A note on the spectrum of composition operators on spaces of real analytic functions

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Abstract

In this paper the spectrum of composition operators on the space of real analytic functions is investigated. In some cases it is completely determined while in some other cases it is only estimated.

1 Introduction

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-constant real analytic map and let $\mathscr{A}(\mathbb{R})$ be the space of real analytic functions defined on $\mathbb{R}$. Each symbol $\varphi : \mathbb{R} \to \mathbb{R}$ defines a composition operator $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ by $C_\varphi(f) := f \circ \varphi, f \in \mathscr{A}(\mathbb{R})$. When $\mathscr{A}(\mathbb{R})$ is endowed with its natural locally convex topology (see e.g. [9]), C_φ is a continuous linear operator on $\mathscr{A}(\mathbb{R})$. In our article [7] we studied the eigenvalues and eigenvectors of composition operators $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$. In this note we complement those results with some examples and remarks concerning the spectrum of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.

Our results in [7] give precise information about the injectivity of the operator $C_\varphi - \lambda I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R}), \lambda \in \mathbb{C} \setminus \{0\}$. In order to investigate the spectrum $\sigma(C_\varphi)$, we have to study when the operator $C_\varphi - \lambda I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is surjective. This operator is surjective if and only if the operator $T_\lambda : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$, defined by $T_\lambda f(x) := f(x) - (1/\lambda)f(\varphi(x)), x \in \mathbb{R}, f \in \mathscr{A}(\mathbb{R})$, is surjective. If we set $g(x,y) := (1/\lambda)y + \gamma(x)$, for $\gamma \in \mathscr{A}(\mathbb{R})$, then $T_\lambda : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is surjective if and only if for each $\gamma \in \mathscr{A}(\mathbb{R})$ there is $f \in \mathscr{A}(\mathbb{R})$ such that $f(x) = g(x, f(\varphi(x)))$, $x \in \mathbb{R}$. We use methods of the paper [4].

Equations of the form $f(x) = g(x, f(\varphi(x))$ with $g : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ real analytic are considered in [4] and [17] (see also [1], [2] or the book [3]). As it is mentioned in the introduction of [4], if u is a fixed point of φ and $\lambda \neq \varphi'(u)^n$ for each $n \in \mathbb{N}_0$, then $f(x) = (1/\lambda)f(\varphi(x)) + \gamma(x)$ has a formal solution $f(x) = u + \sum_{n=1}^{\infty} c_n(x-u)^n$. Smajdor [17] studies conditions to ensure that this series representation has a positive radius of convergence near u.

In the rest of the article we denote $\mathrm{id}\,(x) = x, x \in \mathbb{R}$, and, for a map $\varphi : \mathbb{R} \to \mathbb{R}$, we write $\varphi^{[0]} = \mathrm{id}$ and $\varphi^{[n]}$ for the n-times composition of $\varphi, n \in \mathbb{N}$. By $I : \mathscr{A}(J) \to \mathscr{A}(J)$ we denote the identity operator.

The composition operators on spaces of real analytic functions have been considered in several papers, like [9], [10], [11], [12], [13], [5], [6]. For literature on the space of real analytic functions see a recent survey [9]. For functional analytic tools see [16].

2 Preliminaries

By interval J we mean also unbounded ones (i.e., halflines or the whole real line). It is clear that results on the (point) spectrum of a composition operator on $\mathscr{A}(\mathbb{R})$ can be easily transferred to the case of $\mathscr{A}(J)$ where J is an open interval in $\mathbb{R}$. We recall the following two results from [7].

Proposition 2.1 *Let $\varphi : J \to J$ be a real analytic function on an open interval $J \subset \mathbb{R}$ and let $C_\varphi : \mathscr{A}(J) \to \mathscr{A}(J)$ be the associated composition operator. Then*

(1) ([7, Proposition 1.1 (4)]) $0 \in \sigma(C_\varphi)$ if and only if $\varphi : J \to J$ is not a real analytic diffeomorphism.

(2) ([7, Proposition 2.8 (4)]) If $u \in J$ is a fixed point of φ such that $|\varphi'(u)| \neq 0, 1$, then $\varphi'(u)^n \in \sigma(C_\varphi)$ for each $n \in \mathbb{N}_0$.

We say that some subset A of an open interval $J \subset \mathbb{R}$ is *bounded away from the upper end of J* if there is $\delta \in J$ such that all elements of A are $< \delta$. Analogously we define subsets *bounded away form the lower end of J*. We also use the following description of the point spectrum of C_φ from [7]:

Theorem 2.2 *Let $\varphi : J \to J$ be a real analytic map for some open interval $J \subset \mathbb{R}$.*

(a) If φ has no fixed points and the set of its critical points is bounded away from the upper end of J (in case $\varphi >$ id) or from the lower end of J (in case $\varphi <$ id) then $\sigma_p(C_\varphi) = \mathbb{C} \setminus \{0\}$ and every eigenspace is isomorphic to $\mathscr{A}(\mathbb{T})$.

(b) If φ has a fixed point u then:

(b1) if $\varphi^{[2]}$ has exactly one fixed point and either $0 < |\varphi'(u)| < 1$ or $1 < |\varphi'(u)|$ and φ has no critical points then

$$\sigma_p(C_\varphi) = \{(\varphi'(u))^n : n = 0, 1, \dots\}$$

and the eigenspaces are all one dimensional;

(b2) if $\varphi^{[2]} =$ id $\neq \varphi$ then $\sigma_p(C_\varphi) = \{-1, 1\}$ and the eigenspaces are isomorphic to $\mathscr{A}_+(\mathbb{R})$ the space of even real analytic functions;

(b3) if $\varphi =$ id then $\sigma_p(C_\varphi) = \{1\}$ and the eigenspace is equal to $\mathscr{A}(J)$.

