SOLID HULLS AND CORES OF WEIGHTED H^∞-SPACES.

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Abstract. We determine the solid hull and solid core of weighted Banach spaces H_v^∞ of analytic functions functions f such that $v|f|$ is bounded, both in the case of the holomorphic functions on the disc and on the whole complex plane, for a very general class of radial weights v. Precise results are presented for concrete weights on the disc that could not be treated before. It is also shown that if H_v^∞ is solid, then the monomials are an (unconditional) basis of the closure of the polynomials in H_v^∞. As a consequence H_v^∞ does not coincide with its solid hull and core in the case of the disc. An example shows that this does not hold for weighted spaces of entire functions.

1. Introduction and preliminaries

The solid hulls of weighted H^∞-type Banach spaces H_v^∞ of analytic functions on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ were characterized in [8] for a large class of weight functions v, and a similar study for entire functions was made in [7]. In Theorem 3.1 we extend the results of [8] by means of the calculations of certain numerical constants, which yields many novel, concrete examples of solid hulls. At the same time, we also describe in Theorem 2.4 the solid cores of these Banach spaces in a similar way as we did in Theorem 2.3 (Theorem 2.1 in [7]) for the solid hull. Moreover, we prove that in the case of analytic functions on the disc the Banach space H_v^∞ is always different from both its solid hull and core; see Corollary 5.3. However, an example is given in the case of entire functions to show that H_v^∞ may coincide with both its solid hull and core, in particular it is a solid space. We also prove in Theorem 5.2 that if H_v^∞ coincides with its solid hull, then the monomials are an (unconditional) basis of the closure of the polynomials in H_v^∞.

To describe the results in detail, let us introduce some notation and terminology. We set $R = 1$ (for the case of holomorphic functions on the unit disc) and $R = +\infty$ (for the case of entire functions). A *weight* v is a continuous function $v : [0, R[\to]0, \infty[$, which is non-increasing on $[0, R[$ and satisfies $\lim_{r\to R} r^n v(r) = 0$ for each $n \in \mathbb{N}$. We extend v to $\mathbb{D}$ if $R = 1$ and to $\mathbb{C}$ if $R = +\infty$ by $v(z) := v(|z|)$. For such a weight v, we study the Banach space H_v^∞ of analytic functions f on the disc $\mathbb{D}$ (if $R = 1$) or on the whole complex plane $\mathbb{C}$ (if $R = +\infty$) such that $\|f\|_v := \sup_{|z|<R} v(z)|f(z)| < \infty$. For an analytic function $f \in H(\{z \in \mathbb{C}; |z| < R\})$ and $r < R$, we denote $M(f, r) := \max\{|f(z)| \; ; \; |z| = r\}$. Using the notation O and o of Landau, $f \in H_v^\infty$ if and only if $M(f, r) = O(1/v(r)), r \to R$. It is known that the closure of the polynomials in H_v^∞ coincides with the Banach space H_v^0 of all those analytic functions on $\{z \in \mathbb{C}; |z| < R\}$ such that $M(f, r) = o(1/v(r)), r \to R$. see e.g. [3]. It will be clear from the context in the rest of the article when we refer to analytic functions on the disc or entire functions. Anyway, if it is necessary to distinguish at some point, we will use the notations $H_v^\infty(\mathbb{D})$ and $H_v^\infty(\mathbb{C})$.

We shall identify an analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$ with the sequence of its Taylor coefficients $(a_n)_{n=0}^\infty$. Let A, B and H be vector spaces of complex sequences containing the space of all the sequences with finitely many non-zero coordinates. The space A be is *solid* if $a = (a_n) \in A$ and $|b_n| \le |a_n|$ for each n implies $b = (b_n) \in A$. The *solid hull of* A is

$$S(A) := \{(c_n) \,:\, \exists (a_n) \in A \text{ such that } |c_n| \le |a_n| \ \forall n \in \mathbb{N}\}.$$

The *solid core of* A is

$$s(A) := \{(c_n) \,:\, (c_n a_n) \in A \ \forall (a_n) \in \ell_\infty\}.$$

The *set of multipliers form* A *into* B is

$$(A, B) := \{c = (c_n) \,:\, (c_n a_n) \in B \ \forall (a_n) \in A\}.$$

The following facts are also known, see [1]: 1. A is solid if and only if $\ell_\infty \subset (A, A)$; 2. $A \subset (B, H)$ if and only if $B \subset (A, H)$; 3. the solid core $s(A)$ of A is the largest solid space contained in A, and moreover $s(A) = (\ell_\infty, A)$; 4. the solid hull $S(A)$ of A is the smallest solid space containing A; 5. If X is solid, $(A, X) = (S(A), X)$ and $(X, A) = (X, s(A))$.

