$. That is, (4.6) is actually an equality which, combined with (4.5), establishes the validity of (4.3).

Clearly, (4.4) implies that $(I - \mathsf{C}^{(0,w)})$ has closed range in $c_0(w)$. Conversely, if $(I - \mathsf{C}^{(0,w)})$ has closed range in $c_0(w)$, i.e., $(I - \mathsf{C}^{(0,w)})(c_0(w)) = \overline{(I - \mathsf{C}^{(0,w)})(c_0(w))}$, then (4.4) follows from (4.3).

(ii) It is known that the Cesàro operator $\mathsf{C}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ satisfies $\mathrm{Ker}(I - \mathsf{C}) = \mathrm{span}\{\mathbf{1}\}$ in $\mathbb{C}^{\mathbb{N}}$; see the proof of Proposition 4.1 in [3]. Hence, $\mathrm{Ker}(I - \mathsf{C}^{(0,w)}) = \mathrm{span}\{\mathbf{1}\}$ if and only if $\mathbf{1} \in c_0(w)$. Otherwise, $(I - \mathsf{C}^{(0,w)})$ is injective, i.e., $\mathrm{Ker}(I - \mathsf{C}^{(0,w)}) = \{0\}$. □

Remark 4.2. In relation to the previous result we note that $\mathbf{1} \in c_0(w)$ if and only if $\lim_{n\to\infty} w(n) = 0$.

The following result characterizes the mean ergodicity of $\mathsf{C}^{(0,w)}$. Recall that $T \in \mathcal{L}(X)$, with X a Banach space, is *Cesàro bounded* if $\sup_{n\in\mathbb{N}} \|T_{[n]}\| < \infty$, [18, p.72].

Proposition 4.3. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. Assume that $\mathsf{C}^{(0,w)}$ is Cesàro bounded and τ_s-$\lim_{n\to\infty} \frac{1}{n}(\mathsf{C}^{(0,w)})^n = 0$. Then the following assertions are equivalent.*

(i) *$\mathsf{C}^{(0,w)}$ is mean ergodic.*
(ii) *The constant sequence $\mathbf{1} \in c_0(w)$.*
(iii) *The weight w satisfies $\lim_{n\to\infty} w(n) = 0$.*

Proof. (i)⇒(ii). The discussion prior to Lemma 4.1 (cf. (4.2)) yields

$$c_0(w) = \mathrm{Ker}(I - \mathsf{C}^{(0,w)}) \oplus \overline{(I - \mathsf{C}^{(0,w)})(c_0(w))}.$$

Lemma 4.1(i) reveals that $\overline{(I - \mathsf{C}^{(0,w)})(c_0(w))} = \{x \in c_0(w)\colon x_1 = 0\}$ and so the previous decomposition of $c_0(w)$ shows that $\mathrm{Ker}(I - \mathsf{C}^{(0,w)}) \neq \{0\}$. According to Lemma 4.1(ii) it follows that $\mathbf{1} \in c_0(w)$ and $\mathrm{Ker}(I - \mathsf{C}^{(0,w)}) = \mathrm{span}\{\mathbf{1}\}$.

(ii)⇒(i). Set $f_1 := \mathbf{1}$ and $f_j := f_1 - \sum_{k=1}^{j-1} e_k$ for $j \geq 2$, in which case $\{f_j\}_{j\in\mathbb{N}} \subseteq c_0(w)$. It is clear that $\{f_j\}_{j\geq 2} \subseteq \{x \in c_0(w)\colon x_1 = 0\}$ and hence, via (4.3), that $\{f_j\}_{j\geq 2} \subseteq \overline{\mathrm{span}\{e_r\colon r \geq 2\}}$. So, $e_1 = f_1 - f_2$ belongs to $\mathrm{span}\{f_1\} \oplus \overline{\mathrm{span}\{e_r\colon r \geq 2\}}$ and hence,

$$c_0(w) = \mathrm{span}\{f_1\} \oplus \overline{\mathrm{span}\{e_r\colon r \geq 2\}}$$

as $\{e_r\}_{r\in\mathbb{N}}$ is a basis for $c_0(w)$. According to (4.3)

$$c_0(w) = \mathrm{Ker}(I - \mathsf{C}^{(0,w)}) \oplus \overline{(I - \mathsf{C}^{(0,w)})(c_0(w))}.$$

It follows from this identity, together with the assumptions that $\mathsf{C}^{(0,w)}$ is Cesàro bounded and τ_s-$\lim_{n\to\infty} \frac{1}{n}(\mathsf{C}^{(0,w)})^n = 0$, that $\mathsf{C}^{(0,w)}$ is mean ergodic, [18, p.73 Theorem 1.3].

(ii)⇔(iii). See Remark 4.2. □

Remark 4.4. (i) Every mean ergodic operator $T \in \mathcal{L}(X)$, with X a Banach space, is necessarily Cesàro bounded (by the Principle of Uniform Boundedness). Moreover, (4.1) implies that

$$\frac{1}{n}T^n = T_{[n]} - \frac{n-1}{n}T_{[n-1]}, \quad n \in \mathbb{N}, \tag{4.7}$$

with $T_{[0]} := I$, from which it is clear that τ_s-$\lim_{n\to\infty} \frac{1}{n}T^n = 0$.

An operator $T \in \mathcal{L}(X)$ is called *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. This property clearly implies that T is both Cesàro bounded and satisfies τ_s-$\lim_{n\to\infty} \frac{1}{n}T^n = 0$. It is immediate from (4.7) that if T is Cesàro bounded, then $\sup_{n\in\mathbb{N}} \frac{\|T^n\|}{n} < \infty$. E. Hille exhibited a classical kernel operator T in $L^1([0,1])$ which fails to be power bounded (actually, $\|T^n\| = O(n^{1/4})$) but, nevertheless, is mean ergodic, [17]. There also exist operators $T \in \mathcal{L}(X)$ which are Cesàro bounded but $\frac{1}{n}T^n \not\to 0$ in the strong operator topology, even for X a finite dimensional space, [14, Example 4.7].

Observe that $\sigma(T) \subseteq \{\lambda \in \mathbb{C} \colon |\lambda| \le 1\}$ whenever T satisfies $\sup_{n\in\mathbb{N}} \frac{\|T^n\|}{n^k} < \infty$ for some $k \in \mathbb{N}$ (eg. if τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ or if T is Cesàro bounded). Indeed, suppose there exists $z \in \sigma(T)$ with $|z| > 1$. By the Spectral Mapping Theorem $z^n \in \sigma(T^n)$, $n \in \mathbb{N}$, and so the inequality $\|T^n\| \ge r(T^n) \ge |z|^n$ for all $n \in \mathbb{N}$, [13, p.257 Lemma 1], implies that $\frac{\|T^n\|}{n^k} \ge \frac{|z|^n}{n^k} \uparrow \infty$. This violates the stated assumption on T.