(c) In all other cases $\sigma_p(C_\varphi) = \{1\}$ and the eigenspace consists of constant functions only.

We will use also the following result of Smajdor [17, Theorem p. 40]:

Theorem 2.3 *Let φ be a holomorphic function of one complex variable on a neighbourhood of $u \in \mathbb{C}$, $\varphi(u) = u$, $|\varphi'(u)| < 1$. Let h be a holomorphic function of two complex variables on a neighbourhood of $(u, v) \in \mathbb{C}^2$, $h(u, v) = v$. If there is a formal solution $f(z) = \sum_n f_n(z - u)^n$ of the equation*

$$(1) \qquad\qquad f(z) = h(z, f(\varphi(z))), \qquad f(u) = v,$$

then there is a holomorphic solution of (1) on some neighbourhood of u.
In case

$$(2) \qquad\qquad 1 - \frac{\partial h}{\partial v}(u, v) \cdot (\varphi'(u))^n \neq 0 \qquad \text{for every } n = 0, 1, \dots$$

the formal solution is unique and so it is convergent around u and gives a holomorphic solution of (1) on some neighbourhood of u.

Please note that in the proof of Proposition 4.1 we give a simple example where for $|\varphi'(u)| = 1$ the result above fails.

3 Spectrum

The case of $\varphi : J \to J$ without fixed points is mostly solved.

Corollary 3.1 *Let $\varphi : J \to J$ be real analytic, $J \subset \mathbb{R}$ an open interval, and φ have no fixed points.*

(a) *If φ is a diffeomorphism onto J then $\sigma(C_\varphi) = \sigma_p(C_\varphi) = \mathbb{C} \setminus \{0\}$.*

(b) *If φ is not a diffeomorphism onto J but the set of critical points of φ is bounded away from the upper end of J (in case $\varphi >$ id) or from lower end of J (in case $\varphi <$ id) then $\sigma(C_\varphi) = \mathbb{C}$ and $\sigma_p(C_\varphi) = \mathbb{C} \setminus \{0\}$.*

Proof: Combine Proposition 2.1 and Theorem 2.2. □

Problem 3.2 *Describe $\sigma(C_\varphi)$ if $\varphi > \mathrm{id}$ has no fixed points but the set of critical points of φ is not bounded away from the upper end of the domain.*

Now, we concentrate on the fixed point case. We will need the following standard extension lemma (this procedure for diffeomorphisms was used in [4], we write it precisely in the general case).

Lemma 3.3 *Let $\varphi : (a,b) \to (a,b)$ be a real analytic map on an open interval $(a,b) \subset \mathbb{R}$ and let $\varphi(c,d) \subset (c,d) \subset [c,d] \subset (a,b)$ for some fixed a,b,c,d. Assume that for every $x \in (a,b)$ there is $n \in \mathbb{N}$ such that $\varphi^{[n]}(x) \in (c,d)$ (i.e., (a,b) is the attraction basin for (c,d)).*
If $\tilde{f} \in \mathscr{A}(c,d)$ satisfies

$$(3) \qquad \tilde{f}(\varphi(x)) - \lambda \tilde{f}(x) = \gamma(x) \qquad \text{for every } x \in (c,d),$$

for some fixed $\lambda \in \mathbb{C} \setminus \{0\}$ and $\gamma \in \mathscr{A}(a,b)$, then $\tilde{f}$ extends to $f \in \mathscr{A}(a,b)$ satisfying

$$(4) \qquad f(\varphi(x)) - \lambda f(x) = \gamma(x) \qquad \text{for every } x \in (a,b).$$

Proof: Define
$$J_n := \{x \in (a,b) \mid \varphi^{[n]}(x) \in (c,d)\}.$$

Clearly, (J_n) is an increasing open exhaustion of (a,b). Take any compact increasing exhaustion $([a_n, b_n])_{n \in \mathbb{N}}$ with
$$(c,d) \subset [a_n, b_n] \subset (a,b) \quad \text{and} \quad \bigcup_{n \in \mathbb{N}} [a_n, b_n] = (a,b).$$

Without loss of generality we may assume that $J_n \supset [a_n, b_n]$.

We define inductively:
$$f_0 := \tilde{f}, \qquad f_n(x) := \frac{1}{\lambda}\left(\gamma(x) + f_{n-1}(\varphi(x))\right) \qquad \text{for } x \in J_n.$$

By (3) it is easy to observe that $f_n = \tilde{f}$ on (c,d) and thus f_n extends $\tilde{f}$ on $[a_n, b_n]$. Now, defining $f(x) := f_n(x)$ for $x \in [a_n, b_n]$ we obtain a real analytic function $f \in \mathscr{A}(a,b)$. Since
$$f(\varphi(x)) - \lambda f(x) = \gamma(x) \qquad \text{for every } x \in (c,d)$$

and since both sides of the above equality are analytic on (a,b) so the equality holds everywhere on (a,b). $\qquad \square$

Note that if $\varphi^{[2]}$ has the unique fixed point u then u must be the unique fixed point of φ as well since otherwise $\varphi(u) = w \neq u$ but then $\varphi^{[2]}(w) = w$ and $\varphi^{[2]}$ would have had two fixed points u and w.

Theorem 3.4 *Let $\varphi : J \to J$ be a real analytic function on an open interval $J \subset \mathbb{R}$ and let $C_\varphi : \mathscr{A}(J) \to \mathscr{A}(J)$ the associated composition operator. Suppose that $\varphi^{[2]}$ has a unique fixed point u and $|\varphi'(u)| < 1$. Then*
(i) If $\lambda \neq 0$ and $\lambda \neq \varphi'(u)^n$ for each $n \in \mathbb{N}_0$, then $\lambda \notin \sigma(C_\varphi)$, and
(ii) $\ker(C_\varphi - \lambda I)$ is finite dimensional for all $\lambda \in \mathbb{C}$.