The results on [7] and [8] contain a characterization of the solid hull of H_v^∞, if v satisfies a condition (b), see (2.1) below. This general characterization is in terms of a numerical sequence $(m_n)_{n=0}^\infty$, which depends on the weight and was studied by the second named author in [16]. However, given a concrete weight on the disc like

$$v(r) = \exp\big(-a/(1-r)^b\big) \tag{1.1}$$

the calculation of the numbers m_n is not an easy matter, and in [8], this was only done in the case $0 < b \le 2$. In Theorem 3.1 we calculate these numbers and thus determine the solid hull for weights

$$v(r) = w(r)\exp\big(-a/(1-r)^b\big) \tag{1.2}$$

with any $a, b > 0$, where w is a differentiable positive function with some growth restriction. Moreover, the same theorem also contains the analogous characterization of the solid core. In Section 4 we show how these results can be used also in the case of the variant

$$v(r) = \exp\big(-a/(1-r^2)^b\big) \tag{1.3}$$

of the weight. The general characterization of solid cores for weights satisfying condition (b) is given in Theorem 2.4. As explained above, further interesting related results are presented in Section 5.

Bennet, Stegenga and Timoney in their paper [2] determined the solid hull and the solid core of the weighted spaces $H_v^\infty(\mathbb{D})$ in the case the weight v is doubling. Exponential weights $v(r) = \exp(-a/(1-r)^b)$ with $a, b > 0$ are not doubling. Not much seems to be known about multipliers and solid hulls of weighted spaces of analytic functions on the unit disc in the case of exponential weights. Hadamard multipliers of certain weighted space $H_a^1(\alpha), \alpha > 0$, were completely described by Dostanić in [10] (see also Chapter 13 in [13]). Other aspects of weighted spaces of analytic functions on the unit disc with exponential weights, like integration operators or Bergman projections, have been investigated recently by Constantin, Dostanić, Pau, Pavlović, Peláez and Rättyä, among others; see [9], [11], [17], [18] and [20]. The solid hull and multipliers on spaces of analytic functions on the disc has been investigated by many authors. In addition to [2], we mention for example [1], [5], [6], [12], the books [13] and [19] and the many references therein.

Spaces of type $H_v^\infty(\mathbb{C})$ and $H_v^\infty(\mathbb{D})$ appear in the study of growth conditions of analytic functions and have been investigated in various articles since the work of Shields and Williams, see *e.g.* [3],[4], [15], [16], [21] and the references therein.

In the case of a "standard" weight $v_\alpha(z)) = (1-|z|^2)^\alpha$, where $\alpha \geq 0$, we denote for every $H_\alpha^\infty := H_\alpha^\infty(\mathbb{D}) := H_{v_\alpha}^\infty$. The solid hull $S\big(H_\alpha^\infty\big)$ of H_α^∞ is known: it equals

$$S\big(H_\alpha^\infty\big) = \Big\{(b_m)_{m=0}^\infty \,:\, \sup_{n\in\mathbb{N}_0}\Big(\sum_{m=2^n}^{2^{n+1}-1} |b_m|^2 (m+1)^{-2\alpha}\Big)^{1/2} < \infty\Big\}.$$

This is Theorem 8.2.1 of [13]. Moreover, the solid core $s\big(H_\alpha^\infty\big)$ can also be characterized, see Theorem 8.3.4 of [13]:

$$s\big(H_\alpha^\infty\big) = \Big\{(b_m)_{m=0}^\infty \,:\, \sup_{n\in\mathbb{N}_0}\Big(\sum_{m=2^n}^{2^{n+1}-1} |b_m|(m+1)^{-\alpha}\Big) < \infty\Big\}.$$

2. Solid hull and core for weights with condition (b).

In this section we consider the quite large class of weights on $\mathbb{D}$ or $\mathbb{C}$ satisfying the regularity condition (b). For such a weight, the solid hull was found in the paper [8], and we determine the solid core here.