(ii) *If the weight w is decreasing, then $\mathsf{C}^{(0,w)}$ is mean ergodic if and only if $w \downarrow 0$.* Indeed, in this case we have $\|\mathsf{C}^{(0,w)}\| = 1$ (cf. Corollary 2.3(i)). Hence, $\mathsf{C}^{(0,w)}$ is power bounded and the conclusion follows from Proposition 4.3 and part (i) above.

The weight w given by $w(1) := 1$, $w(2) := \frac{1}{2}$ and $w(n) := \frac{5}{8}$ for $n \ge 3$ is not decreasing but satisfies

$$\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} \le 1, \quad n \in \mathbb{N}.$$

Hence, by Proposition 2.2(i) (with $v = w$) and Corollary 2.3(i), it is again the case that $\|\mathsf{C}^{(0,w)}\| = 1$. In particular, $\mathsf{C}^{(0,w)}$ is power bounded. Since $\mathbf{1} \notin c_0(w)$, it follows from Proposition 4.3 that $\mathsf{C}^{(0,w)}$ is not mean ergodic.

Consider the strictly positive weight w given in Example 2.11. Also this weight is not decreasing and satisfies $\|\mathsf{C}^{(0,w)}\| = 1$, i.e., $\mathsf{C}^{(0,w)}$ is power bounded. Since $\lim_{n\to\infty} w(n) = 0$, it follows from Proposition 4.3 that $\mathsf{C}^{(0,w)}$ is mean ergodic.

Suppose w is a bounded, strictly positive weight such that $a_w := \inf_{n\in\mathbb{N}} w(n) > 0$. Then

$$\sup_{n\in\mathbb{N}} \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \leq \frac{\|w\|_\infty}{a_w} < \infty$$

and so $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$; see Corollary 2.3(i). Since $\mathbf{1} \notin c_0(w)$, the operator $\mathsf{C}^{(0,w)}$ cannot be mean ergodic (whenever $\mathsf{C}^{(0,w)}$ is Cesàro bounded and satisfies $\tau_s\text{-}\lim_{n\to\infty} \frac{1}{n}(\mathsf{C}^{(0,w)})^n = 0$); see Proposition 4.3. In particular, this implies that the classical Cesàro operator $\mathsf{C}^{(0)}$ is not mean ergodic; see also [3, Proposition 4.3]. It is shown in [3, Section 4] that the Cesàro operator also fails to be mean ergodic in the classical Banach sequence spaces c, ℓ_p $(1 < p \leq \infty)$, bv_0 and bv but, that it is mean ergodic in bv_p $(1 < p < \infty)$.

There also exist weights w with $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ and $\|\mathsf{C}^{(0,w)}\| > 1$. Define w via $w(1) := 1$, $w(2) := \frac{1}{2}$, $w(3) := \frac{7}{8}$ and $w(n) := \frac{1}{n}$ for $n \geq 4$. For each $n \geq 4$ we have

$$\begin{aligned} \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} &= \frac{1}{n^2}\left(\frac{29}{7} + \sum_{k=4}^{n} \frac{1}{w(k)}\right) = \frac{1}{n^2}\left(\frac{29}{7} + \sum_{k=4}^{n} k\right) \\ &= \frac{1}{n^2}\left(\frac{29}{7} + \frac{(n-3)(n+4)}{2}\right). \end{aligned}$$

It follows from Corollary 2.3(i) that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. Moreover,

$$\frac{29}{24} = \frac{w(3)}{3} \sum_{k=1}^{3} \frac{1}{w(k)} \leq \left(\sup_{n\in\mathbb{N}} \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)}\right) = \|\mathsf{C}^{(0,w)}\|.$$

Even though $\|\mathsf{C}^{(0,w)}\| > 1$ it is still the case that $\mathsf{C}^{(0,w)}$ is power bounded. To see this consider the weight v given by $v(1) = v(2) := 1$, $v(3) := \frac{7}{8}$ and $v(n) := \frac{1}{n}$ for $n \geq 4$. Since $v \downarrow 0$, Corollary 2.3(i) yields that $\|\mathsf{C}^{(0,v)}\| = 1$ and hence, $\|(\mathsf{C}^{(0,v)})^n\| \leq 1$ for every $n \in \mathbb{N}$. Because of the inequalities

$$\frac{1}{2}v(n) \leq w(n) \leq v(n), \quad n \in \mathbb{N},$$

the identity transformation $I\colon c_0(v) \to c_0(w)$ is a Banach space isomorphism. Moreover, $\mathsf{C}^{(0,w)} = I\mathsf{C}^{(0,v)}I^{-1}$ and hence, $(\mathsf{C}^{(0,w)})^n = I(\mathsf{C}^{(0,v)})^n I^{-1}$ for all $n \in \mathbb{N}$. It follows that

$$\|(\mathsf{C}^{(0,w)})^n\| \leq \|I\| \cdot \|(\mathsf{C}^{(0,v)})^n\| \cdot \|I^{-1}\| \leq \|I\| \cdot \|I^{-1}\|, \quad n \in \mathbb{N},$$

which clearly implies the power boundedness of $\mathsf{C}^{(0,w)}$. It is routine to verify that actually $\|I\| = 1$ and $\|I^{-1}\| = 2$. Since $\mathbf{1} \in c_0(w)$, part (i) of this Remark and Proposition 4.3 yield that $\mathsf{C}^{(0,w)}$ is mean ergodic. Of course, since $\lim_{n\to\infty} \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} = \frac{1}{2}$, it is clear from Corollary 2.3(ii) that $\mathsf{C}^{(0,w)} \notin \mathcal{K}(c_0(w))$. A further example with these features is given in Remark 4.11(ii) below.