Remark 3.5 *The case of φ diffeomorphic onto is proved in [4, Th. 4.4].*

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Proof: (i): Consider $\lambda \neq 0$, $\lambda \neq \varphi'(u)^n$, $n \in \mathbb{N}_0$. By Theorem 2.2, $C_\varphi - \lambda I$ is injective, we will show that it is surjective.

It suffices to solve, for every $\gamma \in \mathscr{A}(J)$, the equation

$$f(x) - (1/\lambda)f(\varphi(x)) = \gamma(x),$$

which is equivalent to

$$f(x) = g(x, f(\varphi(x))),$$

with $g(x, y) = \gamma(x) + (1/\lambda)y$. In order to apply Theorem 2.3 to the equation above, we define $v := \lambda\gamma(u)/(\lambda - 1)$. Clearly, $g(u, v) = v$. Since

$$\frac{\partial g}{\partial v}(u, v) = \frac{1}{\lambda},$$

the condition (2) is satisfied.

Therefore all the assumptions of Theorem 2.3 are satisfied and the unique formal solution f_0 is real analytic on a neighbourhood of u. Hence there is $r > 0$ such that $[u - r, u + r] \subset J$ and $f_0 \in \mathscr{A}(u - r, u + r)$ satisfies

$$f_0(x) - 1/\lambda f(\varphi(x)) = \gamma(x) \qquad \text{for every } x \in (u - r, u + r).$$

Since $|\varphi'(u)| < 1$, we may assume without loss generality that $\varphi(]u - r, u + r[) \subset\,]u - r, u + r[$. In the proof of [7, Theorem 2.6] it is proved that if $\varphi : J \to J$, $J \subset \mathbb{R}$ an open interval, has a fixed point u, $|\varphi'(u)| < 1$, and $\varphi^{[2]}$ has exactly one fixed point, then for every $x \in J$ holds $\varphi^{[n]}(x) \to u$ as $n \to \infty$. Thus $(c, d) = (u - r, u + r)$ satisfies the assumptions of Lemma 3.3. This completes the proof.

(ii): This follows from results of [7, Section 2] transferring them to an arbitrary interval instead of the whole line. For readers convenience we sketch the proof. By assumption $\varphi^{[2]}$ has a unique fixed point in J and we can aply [7, Theorem 2.6] to conclude that $\varphi'(u)^n$ is an eigenvalue of C_φ for each $n \in \mathbb{N}_0$. By [7, Theorem 2.9 (c) and (d)], $\ker(C_\varphi - \lambda I)$ is one dimensional for $\lambda = 0$ (if $\varphi'(u) = 0$) or $\lambda = \varphi'(u)^n$ (if $0 < |\varphi'(u)| < 1$). We can apply [7, Propositions 1.1 and 2.4] to conclude that $\ker(C_\varphi - \lambda I) = \{0\}$ for $\lambda \neq \varphi'(u)^n$, $n \in \mathbb{N}_0$. $\square$

Corollary 3.6 *If $\varphi : J \to J$ is a real analytic on an open interval $J \subset \mathbb{R}$, such that $\varphi^{[2]}$ has the unique fixed point u satisfying $|\varphi'(u)| < 1$, then*

$$\sigma(C_\varphi) = \begin{cases} \{\varphi'(u)^n \mid n \in \mathbb{N}_0\} & \text{if } \varphi \text{ is a diffeomorphism,} \\ \{\varphi'(u)^n \mid n \in \mathbb{N}_0\} \cup \{0\} & \text{otherwise} \end{cases}$$

and

$$\sigma_p(C_\varphi) = \{\varphi'(u)^n \mid n \in \mathbb{N}_0\} \setminus \{0\}$$

Moreover, $\ker(C_\varphi - \lambda I)$ is finite dimensional for all $\lambda \in \mathbb{C}$.

Proof: The description of $\sigma_p(C_\varphi)$ follows from Theorem 2.2. The description of $\sigma(C_\varphi)$ follows combining of Proposition 2.1 and Theorem 3.4. $\square$

Corollary 3.7 *If $\varphi : J \to J$ is a real analytic diffeomorphism on an open interval $J \subset \mathbb{R}$ such that $\varphi^{[2]}$ has a unique fixed point u satisfying $|\varphi'(u)| > 1$, then $\sigma(C_\varphi) = \sigma_p(C_\varphi) = \{\varphi'(u)^n \mid n \in \mathbb{N}_0\}$ and $\ker(C_\varphi - \lambda I)$ is finite dimensional for all $\lambda \in \mathbb{C}$.*

Proof: The map $\psi := \varphi^{-1} : J \to J$ is a real analytic diffeomorphism with a unique fixed point $u \in J$ and $0 < |\psi'(u)| < 1$. The conclusion follows from Corollary 3.6 and the following observation: for $\mu \neq 0$, $\mu \neq \psi'(u)^n = 1/\varphi'(u)^n, n \in \mathbb{N}_0$, the operator $C_\psi - \mu I : \mathscr{A}(J) \to \mathscr{A}(J)$ is surjective/injective if and only if $C_\varphi - (1/\mu)I : \mathscr{A}(J) \to \mathscr{A}(J)$ is surjective/injective. $\qquad\square$

Problem 3.8 *Describe $\sigma(C_\varphi)$ if φ has a fixed point u, $|\varphi'(u)| > 1$, $\varphi^{[2]}$ has a unique fixed point but φ is not a diffeomorphism.*

By [4, Theorem 4.5] we obtain immediately:

Proposition 3.9 *Let $\varphi : j \to J$, $J \subset \mathbb{R}$ open interval, be a real analytic diffeomorphism with fixed points $x_1 < x_2 < ... < x_s$ such that $|\varphi'(x_i)| \neq 1$ for $i = 1, 2, ..., s$, $s > 1$. For each $\lambda \neq 0$, the operator $C_\varphi - \lambda I : \mathscr{A}(J) \to \mathscr{A}(J)$ has closed range, $\ker C_\varphi - \lambda I = \{0\}$ if $\lambda \neq 1$, $\ker C_\varphi - I$ is finite dimensional and the codimension of $\operatorname{im} C_\varphi - \lambda I$ in $\mathscr{A}(J)$ is infinite. In particular, $\sigma_p(C_\varphi) = \{1\}$ and $\sigma(C_\varphi) = \mathbb{C} \setminus \{0\}$.*

Proof: By Proposition 2.1 (1), $0 \notin \sigma(C_\varphi)$ and by Theorem 2.2, $\sigma_p(C_\varphi) = \{1\}$, and $\ker(C_\varphi - I)$ is one dimensional. The rest of the statement is a direct consequence of [4, Theorem 4.5]. $\quad\square$

Example 3.10 (a) If $\varphi(x) = e^{\alpha x}, \alpha > 1/e$, then $\sigma_p(C_\varphi) = \mathbb{C} \setminus \{0\}$ and $\sigma(C_\varphi) = \mathbb{C}$ (use Corollary 3.1).

(b) If φ has no fixed points on $\mathbb{R}$ and φ has only finitely many critical points, then $\sigma_p(C_\varphi) = \mathbb{C} \setminus \{0\}$ and $\sigma(C_\varphi) = \mathbb{C}$ by Corollary 3.1.

(c) If $\varphi(x) = \alpha \arctan(x), 0 < \alpha < 1$, then $\sigma_p(C_\varphi) = \{\alpha^n \mid n \in \mathbb{N}_0\}$ and $\sigma(C_\varphi) = \sigma_p(C_\varphi) \cup \{0\}$. In this case φ has a unique fixed point $u = 0$ and $\varphi'(u) = \alpha$. We have $0 < |\varphi'(u)| < 1$ and φ is not surjective, hence the statement follows from Corollary 3.6.

Before we deal with more examples, we present for the reader's convenience the main result in [4, Section 3] in a way suitable for our purposes explaining details omitted in [4].

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic self map, let $\lambda \in \mathbb{C}, \lambda \neq 0$, and consider the operator $T : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R}), Tf(x) := f(\varphi(x)) - \lambda f(x)$. Let $(U_j)_j$ be an open covering of $\mathbb{R}$ such that $\varphi(U_j) \subset U_j$ for each j and such that each U_j intersects only finitely many other U_k's. Denote by $T_k : \mathscr{A}(U_k) \to \mathscr{A}(U_k)$ and $T_{k,j} : \mathscr{A}(U_k \cap U_j) \to \mathscr{A}(U_k \cap U_j)$ the operator T defined on the corresponding space. With the notation $g|V$ for the restriction of the function g to the set V, define

$$\mathscr{A}(\mathbb{R}, (U_k), T) := \{\gamma \in \mathscr{A}(\mathbb{R}) \ : \ \gamma|U_k \in \operatorname{im}(T_k) \ \forall k\},$$
$$E := \{(f_{k,j})_{k,j} \ : \ f_{k,j} = -f_{j,k}, \ \ f_{k,j} \in \ker(T_{k,j})\},$$
$$F := \{(f_k)_k \ : \ f_k \in \ker(T_k) \ \forall k\},$$

and

$$S : F \to E, \quad S((f_k)_k) := ((f_k - f_j)|U_k \cap U_j)_{k,j}.$$

Theorem 3.11 *([4, Theorem 3.1]) There is a linear surjective map*

$$\mathcal{F} : \mathscr{A}(\mathbb{R}, (U_k), T) \to E/S(F)$$

with kernel equal to $\operatorname{im}(T)$.

In particular, if $S(F) \neq E$, then T is not surjective.

Proof: Given $\gamma \in \mathscr{A}(\mathbb{R}, (U_k), T)$, for each k there is $f_k \in \mathscr{A}(U_k)$ such that $T_k f_k = \gamma|U_k$. Set $f_{k,j} := (f_k - f_j)|U_k \cap U_j$. Clearly $T_{k,j} f_{k,j}(x) = 0$ for each $x \in U_k \cap U_j$ and each k, j. Hence $(f_{k,j})_{k,j} \in E$. We define $\mathcal{F}(\gamma) = (f_{k,j})_{k,j} + S(F)$. To see that $\mathcal{F}$ is well defined, suppose that $T_k g_k = \gamma|U_k$ with $g_k \in \mathscr{A}(U_k)$ and set $g_{k,j} := (g_k - g_j)|U_k \cap U_j$. We have $(f_{k,j} - g_{k,j})_{k,j} = S((f_k - g_k)_k)$ and $T_k(f_k - g_k) = 0$ for each k. Therefore $(f_{k,j})_{k,j} - (g_{k,j})_{k,j} \in S(F)$ and $\mathcal{F}$ is indeed well defined.

We show now that $\mathcal{F}$ is surjective. Fix $(f_{k,j})_{k,j} + S(F) \in E/S(F)$. Our assumptions on the covering $(U_k)_k$ of $\mathbb{R}$ permit us to find open sets V_k in the complex plane with $V_k \cap \mathbb{R} = U_k$ and extensions of the functions $f_{k,j}$ to holomorphic functions $g_{k,j}$ on $V_k \cap V_j$.