Definition 2.1. Let $r_n \in]0,R[$ be a global maximum point of the function $r^m v(r)$ for any $m > 0$. The weight v satisfies the *condition* (b) *if there exist numbers* $b > 2$, $K > b$ *and* $0 < m_1 < m_2 < \dots$ *with* $\lim_{n\to\infty} m_n = \infty$ *such that*

$$b \leq \Big(\frac{r_{m_n}}{r_{m_{n+1}}}\Big)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \leq K. \tag{2.1}$$

Remark 2.2. (1) The second named author introduced the following *condition (B)* on the weight v in [16]:

$$\forall b_1 > 0 \ \exists b_2 > 1 \ \exists c > 0 \ \forall m,n :$$

$$\Big(\frac{r_m}{r_n}\Big)^m \frac{v(r_m)}{v(r_n)} \leq b_1 \ \text{ and } \ |m-n| \geq c \Rightarrow \Big(\frac{r_n}{r_m}\Big)^n \frac{v(r_n)}{v(r_m)} \leq b_2.$$

It was observed in Remark 2.7 of [7] that if a weight v satisfies condition (B), then it also satisfies condition (b) for some $b > 2$, $K > b$ and $0 < m_1 < m_2 < \dots$.

(2) As a consequence of this observation and Section 2 in [16], the following weights satisfy condition (b): For $R = 1$,
(i) $v(r) = (1-r)^\alpha$ with $\alpha > 0$, which are the standard weights on the disc, and
(ii) $v(r) = \exp(-(1-r)^{-1})$.
More examples can be seen in Example 3.3. For $R = +\infty$,
(i) $v(r) = \exp(-r^p)$ with $p > 0$,
(ii) $v(r) = \exp(-\exp r)$, and
(iii) $v(r) = \exp\big(-(\log^+ r)^p\big)$, where $p \geq 2$ and $\log^+ r = \max(\log r, 0)$.

We recall the result [8], Theorem 2.1 for $\mathbb{D}$, or [7], Theorem 2.5 for entire functions:

Theorem 2.3. *If the weight v satisfies (b), we have*

$$S(H_v^\infty) = \Big\{(b_m)_{m=0}^\infty : \sup_n v(r_{m_n})\Big(\sum_{m_n < m \leq m_{n+1}} |b_m|^2 r_{m_n}^{2m}\Big)^{1/2} < \infty\Big\}. \tag{2.2}$$

Now let us prove the following statement.

Theorem 2.4. *For a weight v satisfying (b), we have*

$$s(H_v^\infty) = \Big\{(b_m)_{m=0}^\infty : \sup_n v(r_{m_n})\Big(\sum_{m_n < m \le m_{n+1}} |b_m| r_{m_n}^m\Big) < \infty\Big\}. \tag{2.3}$$

Proof. For a holomorphic function f with $f(z) = \sum_{n=0}^\infty a_n z^n$, we let h_f denote the function defined by

$$h_f(z) = \sum_{n=0}^\infty |a_n| z^n \quad \text{for all } z.$$

It is easy to see that the solid core $s(H_v^\infty)$ of H_v^∞ coincides with the set

$$\{f : f \text{ holomorphic}, h_f \in H_v^\infty\}.$$

Now, let $f \in H_v^\infty$, $f(z) = \sum_{n=0}^\infty b_n z^n$. If

$$h_f(z) = \sum_{n=0}^\infty |b_n| z^n, \tag{2.4}$$

belongs to H_v^∞, it is easily seen that

$$\|h_f\|_v = \sup_{0<r<R} v(r) \sum_{n=0}^\infty |b_n| r^n.$$

This implies

$$\sup_n v(r_{m_n})\Big(\sum_{m_n < m \le m_{n+1}} |b_m| r_{m_n}^m\Big) \le \|h_f\|_v. \tag{2.5}$$

Thus the solid core is contained in the right-hand side of (2.3). Now we proceed with the reverse inclusion.