(iii) If w is a bounded, strictly positive weight such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$, then necessarily

$$\|(\mathsf{C}^{(0,w)})^n\| \geq 1, \quad n \in \mathbb{N}.$$

Indeed, since $1 \in \sigma(\mathsf{C}^{(0,w)})$ (cf. Theorem 3.4(1)), it follows that $1 \in \sigma((\mathsf{C}^{(0,w)})^n)$ for all $n \in \mathbb{N}$ and hence, $\|(\mathsf{C}^{(0,w)})^n\| \geq r((\mathsf{C}^{(0,w)})^n) \geq 1$ for $n \in \mathbb{N}$. In the event that w is *decreasing* it follows from $\|(\mathsf{C}^{(0,w)})^n\| \leq \|\mathsf{C}^{(0,w)}\|^n = 1$ (cf. Corollary 2.3(i)) that actually $\|(\mathsf{C}^{(0,w)})^n\| = 1$ for all $n \in \mathbb{N}$.

A Banach space operator $T \in \mathcal{L}(X)$ is called *uniformly mean ergodic* if there exists $P \in \mathcal{L}(X)$ such that $\lim_{n\to\infty} \|T_{[n]} - P\| = 0$. It is immediate from (4.7) that necessarily $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$.

In order to characterize the uniform mean ergodicity of $\mathsf{C}^{(0,w)}$ some preliminary results are required. For convenience of notation set $X_1(w) := \{x \in c_0(w) \colon x_1 = 0\}$. Clearly the closed subspace $X_1(w) \subseteq c_0(w)$ is invariant for $(I - \mathsf{C}^{(0,w)})$. Moreover, the restriction $(I - \mathsf{C}^{(0,w)})|_{X_1(w)}$ of $(I - \mathsf{C}^{(0,w)})$ to $X_1(w)$ is injective because $\mathbf{1} \notin X_1(w)$ and $\mathrm{Ker}(I - \mathsf{C}) = \mathrm{span}\{\mathbf{1}\}$ when we consider $\mathsf{C} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$.

Lemma 4.5. *Let w be a bounded, strictly positive sequence such that* $\lim_{n\to\infty} w(n) = 0$ *and* $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. *Then*

$$(I - \mathsf{C}^{(0,w)})(X_1(w)) = (I - \mathsf{C}^{(0,w)})(c_0(w)). \tag{4.8}$$

Proof. Clearly the left-side of (4.8) is contained in the right-side. Concerning the reverse inclusion, first observe that

$$(I - \mathsf{C}^{(0,w)})x = \left(0, x_2 - \frac{x_1 + x_2}{2}, x_3 - \frac{x_1 + x_2 + x_3}{3}, \ldots\right), \quad x = (x_n)_n \in c_0(w), \tag{4.9}$$

and, in particular, that

$$(I - \mathsf{C}^{(0,w)})y = \left(0, \frac{y_2}{2}, y_3 - \frac{y_2 + y_3}{3}, y_4 - \frac{y_2 + y_3 + y_4}{4}, \ldots\right), \tag{4.10}$$

for each $y = (0, y_2, y_3, \ldots) \in X_1(w)$.

Fix $x \in c_0(w)$. It follows from (4.9) that

$$x_j - \frac{1}{j}\sum_{k=1}^{j} x_k = \frac{1}{j}\left((j-1)x_j - \sum_{k=1}^{j-1} x_k\right), \quad j \geq 2, \tag{4.11}$$

is the j-th coordinate of $(I - \mathsf{C}^{(0,w)})x$. Set $y_i := x_i - x_1$ for $i \in \mathbb{N}$ and observe that the vector $y := (y_i)_i$ belongs to $X_1(w)$ because $\lim_{n\to\infty} w(n) = 0$ implies that $(0, 1, 1, \ldots) \in c_0(w)$; see Remark 4.2. Direct calculation (via (4.10)) reveals that the j-th coordinate of $(I - \mathsf{C}^{(0,w)})y$ is precisely (4.11) for $j \geq 2$. So, $(I - \mathsf{C}^{(0,w)})y = (I - \mathsf{C}^{(0,w)})x$ which establishes that the right-side of (4.8) is contained in the left-side. □

To proceed further we require some auxiliary operators. So, let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. Define the associated weight $\tilde{w} := (w(n+1))_{n=1}^{\infty}$ and the related left shift operator $S \colon X_1(w) \to c_0(\tilde{w})$ by

$$Sx := (x_2, x_3, x_4, \ldots), \quad x = (x_n)_n \in X_1(w), \tag{4.12}$$

in which case S is an isometric isomorphism of $X_1(w)$ onto $c_0(\tilde{w})$. The inverse transformation $S^{-1} \colon c_0(\tilde{w}) \to X_1(w)$ is the right shift, that is,

$$S^{-1}y := (0, y_1, y_2, y_3, \ldots), \quad y = (y_n)_n \in c_0(\tilde{w}). \tag{4.13}$$

Of course, the norm in $c_0(\tilde{w})$ is given by $\|y\|_{0,\tilde{w}} := \sup_{n\in\mathbb{N}} w(n+1)|y_n|$ for $y \in c_0(\tilde{w})$. For $y \in c_0(\tilde{w})$ it follows from (4.13) that

$$(S \circ (I - \mathsf{C}^{(0,w)})|_{X_1(w)} \circ S^{-1})y = (S \circ (I - \mathsf{C}^{(0,w)})|_{X_1(w)})(0, y_1, y_2, y_3, \ldots).$$

An application of this formula, (4.10) and direct calculation yields

$$(S \circ (I - \mathsf{C}^{(0,w)})|_{X_1(w)} \circ S^{-1})y = \left(\frac{1}{(n+1)} \left(ny_n - \sum_{k=1}^{n-1} y_k \right) \right)_{n\in\mathbb{N}},$$

with $y_0 := 0$. It follows that the lower triangular matrix $A := (a_{nm})_{n,m\in\mathbb{N}}$ of $(S \circ (I - \mathsf{C}^{(0,w)})|_{X_1(w)} \circ S^{-1}) \in \mathcal{L}(c_0(\tilde{w}))$ is given as follows: for each $n \in \mathbb{N}$,

$$a_{nm} := \begin{cases} 0 & \text{if } m > n \\ \frac{n}{(n+1)} & \text{if } m = n \\ -\frac{1}{(n+1)} & \text{if } 1 \le m < n. \end{cases} \tag{4.14}$$

Since S^{-1}, S and $(I - \mathsf{C}^{(0,w)})|_{X_1(w)}$ are all injective, so is the operator $(S \circ (I - \mathsf{C}^{(0,w)})|_{X_1(w)} \circ S^{-1}) \in \mathcal{L}(c_0(\tilde{w}))$ determined by A. It is clear from (4.14) that the operator $A\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is also injective. Moreover, considered in the space $\mathbb{C}^{\mathbb{N}}$, it is routine to check that A is also surjective and its inverse transformation $B\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is determined by the lower triangular matrix $B := (b_{nm})_{n,m\in\mathbb{N}}$ given as follows: for each fixed $n \in \mathbb{N}$,

$$b_{nm} = \begin{cases} 0 & \text{if } m > n \\ \frac{(n+1)}{n} & \text{if } m = n \\ \frac{1}{m} & \text{if } 1 \le m < n. \end{cases} \tag{4.15}$$