Indeed, let $f_{k,j} \in \mathscr{A}(U_k \cap U_j)$ then there is an open complex neighbourhood W of $U_k \cap U_j$, $W \cap \mathbb{R} = U_k \cap U_j$, such that $f_{k,j}$ extends holomorphically as a function $g_{k,j}$ on W. We define:

$$\tau_j(x) := \begin{cases} \min(d(x, \partial_\mathbb{R} U_j), d(x, \partial_\mathbb{C} W)) & \text{if } x \in U_k \cap U_j \text{ and } d(x, \partial_\mathbb{R} U_j) \leq d(x, \partial_\mathbb{R} U_k),\ k \neq j; \\ d(x, \partial_\mathbb{R} U_j) & \text{otherwise.} \end{cases}$$

We define τ_k analogously. Then we define open sets V_j and V_k, for instance,

$$V_j := \{x + iy \in \mathbb{C}^d : x \in U_j, |y| < \tau_j(x)\}.$$

Then

$$V_k \cap V_j \subset W, \quad V_k \cap V_j \cap \mathbb{R} = U_k \cap U_j, \quad V_j \cap \mathbb{R} = U_j, \quad V_k \cap \mathbb{R} = U_k.$$

If U_k intersects more sets U_j (but finitely many) then we can take intersections of the obtained sets V_k. In that way we get $g_{k,j}$ holomorphic on $V_k \cap V_j$.

Since for the orthogonal projection $p : \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}$ we have $p(V_j) = V_j \cap \mathbb{R} = U_j$ (see the definition of V_j), thus

$$p(V_j \cap V_k \cap V_l) = U_j \cap U_k \cap U_l$$

and every connected component of $V_j \cap V_k \cap V_l$ contains a non-empty open part of $U_j \cap U_k \cap U_l$. As easily seen on the intersection $U_j \cap U_k \cap U_l$ it holds $f_{j,k} + f_{k,l} + f_{l,j} = 0$ by the very definition of the functions. Thus also extensions satisfy $g_{jk} + g_{kl} + g_{lj} = 0$ on $V_j \cap V_k \cap V_l$ for any triple $\{j, k, l\}$.

By the Cartan-Grauert result (see [8, Prop. 1]) every open set in $\mathbb{R}^d$ has a basis of complex neighbourhoods being domains of holomorphy. So there is a domain of holomorphy $V \subset \bigcup V_k$, $V \cap J = J$. Thus we can apply the solution to the first Cousin problem [15, Th. 5.5.1] for the covering $(V_k \cap V)$ of V and Cousin data $(g_{j,k})$ to find, after taking restrictions, $(f_k)_k$ with $f_k \in \mathcal{A}(U_k)$ such that $f_{k,j} = f_k - f_j$ for each k, j. Define $\gamma_k := T_k f_k \in \mathcal{A}(U_k)$. Since $f_{k,j} \in \ker(T_{k,j})$, $\gamma_k(x) = \gamma_j(x)$ for each $x \in U_k \cap U_j$, and we can find $\gamma \in \mathscr{A}(\mathbb{R})$ such that $\gamma|U_k = \gamma_k = T_k f_k$ for each k. This means $\gamma \in \mathscr{A}(\mathbb{R}, (U_k), T)$. Moreover $\mathcal{F}(\gamma) = (f_{k,j})_{k,j} + S(F)$.

It remains to show that $\ker(\mathcal{F}) = \operatorname{im}(T)$. If $\gamma \in \operatorname{im}(T)$, there is $f \in \mathscr{A}(\mathbb{R})$ such that $Tf = \gamma$. Then $\gamma \in \mathscr{A}(\mathbb{R}, (U_k), T)$ since $\gamma|U_k = T_k(f|U_k)$ for each k. Moreover, $\mathcal{F}(\gamma) = 0 \in E/S(F)$ and $\gamma \in \ker(\mathcal{F})$.

To prove the reverse inclusion, take $\gamma \in \mathscr{A}(\mathbb{R}, (U_k), T)$ with $\mathcal{F}(\gamma) = 0 \in E/S(F)$. For each k there is $f_k \in \mathscr{A}(U_k)$ such that $T_k f_k = \gamma|U_k$. By assumption $((f_k - f_j)|U_k \cap U_j)_{k,j} \in S(F)$. Therefore we can find $(g_k)_k$, $g_k \in \mathscr{A}(U_k)$, $T_k g_k = 0$ such that $f_k - f_j = g_k - g_j$ on $U_k \cap U_j$ for each k, j. Hence $f_k - g_k = f_j - g_j$ on $U_k \cap U_j$ for each k, j. Accordingly, there is $f \in \mathscr{A}(\mathbb{R})$ such that $f|U_k = f_k - g_k$ for each k. If $x \in U_k$, we get $Tf(x) = T_k f_k(x) - T_k g_k(x) = \gamma(x)$. Since (U_k) is an open covering of $\mathbb{R}$ we get $Tf = \gamma$ and $\gamma \in \operatorname{im}(T)$. $\square$

Proposition 3.12 *If $\varphi(x) = x^s, s \in \mathbb{N}, s > 1$, then for $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ we have that $\sigma_p(C_\varphi) = \{1\}$, $\ker(C_\varphi - I)$ is one dimensional and its elements are the constants, but $\sigma(C_\varphi) = \mathbb{C}$.*

Proof: The statement on eigenvalues and eigenspaces follows by Theorem 2.2.

Now, we consider the spectrum. First of all, since φ is not a diffeomorphism, $0 \in \sigma(C_\varphi)$ by Proposition 2.1. The map φ has three fixed points $0, 1, -1$ if s is odd and two $0, 1$ if s is even. Anyway, $\varphi'(1) = s$ (which coincides with $\varphi'(-1)$ if s is odd). By Proposition 2.1, as $s \neq 1$, $s^n \in \sigma(C_\varphi)$ for each $n \in \mathbb{N}_0$. It remains to show that

$$T := C_\varphi - \lambda I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$$

is not surjective for $\lambda \neq 0, \lambda \neq s^n, n \in \mathbb{N}_0$.