According to [16], Proposition 5.2., there are numbers $\beta_m \in [0,1]$ and a constant $c > 0$ (independent of f) such that

$$\|h_f\|_v \le c \sup_n \sup_{r_{m_{n-1}} \le |z| \le r_{m_{n+1}}} v(z) \Big| \sum_{m_{n-1} < m \le m_{n+1}} \beta_m |b_m| z^m \Big|.$$

This implies

$$\|h_f\|_v \le c \sup_n \sup_{r_{m_{n-1}} \le r \le r_{m_{n+1}}} v(r) \Big(\sum_{m_{n-1} < m \le m_{n+1}} |b_m| r^m\Big). \tag{2.6}$$

For $r_{m_{n-1}} \le r \le r_{m_{n+1}}$ we have

$$\begin{aligned}
& v(r) \sum_{m_{n-1} < m \le m_n} |b_m| r^m \\
&= \frac{v(r)}{v(r_{m_{n-1}})} v(r_{m_{n-1}}) \sum_{m_{n-1} < m \le m_n} |b_m| r_{m_{n-1}}^m \Big(\frac{r}{r_{m_{n-1}}}\Big)^m \\
&\le \Big(\frac{r}{r_{m_{n-1}}}\Big)^{m_n} \frac{v(r)}{v(r_{m_{n-1}})} v(r_{m_{n-1}}) \sum_{m_{n-1} < m \le m_n} |b_m| r_{m_{n-1}}^m \\
&\le \Big(\frac{r_{m_n}}{r_{m_{n-1}}}\Big)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n-1}})} v(r_{m_{n-1}}) \sum_{m_{n-1} < m \le m_n} |b_m| r_{m_{n-1}}^m
\end{aligned}$$

$$\leq Kv(r_{m_{n-1}}) \sum_{m_{n-1}<m\leq m_n} |b_m| r_{m_{n-1}}^m. \tag{2.7}$$

Here we used that r_{m_n} is a global maximum point for $r^{m_n}v(r)$.

Similarly, for $r_{m_{n-1}} \leq r \leq r_{m_n}$ we have

$$\begin{aligned}
& v(r) \sum_{m_n<m\leq m_{n+1}} |b_m| r^m \\
&= \frac{v(r)}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n<m\leq m_{n+1}} |b_m| r_{m_n}^m \Big(\frac{r}{r_{m_n}}\Big)^m \\
&\leq \Big(\frac{r}{r_{m_n}}\Big)^{m_n} \frac{v(r)}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n<m\leq m_{n+1}} |b_m| r_{m_n}^m \\
&\leq v(r_{m_n}) \sum_{m_n<m\leq m_{n+1}} |b_m| r_{m_n}^m. \qquad (2.8)
\end{aligned}$$

Finally, if $r_{m_n} \leq r \leq r_{m_{n+1}}$ then

$$\begin{aligned}
& v(r) \sum_{m_n<m\leq m_{n+1}} |b_m| r^m \\
&= \frac{v(r)}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n<m\leq m_{n+1}} |b_m| r_{m_n}^m \Big(\frac{r}{r_{m_n}}\Big)^m \\
&\leq \Big(\frac{r}{r_{m_n}}\Big)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n<m\leq m_{n+1}} |b_m| r_{m_n}^m \\
&\leq \Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)^{m_{n+1}} \frac{v(r_{m_{m+1}})}{v(r_{m_n})} v(r_{m_n}) \sum_{m_n<m\leq m_{n+1}} |b_m| r_{m_n}^m \\
&\leq Kv(r_{m_n}) \sum_{m_n<m\leq m_{n+1}} |b_m| r_{m_n}^m. \qquad (2.9)
\end{aligned}$$

Hence, according to (2.6), with (2.7), (2.8) and (2.9),

$$\|h_f\|_v \leq c2K \sup_n v(r_{m_n}) \Big(\sum_{m_n<m\leq m_{n+1}} |b_m| r_{m_n}^m \Big).$$