Recall the isometric isomorphism $\Phi_{\tilde{w}}\colon c_0(\tilde{w}) \to c_0$ given by (1.2), with $\tilde{w}$ in place of w, whose inverse $\Phi_{\tilde{w}}^{-1}\colon c_0 \to c_0(\tilde{w})$ is given by $\Phi_{\tilde{w}}^{-1}(z) := \left(\frac{z_n}{\tilde{w}(n)}\right)_{n\in\mathbb{N}} = \left(\frac{z_n}{w(n+1)}\right)_{n\in\mathbb{N}}$ for $z \in c_0$. Of course, both $\Phi_{\tilde{w}}$ and $\Phi_{\tilde{w}}^{-1}$ can be extended to isomorphisms between $\mathbb{C}^{\mathbb{N}}$; we denote these extensions by the same symbols as no confusion can occur. The lower triangular matrix D corresponding to the linear transformation $\Phi_{\tilde{w}} \circ B \circ \Phi_{\tilde{w}}^{-1}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is given by

$$D = \left(\frac{\tilde{w}(n) b_{nm}}{\tilde{w}(m)} \right)_{n,m\in\mathbb{N}} = \left(\frac{w(n+1) b_{nm}}{w(m+1)} \right)_{n,m\in\mathbb{N}}.$$

Clearly D maps c_0 continuously into c_0 if and only if B maps $c_0(\tilde{w})$ continuously into $c_0(\tilde{w})$.

In order to determine when $D \in \mathcal{L}(c_0)$, assume now that $\lim_{n\to\infty} w(n) = 0$ (equivalently, $\lim_{n\to\infty} \tilde{w}(n) = 0$). For fixed $m \in \mathbb{N}$ it follows from (4.15) that

$$\lim_{n\to\infty} \frac{w(n+1) b_{nm}}{w(m+1)} = \frac{1}{mw(m+1)} \lim_{n\to\infty} w(n+1) = 0. \tag{4.16}$$

On the other hand, for fixed $n \in \mathbb{N}$ it also follows from (4.15) that

$$\sum_{m=1}^{\infty} \frac{w(n+1)b_{nm}}{w(m+1)} = \frac{(n+1)}{n} + w(n+1)\sum_{m=1}^{n-1} \frac{1}{mw(m+1)} \leq 2 + w(n+1)\sum_{m=1}^{n-1} \frac{1}{mw(m+1)}.$$

Since also

$$w(n+1)\sum_{m=1}^{n-1} \frac{1}{mw(m+1)} \leq \sum_{m=1}^{\infty} \frac{w(n+1)b_{nm}}{w(m+1)}, \quad n \in \mathbb{N},$$

it is clear that

$$\sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} \frac{w(n+1)b_{nm}}{w(m+1)} < \infty \tag{4.17}$$

if and only if

$$\sup_{n\in\mathbb{N}} w(n+1)\sum_{m=1}^{n-1} \frac{1}{mw(m+1)} < \infty. \tag{4.18}$$

According to (4.16), it follows from Lemma 2.1 (with D in place of T) that $D \in \mathcal{L}(c_0)$ precisely when (4.17), equivalently (4.18), holds.

The previous discussion establishes most of the following result.

Lemma 4.6. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. Suppose that w satisfies both $\lim_{n\to\infty} w(n) = 0$ and the condition (4.18). Then the matrix B (given by (4.15)) maps $c_0(\tilde{w})$ continuously into $c_0(\tilde{w})$. Moreover, the matrix A (given by (4.14)) determines the Banach space isomorphism $(S\circ(I-\mathsf{C}^{(0,w)})|_{X_1(w)}\circ S^{-1}) \in \mathcal{L}(c_0(\tilde{w}))$, with B as its inverse. Furthermore,*

$$(I - \mathsf{C}^{(0,w)})(X_1(w)) = X_1(w). \tag{4.19}$$

Proof. It only remains to verify (4.19). Clearly, $(I-\mathsf{C}^{(0,w)})(X_1(w)) \subseteq X_1(w)$ and so it suffices to establish the reverse inclusion. To this effect, fix $y \in X_1(w)$. Then $Sy \in c_0(\tilde{w})$. Since $(S \circ (I - \mathsf{C}^{(0,w)})|_{X_1(w)} \circ S^{-1}) \in \mathcal{L}(c_0(\tilde{w}))$ is an isomorphism, there exists $z \in c_0(\tilde{w})$ such that

$$Sy = (S \circ (I - \mathsf{C}^{(0,w)})|_{X_1(w)} \circ S^{-1})z.$$

Via the injectivity of S it follows that $y = (I - \mathsf{C}^{(0,w)})|_{X_1(w)}x$ with $x := S^{-1}z \in X_1(w)$. That is, $y \in (I - \mathsf{C}^{(0,w)})(X_1(w))$ as required. □

The following result characterizes the uniform mean ergodicity of $\mathsf{C}^{(0,w)}$. It relies on a well known result of M.Lin, [20], stating that a Banach space operator $T \in \mathcal{L}(X)$ satisfying $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$ is uniformly mean ergodic if and only if $(I-T)(X)$ is a *closed* subspace of X.

Proposition 4.7. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ and $\lim_{n\to\infty} \frac{\|(\mathsf{C}^{(0,w)})^n\|}{n} = 0$. Then $\mathsf{C}^{(0,w)}$ is uniformly mean ergodic if and only if w satisfies both of the conditions*

(i) $\lim_{n\to\infty} w(n) = 0$, *and*
(ii) $\sup_{n\in\mathbb{N}} w(n+1)\sum_{m=1}^{n-1} \frac{1}{mw(m+1)} < \infty$.

Proof. Assume that (i) and (ii) are satisfied. According to Lemma 4.6 (see (4.19)) the subspace $(I - \mathsf{C}^{(0,w)})(X_1(w))$ is closed in $c_0(w)$ and hence, Lemma 4.5 yields that $(I - \mathsf{C}^{(0,w)})(c_0(w))$ is closed in $c_0(w)$. Since $\lim_{n\to\infty} \frac{\|(\mathsf{C}^{(0,w)})^n\|}{n} = 0$, the above mentioned result of Lin implies that $\mathsf{C}^{(0,w)}$ is uniformly mean ergodic.