We consider first the case when s is odd.

Set $U_{-1} := (-\infty, 0)$, $U_0 := (-1, 1)$, $U_1 := (0, \infty)$, that are invariant sets under φ and they cover $\mathbb{R}$. Denote by T_k the restriction of $T = C_\varphi - \lambda I$ to $\mathscr{A}(U_k)$, $k = -1, 0, 1$. By Theorem 3.4, $\ker T_0$ is finite dimensional, since φ and $\varphi^{[2]}$ have on U_0 only one fixed point. On the other hand, since φ is a real analytic diffeomorphism both in U_{-1} and in U_1, we can apply Corollary 3.7 to conclude that $\ker T_i$ is finite dimensional for $i = -1, 1$. Now set $U_{-1,0} := U_{-1} \cap U_0 = (-1, 0)$ and $U_{0,1} := U_0 \cap U_1 = (0, 1)$ and $T_{i,k}$ for the operator T restricted to $\mathscr{A}(U_{i,k})$. Each $U_{i,k}$ is invariant under φ and φ is a real analytic diffeomorphism on each $U_{i,k}$ without fixed points. We can apply Theorem 2.2 to obtain that each $\ker T_{i,k}$ is infinite dimensional.

In the notation of Theorem 3.11, we have that $S : F \to E$ is not surjective, because the domain is finite dimensional and the range is infinite dimensional. We can apply Theorem 3.11 to conclude that $C_\varphi - \lambda I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is not surjective (even more is true: $\operatorname{codim} \operatorname{im} C_\varphi - \lambda I = \infty$).

Now suppose that s is even. We define $U_0 = \mathbb{R} \setminus \{0\}$, $U_1 = (-1, 1)$. They are invariant with respect to φ. We will describe $\ker T|_{\mathscr{A}(U_i)}$.

First, if $f \in \ker T|_{\mathscr{A}(U_0)}$, then for $x < 0$ we can calculate $f(x) = (1/\lambda) f(x^s)$ so it is uniquely determined by $f|_{\mathbb{R}_+}$ and the latter function belongs to $\ker T|_{\mathscr{A}(\mathbb{R}_+)}$. Since φ is a diffeomorphism and $\varphi^{[2]}$ has one fixed point on $\mathbb{R}_+$ thus by Corollary 3.7, the kernel is finite dimensional. Now, $\ker T|_{\mathscr{A}(U_1)}$ is finite dimensional by Theorem 3.4.

For the space $\mathscr{A}(U_0 \cap U_1)$ the kernel of T_λ is infinite dimensional, since again its elements f are determined uniquely by $f|_{(0,1)}$ and the latter belongs to $\ker T|_{\mathscr{A}(0,1)}$. By Theorem 2.2, this kernel is infinite dimensional.

We apply again Theorem 3.11 to conclude that $T = C_\varphi - \lambda I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is not surjective, since the map $S : F \to E$ is not surjective. Indeed, the domain is finite dimensional but the range space is infinite dimensional and so $\operatorname{codim} \operatorname{im} C_\varphi - \lambda I = \infty$. $\qquad \square$

In order to get more examples, the following observation is useful. The proof is easy.

Lemma 3.13 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic self map and let $\delta : \mathbb{R} \to \mathbb{R}$ be a real analytic diffeomorphism. The real analytic self map $\psi : \mathbb{R} \to \mathbb{R}, \psi := \delta^{-1} \circ \varphi \circ \delta$, satisfies $\sigma_p(C_\psi) = \sigma_p(C_\varphi)$ and $\sigma(C_\psi) = \sigma(C_\varphi)$. Moreover, u is a fixed point of φ if and only if $\delta^{-1}(u)$ is a fixed point of ψ. If u is a fixed point of φ, then $\psi'(\delta^{-1}(u)) = \varphi'(u)$. Moreover, a subset A of $\mathbb{R}$ is invariant by φ if and only if $\delta^{-1}(A)$ is invariant for ψ.*

As an immediate consequence of Lemma 3.13 and Example 3.12, we get

Example 3.14 Let $\delta : \mathbb{R} \to \mathbb{R}$ be a real analytic diffeomorphism and let $s \in \mathbb{N}$, $s > 1$. The real analytic self map $\varphi(x) := \delta^{-1}(\delta(x)^s), x \in \mathbb{R}$, satisfies $\sigma_p(C_\varphi) = \{1\}$ and $\sigma(C_\varphi) = \mathbb{C}$.

To mention a concrete explicit example, if $\delta(x) = e^x - e^{-x}$, then $\delta^{-1}(x) = \log(\frac{1}{2}((y^2 + 4)^{1/2} - y))$.

4 Quadratic function

We now investigate the spectrum $\sigma(C_\varphi)$ of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$, when $\varphi(x) = ax^2 + bx + c, a \neq 0$. We distinguish several cases, depending on the number of fixed points on φ. Lemma 3.13 will be used to reduce the problem to the study of certain parameter family.

Case 1. The symbol φ has no fixed points. In this case, since φ has only one critical point, it follows from Corollary 3.1 that $\sigma(C_\varphi) = \mathbb{C}$ and $\sigma_p(C_\varphi) = \mathbb{C} \setminus \{0\}$.

Case 2 and 3. The symbol has fixed points v, u (possibly $u = v$) and

$$\varphi(x) = x + a(x - u)(x - v), \qquad a \neq 0.$$

Choose $u \leq v$ if $a < 0$ or $u > v$ if $a > 0$. then taking

$$\delta(x) = -\frac{1}{a}x + u$$

we will obtain

$$\psi(x) = \delta^{-1} \circ \varphi \circ \delta(x) = [1 + a(u - v)]\, x - x^2.$$

Clearly, u, v were selected to ensure that $\mu := 1 + a(u - v) \geq 1$. By Lemma 3.13, it is enough to consider

$$\varphi(x) = -x^2 + \mu x \qquad \text{with } \mu \geq 1.$$

Case 2. The symbol φ has only one fixed point u. Then the behaviour of C_φ is the same as C_ψ for $\psi(x) = -x^2 + x$.