This together with (2.1) yields (2.3). □

3. A weight of the form $w(r)\exp(-a/(1-r)^b)$.

In this section we only deal with weights defined on the unit disc $\mathbb{D}$. Our purpose is to improve the results of [8] by calculating the solid hulls and cores for a larger class of concrete examples, namely, for weights of the form

$$v(z) = w(r) \exp\Big(-\frac{a}{(1-r)^b}\Big), \quad z \in \mathbb{D}, \tag{3.1}$$

where $a, b > 0$ are given constants and $w : [0,1[\to]0,\infty[$ is a differentiable function, extended to $\mathbb{D}$ by $w(z) = w(|z|)$. We remark the examples in [8] only contain the case $b \leq 2$, $w \equiv 1$.

We will prove

Theorem 3.1. *Let $w'(r)/w(r)$ be a decreasing function and assume that there are $n_0 > 0$ and $\alpha \in]0, 1 + b/2[$ such that*

$$(1-r)^\alpha \frac{w'(r)}{w(r)} \text{ is bounded on } [0,1[\tag{3.2}$$

$$\frac{1}{e} \le \frac{w(1-(\frac{a}{bn^2})^{1/b})}{w(1-(\frac{a}{b(n+1)^2})^{1/b})} \le e \text{ for } n \ge n_0. \tag{3.3}$$

Then, the solid hull of H_v^∞ is equal to

$$\Big\{(b_m)_{m=0}^\infty : \sup_n w\Big(1-\Big(\frac{a}{bn^2}\Big)^{1/b}\Big)e^{-bn^2}\Big(\sum_{\substack{m\in\mathbb{N}\\ m_n<m\le m_{n+1}}} |b_m|^2\Big(1-\Big(\frac{a}{bn^2}\Big)^{1/b}\Big)^{2m}\Big)^{1/2} < \infty\Big\},$$

where

$$m_n = b\left(\frac{b}{a}\right)^{1/b} n^{2+2/b} - bn^2 - \Big(1-\Big(\frac{a}{bn^2}\Big)^{1/b}\Big)\frac{w'(1-(\frac{a}{bn^2})^{1/b})}{w(1-(\frac{a}{bn^2})^{1/b})}$$

(which reduces to $m_n = b^{1+1/b}a^{-1/b}n^{2+2/b} - bn^2$, if $w \equiv 1$).

Moreover, the solid core of H_v^∞ is equal to

$$\Big\{(b_m)_{m=0}^\infty : \sup_n w\Big(1-\Big(\frac{a}{bn^2}\Big)^{1/b}\Big)e^{-bn^2}\sum_{\substack{m\in\mathbb{N}\\ m_n<m\le m_{n+1}}} |b_m|\Big(1-\Big(\frac{a}{bn^2}\Big)^{1/b}\Big)^{m} < \infty\Big\}.$$

We postpone the proof a bit and consider some remarks and examples. Let us start with the following quite trivial observation which, however, is very useful to simplify the presentations of the solid hulls and cores.

Lemma 3.2. *Let $1 \le p < \infty$, and let $(K_n)_{n\in\mathbb{N}_0}$, $(\tilde K_n)_{n\in\mathbb{N}_0}$, $(L_m)_{n\in\mathbb{N}_0}$ and $(\tilde L_m)_{n\in\mathbb{N}_0}$ be sequences of positive numbers. Assume that there are given two increasing, unbounded sequences $(m_n)_{n\in\mathbb{N}_0}$ and $(\tilde m_n)_{n\in\mathbb{N}_0}$ of positive real numbers such that*

$$m_n < \tilde m_n < m_{n+1} \quad \forall\, n \in \mathbb{N}_0 \tag{3.4}$$

and such that for some constants $C > c > 0$, for all $n \in \mathbb{N}$,

$$\begin{aligned} cK_n^pL_m \le \tilde K_{n-1}^p\tilde L_m \le CK_n^pL_m \quad &\forall\, m \text{ with } m_n < m \le \tilde m_n \\ cK_n^pL_m \le \tilde K_n^p\tilde L_m \le CK_n^pL_m \quad &\forall\, m \text{ with } \tilde m_n < m \le m_{n+1}. \end{aligned} \tag{3.5}$$