Conversely, suppose that $\mathsf{C}^{(0,w)}$ is uniformly mean ergodic. Then it is surely Cesàro bounded and $\lim_{n\to\infty} \frac{\|(\mathsf{C}^{(0,w)})^n\|}{n} = 0$. Hence, Proposition 4.3 ensures that condition (i) is satisfied.

It remains to establish condition (ii). By the mentioned result of Lin the subspace $(I - \mathsf{C}^{(0,w)})(c_0(w))$ is closed in $c_0(w)$. Then Lemma 4.1 (cf. (4.4)) yields that $(I - \mathsf{C}^{(0,w)})(c_0(w)) = X_1(w)$. Since condition (i) holds, it is permissible to apply Lemma 4.5 to conclude that that $(I - \mathsf{C}^{(0,w)})(X_1(w)) = X_1(w)$ and hence, $(I - \mathsf{C}^{(0,w)})|_{X_1(w)} \in \mathcal{L}(X_1(w))$ is surjective. It was noted just prior to Lemma 4.5 that $(I - \mathsf{C}^{(0,w)})|_{X_1(w)}$ is also injective and hence, it is a Banach space isomorphism of $X_1(w)$ onto itself. In particular, there exists $L \in \mathcal{L}(X_1(w))$ satisfying $L \circ (I - \mathsf{C}^{(0,w)})|_{X_1(w)} = I_{X_1(w)} = (I - \mathsf{C}^{(0,w)})|_{X_1(w)} \circ L$. This implies that $A \in \mathcal{L}(c_0(\tilde{w}))$ is an isomorphism (as $A = S \circ (I - \mathsf{C}^{(0,w)})|_{X_1(w)} \circ S^{-1}$), and so there exists $R \in \mathcal{L}(c_0(\tilde{w}))$ with $R \circ A = I_{c_0(\tilde{w})} = A \circ R$. But, $B\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ satisfies $B \circ A = I_{\mathbb{C}^{\mathbb{N}}} = A \circ B$ and so the restriction $B|_{c_0(\tilde{w})}$ of B to $c_0(\tilde{w})$ must coincide with R, i.e., $B|_{c_0(\tilde{w})} \in \mathcal{L}(c_0(\tilde{w}))$. According to the discussion after (4.15) this is equivalent to the requirement that D maps c_0 continuously into c_0. In particular, (4.17) holds which is equivalent to (4.18). This is precisely the validity of condition (ii). □

Proposition 4.8. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$. The following statements are equivalent.*

(i) $\mathsf{C}^{(0,w)}$ *is power bounded.*
(ii) $\lim_{n\to\infty} \frac{\|(\mathsf{C}^{(0,w)})^n\|}{n} = 0$.
(iii) $\mathsf{C}^{(0,w)}$ *is uniformly mean ergodic.*

Proof. (i)⇒(ii). Obvious.

(ii)⇒(iii). The compactness of $\mathsf{C}^{(0,w)}$ ensures that $(I - \mathsf{C}^{(0,w)})(c_0(w))$ is closed in $c_0(w)$, [21, Lemma 3.4.20], [27, Theorem 5.5-D, p.279]. The above mentioned result of Lin then guarantees that $\mathsf{C}^{(0,w)}$ is uniformly mean ergodic.

(iii)⇒(i). Let $X_1 := \mathrm{Ker}(I - \mathsf{C}^{(0,w)})$ and $X_2 := (I - \mathsf{C}^{(0,w)})(c_0(w))$. As noted in (ii)⇒(iii), the space X_2 is closed in $c_0(w)$. Also, $X_1 = \mathrm{span}\{\mathbf{1}\}$ is a 1-dimensional subspace of $c_0(w)$. The hypothesis of (iii) yields that

$$c_0(w) = X_1 \oplus X_2.$$

Moreover, both X_1 and X_2 are invariant subspaces for $\mathsf{C}^{(0,w)}$. Let C_j denote the restriction of $\mathsf{C}^{(0,w)}$ to X_j, $j = 1, 2$. Theorem 2.12 of [12, p.49] ensures that $C_2 \in \mathcal{L}(X_2)$ is compact and hence, $\sigma(C_2) = \{0\} \cup \sigma_{pt}(C_2)$. But, $\sigma_{pt}(C_2) \subseteq \sigma_{pt}(\mathsf{C}^{(0,w)}) = \Sigma$, [12, Theorem 1.30(i), p.20]. If $1 \in \sigma_{pt}(C_2)$, then $1 \in \sigma_{pt}(\mathsf{C}; \mathbb{C}^{\mathbb{N}})$ and, since $\mathrm{Ker}(I - \mathsf{C}) = \mathrm{span}\{\mathbf{1}\}$ in $\mathbb{C}^{\mathbb{N}}$, it follows that $\mathbf{1} \in X_2$: a contradiction by Lemma 4.1(i). Accordingly, $1 \notin \sigma_{pt}(C_2)$ and so $\sigma_{pt}(C_2) \subseteq \{\frac{1}{m} \colon m \in \mathbb{N},\ m \geq 2\}$. This implies that the spectral radius $r(C_2) \leq \frac{1}{2} < 1$. Since also $\lim_{n\to\infty} \frac{\|(\mathsf{C}^{(0,w)})^n\|}{n} = 0$ (see (4.7)), we can conclude that $\mathsf{C}^{(0,w)}$ is power bounded, [24, Theorem 2.10]. □

Remark 4.9. (i) Suppose that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$. Then condition (i) of Proposition 4.7 is automatically satisfied, i.e., $\lim_{n\to\infty} w(n) = 0$; see Proposition 3.9. If, in addition, $\lim_{n\to\infty} \frac{\|(\mathsf{C}^{(0,w)})^n\|}{n} = 0$, then Proposition 4.8 implies that condition (ii) of Proposition 4.7 is also satisfied.

(ii) It is routine to check that condition (ii) of Proposition 4.7 is equivalent to the (more symmetric) condition

$$\sup_{n\in\mathbb{N}} w(n) \sum_{m=1}^{n} \frac{1}{mw(m)} < \infty. \tag{4.20}$$

Define $a_n := \frac{1}{w(n)}$ for $n \in \mathbb{N}$. Then (4.20) takes the equivalent form

$$\sup_{n\in\mathbb{N}} \frac{1}{a_n} \sum_{k=1}^{n} \frac{a_k}{k} < \infty. \tag{4.21}$$

Proposition 4.10. *Let w be a bounded, strictly positive sequence such that*

$$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} \in [0,1). \tag{4.22}$$

Then $\mathsf{C}^{(0,w)}$ is compact, power bounded and uniformly mean ergodic.