Proposition 4.1 *In Case 2, $\varphi(x) = -x^2 + x$, $\sigma_p(C_\varphi) = \{1\}$ but $\sigma(C_\varphi) \supseteq \{0\} \cup [1, +\infty)$.*

Proof: Clearly, by Theorem 2.2 and Proposition 2.1 we have

$$\sigma_p(C_\varphi) = \{1\}, \qquad \sigma(C_\varphi) \supset \{1, 0\}.$$

Now, we consider the map $T : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$, $T(f) = C_\varphi(f) - \lambda f$, and we will show that the function id, id$(x) = x$, does not belong to the image of T for $\lambda \in (1, +\infty)$.

It is a simple calculation that there is the unique formal solution $f(z) = \sum_{n=0}^{\infty} f_n z^n$ of the equation $T(f) = $ id, with

$$(5) \qquad f_0 = 0, \quad f_1 = \frac{1}{1 - \lambda}, \quad f_n = \frac{1}{1 - \lambda} \sum_{j=1}^{\left[\frac{n}{2}\right]} \binom{n - j}{j} (-1)^{j-1} f_{n-j}.$$

It is clear that for $1 - \lambda < 0$, $f_1 < 0$ and $f_2 > 0$. Then one shows inductively by (5) that $f_n(-1)^{n-1} > 0$. Therefore

$$|f_n| > \frac{1}{|1 - \lambda|} \binom{n - 1}{1} |f_{n-1}| = \frac{n - 1}{|1 - \lambda|} |f_{n-1}|.$$

Again inductively

$$|f_n| > \frac{(n - 1)!}{|1 - \lambda|^n}$$

and the series $\sum_{n=0}^{\infty} f_n z^n$ is nowhere convergent. So there is no real analytic solution f for the equation $T(f) = $ id for any $\lambda > 1$. $\qquad \square$

In Case 2. we also get the following partial positive step.

Proposition 4.2 *Let $\varphi(x) = -x^2 + x$, $\lambda \neq 0$, and assume that the equation*

(6)
$$f(\varphi(x)) - \lambda f(x) = \gamma(x)$$

has an analytic solution f on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and some fixed $\gamma \in \mathscr{A}(\mathbb{R})$. Then this solution extends to a solution on $\mathbb{R}$.

Remark 4.3 *Note that (6) has always the unique formal solution — we do not know for precisely which λ and γ it converges around zero.*

Proof: We observe that for $x \in (0, 1)$ $\varphi^{[n]}(x) \to 0$ as $n \to \infty$ so, by Lemma 3.3, we extend f to $(-\varepsilon, 1)$.

Since $\varphi : (-\infty, 1/2) \to (-\infty, 1/4)$ is a diffeomorphism onto, its inverse ψ is a diffeomorphism $(-\infty, 1/4) \mapsto (-\infty, 1/2)$. Moreover, for $x \in (-\infty, 1/2)$ the equation

$$f(\varphi(x)) - \lambda f(x) = \gamma(x)$$

is equivalent to

$$f(\psi(x)) - (1/\lambda)f(x) = -(1/\lambda)\gamma(\psi(x)).$$

Since for $x \in (-\infty, 0)$ $\psi^{[n]}(x) \to 0$ as $n \to \infty$ we can extend f onto $(-\infty, 0)$ by Lemma 3.3.

Summarizing, f is a solution of (6) on $(-\infty, 1)$.

Since for $x \in (0, 1/2)$ and $y = 1 - x \in (1/2, 1)$ we have

$$\lambda f(x) + \gamma(x) = f(\varphi(x)) = f(\varphi(y)) = \lambda f(y) + \gamma(y)$$

we have

$$f(y) = f(1 - y) + \frac{\gamma(1 - y) - \gamma(y)}{\lambda}.$$

This formula extends f analytically onto $(1/2, +\infty)$ and so on the whole real line. Clearly, this extension satisfies (6) everywhere. $\qquad\square$

Case 3. The symbol φ has exactly two different fixed points u, v. Then the behaviour of C_φ is the same as C_ψ for

$$\psi(x) = -x^2 + \mu x, \qquad \mu > 1.$$

The map ψ has one critical point $\mu/2$ and fixed points 0 and $\mu - 1$.

Case 3.1. Critical point is outside the interval joining fixed points, i.e., $\mu/2 > \mu - 1$ so $1 < \mu < 2$.

Case 3.2. Critical point is equal to one fixed point, i.e., $\mu/2 = \mu - 1$ so $\mu = 2$.

Case 3.3. Critical point is between fixed points, i.e., $\mu/2 < \mu - 1$ so $\mu > 2$.

Proposition 4.4 *In Case 3.1 and 3.2, $\varphi(x) = -x^2 + \mu x$, $1 < \mu \leq 2$, $\sigma(C_\varphi) = \mathbb{C}$ but $\sigma_p(C_\varphi) = \{1\}$ and for every $\lambda \in \mathbb{C}$ kernel of $C_\varphi - \lambda I$ is one-dimensional.*

Proof: We define

$$U_1 := (-\infty, \mu - 1) \cup (1, +\infty), \qquad U_0 := (0, \mu).$$

Observe that for every $x \in (-\infty, \mu - 1)$ there is exactly one point $y \in (1, \infty)$ such that $\varphi(x) = \varphi(y)$ ($y = \mu - x$). Thus $f \in \mathscr{A}(U_1)$ belongs to the kernel of $C_\varphi - \lambda I$, $\lambda \neq 0$ if and only if $f|_{(-\infty, \mu - 1)} \in \ker(C_\varphi - \lambda I)$ on $\mathscr{A}(-\infty, \mu - 1)$ and

$$f(y) = f(\mu - y), \qquad y \in (1, +\infty).$$

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Therefore the kernels of $C_\varphi - \lambda I$ on $\mathscr{A}(U_1)$ and on $\mathscr{A}(-\infty, \mu - 1)$ have the same dimension. In the latter case φ and $\varphi^{[2]}$ have only one fixed point 0 on $(-\infty, \mu - 1)$. Hence, by Theorem 2.2, $\ker C_\varphi - \lambda I$ is one dimensional in $\mathscr{A}(U_1)$.