Then, we have

$$\begin{aligned} &\Big\{(b_m)_{m=0}^\infty \,:\, \sup_{n\in\mathbb{N}_0} K_n\Big(\sum_{m_n<m\le m_{n+1}} |b_m|^pL_m\Big)^{1/p} < \infty\,\Big\} \\ &= \Big\{(b_m)_{m=0}^\infty \,:\, \sup_{n\in\mathbb{N}_0} \tilde K_n\Big(\sum_{\tilde m_n<m\le \tilde m_{n+1}} |b_m|^p\tilde L_m\Big)^{1/p} < \infty\,\Big\} \end{aligned} \tag{3.6}$$

Proof. We have, by (3.5),

$$\begin{aligned} &\sup_{n\in\mathbb{N}_0}\Big(\sum_{m_n<m\le m_{n+1}} K_n^p|b_m|^pL_m\Big)^{1/p} \\ &\le \sup_{n\in\mathbb{N}_0}\Big(\Big(\sum_{m_n<m\le\tilde m_n} K_n^p|b_m|^pL_m\Big)^{1/p} + \Big(\sum_{\tilde m_n<m\le m_{n+1}} K_n^p|b_m|^pL_m\Big)^{1/p}\Big) \end{aligned}$$

$$\leq \frac{1}{c}\sup_{n\in\mathbb{N}_0}\Big(\Big(\sum_{m_n<m\leq\tilde m_n}\tilde K_{n-1}^p|b_m|^p\tilde L_m\Big)^{1/p}+\Big(\sum_{\tilde m_n<m\leq m_{n+1}}\tilde K_n^p|b_m|^p\tilde L_m\Big)^{1/p}\Big)$$
$$\leq \frac{2}{c}\sup_{n\in\mathbb{N}_0}\Big(\sum_{\tilde m_n<m\leq\tilde m_{n+1}}\tilde K_n^p|b_m|^p\tilde L_m\Big)^{1/p}.$$

The converse inequality can be shown in the same way, using the other inequalities in (3.5). □

It is obvious that the representations of the solid hull and core of a weighted space are by no means unique: the sequence m_n and even the coefficients and exponents can be chosen in many ways. We discuss this in the first example.

Example 3.3. (i): $a=b=1$, $w=1$. Here $m_n=n^4-n^2$. However, in [8] the representation of this solid hull was found with the more simple numbers $\tilde m_n=n^4$ instead:

$$S(H_v^\infty)=\Big\{(b_m)_{m=0}^\infty:\sup_n e^{-n^2}\Big(\sum_{m=n^4+1}^{(n+1)^4}|b_m|^2\big(1-n^{-2}\big)^{2m}\Big)^{1/2}<\infty\Big\},$$
$$s(H_v^\infty)=\Big\{(b_m)_{m=0}^\infty:\sup_n e^{-n^2}\sum_{m=n^4+1}^{(n+1)^4}|b_m|\big(1-n^{-2}\big)^{m}<\infty\Big\}.$$

Let us verify that the condition (3.5) of Lemma 3.2 is satisfied with $p=2$; the proof for the case $p=1$ follows by taking square roots. We may obviously assume $n\geq 2$. For m with $m_n<m\leq\tilde m_n$, i.e.,

$$n^4-n^2<m\leq n^4, \tag{3.7}$$

we have $K_n=e^{-n^2}=\tilde K_n$, $L_m=(1-n^{-2})^{2m}$ and $\tilde L_m=(1-(n-1)^{-2})^{2m}$, hence,

$$\begin{aligned}K_n^2L_m&=e^{-2n^2}\Big(1-\frac{1}{n^2}\Big)^{2m}\geq e^{-2n^2}\Big(1-\frac{1}{n^2}\Big)^{n^2 2n^2}\\&\geq Ce^{-2n^2}e^{-2n^2}=Ce^{-4n^2},\end{aligned}\tag{3.8}$$

$$\begin{aligned}K_n^2L_m&\leq e^{-2n^2}\Big(1-\frac{1}{n^2}\Big)^{2n^4-2n^2}\\&\leq e^{-2n^2}\Big(1-\frac{1}{n^2}\Big)^{n^2 2n^2}\Big(1-\frac{1}{n^2}\Big)^{-2n^2}\\&\leq Ce^{-4n^2}\Big(1-\frac{1}{n^2}\Big)^{-2n^2}\leq C'e^{-4n^2},\end{aligned}\tag{3.9}$$