Proof. According to Proposition 2.7 the condition (4.22) implies that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$.

To establish the power boundedness of $\mathsf{C}^{(0,w)}$ recall (cf. the proof of Proposition 2.7) that $\{w(n)\}_{n\geq n_0}$ is decreasing (to 0). Let $0 < \alpha := \min\{w(k) \colon 1 \leq k \leq n_0\}$ and $\beta := \max\{w(k) \colon 1 \leq k \leq n_0\}$ in which case

$$0 < \frac{w(n)}{w(m)} \leq \frac{\beta}{\alpha}, \quad 1 \leq m,\ n \leq n_0.$$

Define $v(n) := \sum_{k=n}^{n_0} w(k)$ for $1 \leq n \leq n_0$ and $v(n) := w(n)$ for $n > n_0$. Then the strictly positive sequence $v \downarrow 0$ and satisfies $w(n) \leq v(n)$ for all $n \in \mathbb{N}$. Moreover, for each $n \in \{1, \ldots, n_0\}$ we have

$$\begin{aligned} v(n) &= \sum_{k=n}^{n_0} w(k) = w(n) + \sum_{k=n+1}^{n_0} w(n) \cdot \frac{w(k)}{w(n)} \\ &\leq w(n) \cdot \frac{\beta}{\alpha} + \sum_{k=n+1}^{n_0} w(n) \cdot \frac{\beta}{\alpha} \leq \frac{\beta n_0}{\alpha} w(n), \end{aligned}$$

that is, $\frac{\alpha}{\beta n_0} v(n) \leq w(n)$. Since $\frac{\alpha}{\beta n_0} \leq 1$, it is also the case that $\frac{\alpha}{\beta n_0} v(n) \leq v(n) = w(n)$ for $n > n_0$. Accordingly,

$$\frac{\alpha}{\beta n_0} v(n) \leq w(n) \leq v(n), \quad n \in \mathbb{N}. \tag{4.23}$$

Because v is decreasing, Remark 4.4(iii) yields that that $\|(\mathsf{C}^{(0,v)})^n\| = 1$ for all $n \in \mathbb{N}$. Then (4.23) ensures that the argument in the last paragraph of Remark 4.4(ii) can also be applied here to conclude that $\mathsf{C}^{(0,w)}$ is power bounded.

That $\mathsf{C}^{(0,w)}$ is also uniformly mean ergodic is now clear from Proposition 4.8. □

Remark 4.11. (i) Whenever a bounded, strictly positive sequence w has the property that $\{w(n)\}_{n\geq n_0}$ is decreasing for some $n_0 \in \mathbb{N}$, then $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ and $\mathsf{C}^{(0,w)}$ is power bounded. Indeed, if $n > n_0$, then

$$\begin{aligned}\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} &= \left(\frac{w(n)}{n}\sum_{k=1}^{n_0}\frac{1}{w(k)}\right) + \frac{w(n)}{n}\sum_{k=n_0+1}^{n}\frac{1}{w(k)} \\ &\leq \|w\|_\infty\left(\sum_{k=1}^{n_0}\frac{1}{w(k)}\right) + \frac{\|w\|_\infty}{n}\cdot\frac{n-n_0}{w(n_0+1)} \\ &\leq \|w\|_\infty\left(\frac{1}{w(n_0+1)} + \sum_{k=1}^{n_0}\frac{1}{w(k)}\right)\end{aligned}$$

which clearly implies (2.5) and hence, $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. The proof of Proposition 4.10 shows that (4.23) holds which, in turn, implies the power boundedness of $\mathsf{C}^{(0,w)}$.

The weight w in Example 3.12 shows that $\mathsf{C}^{(0,w)}$ can be power bounded even when $\{w(n)\}_{n\geq n_0}$ fails to be decreasing for *every* $n_0 \in \mathbb{N}$; see also part (ii)(c) of this Remark.

(ii)(a) The condition (4.22) is *not* necessary for $\mathsf{C}^{(0,w)}$ to be compact, power bounded and uniformly mean ergodic. Indeed, define $w(2n-1) = w(2n) = \frac{1}{2^{n-1}}$ for $n \in \mathbb{N}$ in which case $w \downarrow 0$. In particular (as seen before), $\mathsf{C}^{(0,w)}$ is power bounded.

To show that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$, set $A_m := \frac{w(m)}{m}\sum_{k=1}^{m}\frac{1}{w(k)}$ for $m \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we have

$$A_{2n} = \frac{w(2n)}{2n}\sum_{k=1}^{2n}\frac{1}{w(k)} = \frac{1}{2n2^{n-1}}\cdot 2\sum_{k=1}^{n}2^{k-1} = \frac{2(2^n-1)}{2n2^{n-1}} = \frac{2(2^n-1)}{n2^n}$$

and so $A_{2n} \leq \frac{2}{n}$. Moreover,

$$A_{2n+1} = \frac{w(2n+1)}{2n+1}\left(\frac{1}{w(2n+1)} + \sum_{k=1}^{2n}\frac{1}{w(k)}\right) = \frac{1}{(2n+1)2^n}(2^n + 2(2^n-1))$$

and so $A_{2n+1} \leq \frac{3}{2n+1} \leq \frac{2}{n}$. Hence, $(A_m)_{m\in\mathbb{N}} \in c_0$ and so indeed $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$; see Corollary 2.3(ii).

According to Proposition 4.8 the operator $\mathsf{C}^{(0,w)}$ is also uniformly mean ergodic.

Finally, observe that $\frac{w(n+1)}{w(n)} = 1$ for each odd $n \in \mathbb{N}$ and $\frac{w(n+1)}{w(n)} = \frac{1}{2}$ for each even $n \in \mathbb{N}$. It follows that $\limsup_{n\to\infty}\frac{w(n+1)}{w(n)} = 1$ and so (4.22) is *not* satisfied.