Since $\varphi : (0, \mu) \to (0, \mu)$ and its square have exactly one fixed point $\mu - 1$, $\ker C_\varphi - \lambda I$ is one dimensional in $\mathscr{A}(U_0)$ by Theorem 2.2.

Finally, $U_1 \cap U_0 = (0, \mu - 1) \cup (1, \mu)$. Again as in case of U_1 one can prove that the kernels of $C_\varphi - \lambda I$ in $\mathscr{A}(U_1 \cap U_0)$ and in $\mathscr{A}(0, \mu - 1)$ are isomorphic. Since φ has no fixed point in $(0, \mu - 1)$ and only one critical point, the required kernel is infinite dimensional by Theorem 2.2.

We apply Theorem 3.11 to conclude that, for every $\lambda \neq 0$, the map $C_\varphi - \lambda I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is not surjective. By Proposition 2.1, $\sigma(C_\varphi) = \mathbb{C}$.

The remaining parts of the Proposition follows from Theorem 2.2. $\qquad\square$

In Case 3.3 we have only the following partial result:

Proposition 4.5 *If $\varphi(x) = -x^2 + \mu x$, $\mu > 2$, then $\sigma(C_\varphi) \supset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \cup \{\mu^n\}_n \cup \{(2 - \mu)^n\}_n$.*

Proof: Clearly, $\{0, 1\} \cup \{\mu^n\}_n \cup \{(2 - \mu)^n\}_n \subset \sigma(C_\varphi)$ by Propositions 2.1 and 2.2.

Fix $\lambda \neq 1$, $0 < |\lambda| \leq 1$. Since $\varphi(1) = \mu - 1$, we can select a sequence $x_n \in (0, 1) \subset (0, \mu)$, $x_1 = 1$, such that

$$\varphi^{[n]}(x_n) = \mu - 1, \quad n \in \mathbb{N}$$

Such x_n exists because $\varphi(\mu/2) = \mu^2/4 \geq \mu$ and $1 < \mu/2$. In fact, $x_n = \frac{\mu - \sqrt{\mu^2 - 4x_{n-1}}}{2} = \frac{2x_{n-1}}{\mu + \sqrt{\mu^2 - 4x_{n-1}}}, n \in \mathbb{N}$. Hence $x_n/x_{n-1} < 2/\mu < 1, n \in \mathbb{N}$, and $(x_n)_{n \in \mathbb{N}}$ is decreasing to zero as $n \to \infty$. Moreover, x_n/x_{n-1} tends to $1/\mu$ as $n \to \infty$.

If $C_\varphi f - \lambda f = \gamma$ for some $\gamma \in \mathscr{A}(\mathbb{R})$, then

$$\lambda f(\mu - 1) + \gamma(\mu - 1) = f(\mu - 1), \quad \text{so} \quad f(\mu - 1) = \gamma(\mu - 1)/(1 - \lambda),$$

and analogously

$$f(0) = \frac{\gamma(0)}{1 - \lambda},$$

because both 0 and $\mu - 1$ are fixed points for φ. Since

$$f \circ \varphi^{[n]}(x) = \lambda^n f(x) + \lambda^{n-1}\gamma(x) + \lambda^{n-2}\gamma(\varphi(x)) + \cdots + \gamma(\varphi^{[n-1]}(x)),$$

we have

(7) $$\frac{\gamma(\mu - 1)}{1 - \lambda} = \lambda^n f(x_n) + \lambda^{n-1}\gamma(x_n) + \cdots + \lambda\gamma(\varphi^{[n-2]}(x_n)) + \gamma(1).$$

Fix $c < 1/6$. If $\gamma(0) = 0$, we can choose $n_1 > 2$ so that, for $n \geq n_1$, $|x_n| \leq c$ and $|f(x_n)| \leq c$, since $f(x_n)$ tends to $f(0) = \gamma(0)/(1 - \lambda) = 0$.

Take $\gamma(x) = x^k \gamma_0(x)$, with $\gamma_0 \in \mathscr{A}(\mathbb{R})$ satisfying $\gamma_0(\mu - 1) = 0$ and $1/2 < |\gamma_0(x)| < 1$ for $x \in [0, 1)$, $\gamma_0(1) = 1$. Since x_n/x_{n-1} tends to $1/\mu$ as $n \to \infty$, there is $n_0 > n_1$ such that $x_n < (1/2)x_{n-1}$ for $n > n_0$. We can take k big enough to ensure that $|\gamma(x_j)| < c/n_0$ for $2 \leq j \leq n_0$. Then, for $n > n_0$, we have $|\gamma(x_n)| \leq 2c(1/2)^{n-n_0}$. Summarizing:

$$|\lambda^n f(x_n) + \lambda^{n-1}\gamma(x_n) + \cdots + \lambda\gamma(\varphi^{[n-2]}(x_n))| \leq 2c + 2c\frac{1}{1 - (1/2)^k} \leq 6c.$$

As $c < 1/6$, we reach a contradiction with (7). Therefore $C_\varphi - \lambda I$ is not surjective. $\qquad\square$

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