(b) The example in (a) can be modified. Fix $\alpha > 1$ and define $\tilde{w}$ via $\tilde{w}(2n) := \frac{\alpha}{2^{n-1}}$ and $\tilde{w}(2n-1) := \frac{1}{2^{n-1}}$ for $n \in \mathbb{N}$. A similar calculation as for w in (a) shows that

$$A_{2n} = \frac{(\alpha+1)(2^n-1)}{n2^n} \leq \frac{(\alpha+1)}{n}, \quad n \in \mathbb{N},$$

and also (using $\left(1+\frac{1}{\alpha}\right) \leq 2$) that

$$A_{2n+1} = \frac{\left(2^n + \left(1+\frac{1}{\alpha}\right)(2^n-1)\right)}{(2n+1)2^n} \leq \frac{3}{2n+1}, \quad n \in \mathbb{N}.$$

Hence, $(A_m)_{m\in\mathbb{N}} \in c_0$ and so $\mathsf{C}^{(0,\tilde{w})} \in \mathcal{K}(c_0(\tilde{w}))$. Observe that $\|e_1\|_{0,\tilde{w}} = 1$ and $\|\mathsf{C}^{(0,\tilde{w})}e_1\|_{0,\tilde{w}} = \max\{1, \frac{\alpha}{2}\}$. In particular, $\|\mathsf{C}^{(0,\tilde{w})}\| \geq \frac{\alpha}{2} > 1$ whenever $\alpha > 2$. Moreover, $\{\tilde{w}(n)\}_{n\geq n_0}$ fails to be decreasing for *every* $n_0 \in \mathbb{N}$. Nevertheless, $\tilde{w}$ satisfies

$$w(n) \leq \tilde{w}(n) \leq \alpha w(n), \quad n \in \mathbb{N},$$

with w as in (a) decreasing, and so the argument immediately after (4.23) can be applied to conclude that $\mathsf{C}^{(0,\tilde{w})}$ is power bounded. Then Proposition 4.8 shows that $\mathsf{C}^{(0,\tilde{w})}$ is also uniformly mean ergodic. Finally, observe that $\frac{\tilde{w}(n+1)}{\tilde{w}(n)} = \alpha$ for each odd $n \in \mathbb{N}$ and $\frac{\tilde{w}(n+1)}{\tilde{w}(n)} = \frac{1}{2\alpha} < \frac{1}{2}$ for each even $n \in \mathbb{N}$ and hence, $\limsup_{n\to\infty} \frac{\tilde{w}(n+1)}{\tilde{w}(n)} = \alpha > 1$. In particular, (4.22) again *fails* to hold.

This example shows there exist weights w such that $\|\mathsf{C}^{(0,w)}\|$ can be larger than any *a priori* given positive number and yet $\mathsf{C}^{(0,w)}$ is power bounded.

(c) If $\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} = \infty$, then "all possibilities" may occur. Fix $\alpha > 0$ and define $w(2n) = \frac{n^\alpha}{2^n}$ and $w(2n-1) = \frac{1}{2^n}$ for $n \in \mathbb{N}$. It is routine to check that $w \in s$ but w is *not* decreasing. Actually, there exist *no* decreasing sequence v and constants $A,\ B > 0$ such that

$$Av(n) \leq w(n) \leq Bv(n), \quad n \in \mathbb{N}.$$

For, if so, then these inequalities and the fact that v is decreasing yield

$$Av(2n) \leq \frac{n^\alpha}{2^n} \leq Bv(2n) \leq Bv(2n-1) \leq \frac{B}{A}w(2n-1) = \frac{B}{A}\cdot\frac{1}{2^n},$$

for every $n \in \mathbb{N}$ which is clearly impossible as $\alpha > 0$.

It is possible to verify that $S_w = \emptyset$ and $R_w = \mathbb{R}$.

Since $\frac{w(2n)}{w(2n-1)} = n^\alpha$ for $n \geq 2$, it is clear that $\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} = \infty$. On the other hand, $\frac{w(2n+1)}{w(2n)} = \frac{1}{2n^\alpha} \to 0$ for $n \to \infty$ and so the sequence $\left\{\frac{w(n+1)}{w(n)}\right\}_{n\in\mathbb{N}}$ is not convergent in $[0,\infty]$.

Set $A_m := \frac{w(m)}{m}\sum_{k=1}^m \frac{1}{w(k)}$ for $m \in \mathbb{N}$. Then

$$A_{2n} = \frac{1}{2n}\cdot\frac{n^\alpha}{2^n}\left(\sum_{k=1}^n 2^k + \sum_{k=1}^n \frac{2^k}{k^\alpha}\right), \quad n \in \mathbb{N}, \tag{4.24}$$

and hence,

$$A_{2n} \leq \frac{1}{n^{1-\alpha}}\cdot\frac{1}{2^{n+1}}\cdot 2\sum_{k=1}^n 2^k = \frac{2(2^{n+1}-2)}{n^{1-\alpha}2^{n+1}} \leq \frac{2}{n^{1-\alpha}}, \tag{4.25}$$

for $n \in \mathbb{N}$. On the other hand, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} A_{2n+1} &= \frac{1}{(2n+1)}\cdot\frac{1}{2^{n+1}}\left(2^{n+1} + \sum_{k=1}^n 2^k + \sum_{k=1}^n \frac{2^k}{k^\alpha}\right) \\ &\leq \frac{1}{(2n+1)2^{n+1}}\left(2^{n+1} + 2(2^{n+1}-2)\right) \leq \frac{3}{(2n+1)}. \end{aligned} \tag{4.26}$$

We now consider three cases.

Case (i). $0 < \alpha < 1$.

It is clear from (4.25) and (4.26) that $\{A_n\}_{n\in\mathbb{N}} \in c_0$ and so $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$.

Since $A_1 = 1$, it follows from (4.26) that $A_m \leq 1$ for every odd integer $m \in \mathbb{N}$. Moreover, $A_2 = \frac{w(2)}{2}\left(\frac{1}{w(1)} + \frac{1}{w(2)}\right) = 1$ and $A_4 = \frac{w(4)}{4}\sum_{k=1}^{4}\frac{1}{w(k)} = \frac{1}{4} + \frac{2^\alpha}{2}$. Accordingly, $A_4 \leq 1$ precisely when $2^\alpha \leq \frac{3}{2}$, i.e., when $\alpha \leq \left(\frac{\log(3)}{\log(2)} - 1\right)$. Moreover, $\frac{2}{n^{1-\alpha}} \leq 1$ if and only if $\log(n) \geq \frac{\log(2)}{1-\alpha}$ which is true for all $n \geq 3$ provided α satisfies $\log(3) \geq \frac{\log(2)}{1-\alpha}$, that is, $\alpha \leq \left(1 - \frac{\log(2)}{\log(3)}\right)$. It follows from (4.25) that $A_{2n} \leq 1$ for all $n \geq 3$ provided that $\alpha \leq \left(1 - \frac{\log(2)}{\log(3)}\right)$. Since $\left(1 - \frac{\log(2)}{\log(3)}\right) < \left(\frac{\log(3)}{\log(2)} - 1\right)$ we can conclude that $A_m \leq 1$ for every even integer $m \in \mathbb{N}$ *provided* α satisfies

$$0 < \alpha \leq \left(1 - \frac{\log(2)}{\log(3)}\right). \tag{4.27}$$

Accordingly, $\|\mathsf{C}^{(0,w)}\| = 1$ whenever α satisfies (4.27), that is, $\mathsf{C}^{(0,w)}$ is power bounded. Proposition 4.8 ensures that $\mathsf{C}^{(0,w)}$ is then also uniformly mean ergodic.

Case (ii). $\alpha = 1$.

It follows from (4.25) and (4.26) that $\{A_n\}_{n\in\mathbb{N}} \in \ell_\infty$ and so $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. However, (4.24) with $\alpha = 1$ yields that

$$A_{2n} \geq \frac{1}{2^{n+1}}\sum_{k=1}^{n} 2^k = \frac{(2^{n+1}-2)}{2^{n+1}} \geq \frac{1}{2}, \quad n \in \mathbb{N}.$$

Hence, $\{A_m\}_{m\in\mathbb{N}} \notin c_0$ and so $\mathsf{C}^{(0,w)}$ is *not* compact. Nevertheless, as already noted, $S_w = \emptyset$. It follows from (4.26) that $A_{2n+1} \to 0$ for $n \to \infty$. Combined with $A_{2n} \geq \frac{1}{2}$, for $n \in \mathbb{N}$, we see that the limit u in condition (iii) (immediately after Example 3.12) fails to exist.

Case (iii). $\alpha > 1$.

It is immediate from (4.24) that

$$A_{2n} \geq \frac{n^{\alpha-1}}{2^{n+1}}\sum_{k=1}^{n} 2^k = \frac{n^{\alpha-1}(2^{n+1}-2)}{2^{n+1}} \geq \frac{n^{\alpha-1}}{2}, \quad n \in \mathbb{N}.$$

Accordingly, $\{A_m\}_{m\in\mathbb{N}} \notin \ell_\infty$ and so C does not act continuously in $c_0(w)$, even though $w \in s$.

Example 4.12. (i) Suppose that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. Then $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$ if and only if (2.6) is satisfied, i.e., $\lim_{n\to\infty}\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} = 0$; see Corollary 2.3(ii). It can happen that

$$\lim_{n\to\infty}\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} = u \tag{4.28}$$

exists but $u \neq 0$. Indeed, fix $\beta \geq 0$, $\gamma \geq 0$ and let $w(n) = \frac{1}{n^\beta \log^\gamma(n+1)}$ for $n \in \mathbb{N}$. It was shown in Remark 3.13 that $u = \frac{1}{1+\beta}$ (cf. (4.28)). In particular, $\mathsf{C}^{(0,w)}$ *fails* to be compact.

Since $w \downarrow 0$, Remark 4.9(ii) shows that $\mathsf{C}^{(0,w)}$ *is* mean ergodic. Consider now the case when $\beta = 0$ and $\gamma \geq 1$. The claim is that $\mathsf{C}^{(0,w)}$ *fails* to be uniformly mean ergodic. To establish this it suffices, via (4.21) and condition (ii) of Proposition 4.7, to show that

$$\lim_{n\to\infty} \frac{1}{a_n}\sum_{k=1}^{n}\frac{a_k}{k} = \infty, \tag{4.29}$$

where (in the notation of Remark 4.9(ii)) $a_n = \log^{\gamma}(n+1)$ for $n \in \mathbb{N}$. Direct calculations show that

$$\frac{a_n}{n(a_n - a_{n-1})} = \frac{\log^{\gamma}(n+1)}{n(\log^{\gamma}(n+1) - \log^{\gamma}(n))} = \frac{d_n}{n\gamma}\left(\frac{\log(n+1)}{\log(d_n)}\right)^{\gamma-1}\log(n+1),$$

for some $d_n \in (n,(n+1))$ and each $n \geq 2$. Since

$$0 < \sup_{n\in\mathbb{N}} \frac{d_n}{n\gamma}\left(\frac{\log(n+1)}{\log(d_n)}\right)^{\gamma-1} < \infty,$$

it follows from the previous equality that $\lim_{n\to\infty}\frac{a_n}{n(a_n-a_{n-1})} = \infty$. By the Stolz-Cesàro criterion the identity (4.29) follows.

(ii) There exist weights $w \downarrow 0$ such that $\mathsf{C}^{(0,w)}$ is uniformly mean ergodic but $\mathsf{C}^{(0,w)} \notin \mathcal{K}(c_0(w))$. Indeed, fix $\alpha \geq 1$ and set $w(n) := \frac{1}{n^{\alpha}}$ for $n \in \mathbb{N}$. Again in the notation of Remark 4.9(ii) we have $a_n = n^{\alpha}$ for $n \in \mathbb{N}$. Moreover,

$$\frac{1}{a_n}\sum_{k=1}^{n}\frac{a_k}{k} = \frac{1}{n^{\alpha}}\sum_{k=1}^{n}k^{\alpha-1} \simeq \frac{1}{n^{\alpha}}\cdot\frac{n^{\alpha}}{\alpha}, \quad n \in \mathbb{N},$$

and so (4.21) is satisfied. Since $w \downarrow 0$, the operator $\mathsf{C}^{(0,w)}$ is power bounded (via Remark 4.11(i)) and hence, $\lim_{n\to\infty}\frac{\|(\mathsf{C}^{(0,w)})^n\|}{n} = 0$. The uniform mean ergodicity of $\mathsf{C}^{(0,w)}$ then follows from Proposition 4.7. On the other hand, $\mathsf{C}^{(0,w)}$ *fails* to be compact; see Remark 2.4(ii).

To end this section recall that a Banach space operator $T \in \mathcal{L}(X)$, with X separable, is called *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X. If, for some $z \in X$, the projective orbit $\{\lambda T^n z \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Clearly, hypercyclicity always implies supercyclicity.

Proposition 4.13. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. Then $\mathsf{C}^{(0,w)}$ is not supercyclic and hence, also not hypercyclic.*

Proof. According to Step 2 in the proof of Theorem 3.4 the *infinite* set $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(0,w)})')$. Then, by a result of Ansari and Bourdon [6, Theorem 3.2], $\mathsf{C}^{(0,w)}$ is *not supercyclic.* □

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