Furthermore, we write

$$n^4=(n-1)^4+4(n-1)^3+\rho(n)=(n-1)^2(n-1)^2+4(n-1)^3+\rho(n),$$

where $\rho(n)=6n^2-8n+3$. Using the trivial estimates $\rho(n)-n^2\geq -60(n-1)^2$ and $\rho(n)\leq 50(n-1)^2$ for all $n\geq 2$, we obtain

$$\begin{aligned}\tilde K_{n-1}^2\tilde L_m&=e^{-2(n-1)^2}\Big(1-\frac{1}{(n-1)^2}\Big)^{2m}\leq e^{-2(n-1)^2}\Big(1-\frac{1}{(n-1)^2}\Big)^{2n^4-2n^2}\\&=e^{-2(n-1)^2}\Big(1-\frac{1}{(n-1)^2}\Big)^{2(n-1)^2(n-1)^2+8(n-1)^3+2(\rho(n)-n^2)}\end{aligned}$$

$$\le e^{-2(n-1)^2}\Big(\Big(1-\frac{1}{(n-1)^2}\Big)^{(n-1)^2}\Big)^{2(n-1)^2+8(n-1)-120}$$

$$\le Ce^{-4(n-1)^2}e^{-8(n-1)} = Ce^{-4n^2+8n-4}e^{-8(n-1)} \le C'e^{-4n^2}. \tag{3.10}$$

Similarly,

$$\tilde K_{n-1}^2\tilde L_m = e^{-2(n-1)^2}\Big(1-\frac{1}{(n-1)^2}\Big)^{2m} \ge e^{-2(n-1)^2}\Big(1-\frac{1}{(n-1)^2}\Big)^{2n^4}$$

$$= e^{-2(n-1)^2}\Big(1-\frac{1}{(n-1)^2}\Big)^{2(n-1)^2(n-1)^2+8(n-1)^3+2\rho(n)}$$

$$\ge e^{-2(n-1)^2}\Big(\Big(1-\frac{1}{(n-1)^2}\Big)^{(n-1)^2}\Big)^{2(n-1)^2+8(n-1)+100}$$

$$\ge Ce^{-4(n-1)^2}e^{-8(n-1)} \ge C'e^{-4n^2}. \tag{3.11}$$

We thus see that the first pair of inequalities (3.5) holds. The second one is trivial since for $\tilde m_n < m \le m_{n+1}$, i.e.,

$$n^4 < m \le (n+1)^4 - (n+1)^2,$$

we have $L_m = (1-n^{-2})^{2m} = \tilde L_m$.

(ii): $a = 1$, $b = 2$, $w(r) = 1 - r$. Here

$$m_n = 2^{3/2}n^3 - 2n^2 + 2^{1/2}n - 1$$

and

$$S(H_v^\infty) = \Big\{(b_m)_{m=0}^\infty : \sup_n \frac{e^{-2n^2}}{\sqrt{2}n}\Big(\sum_{\substack{m\in\mathbb N\\ m_n<m\le m_{n+1}}} |b_m|^2\big(1-(\sqrt{2}n)^{-1}\big)^{2m}\Big)^{1/2} < \infty\Big\},$$

$$s(H_v^\infty) = \Big\{(b_m)_{m=0}^\infty : \sup_n \frac{e^{-2n^2}}{\sqrt{2}n}\sum_{\substack{m\in\mathbb N\\ m_n<m\le m_{n+1}}} |b_m|\big(1-(\sqrt{2}n)^{-1}\big)^{m} < \infty\Big\}.$$

(iii): $a = b = 1$, $w(r) = (1-\log(1-r))^{-1}$. Here, a direct calculation yields

$$m_n = n^4 - n^2 + \frac{n^2-1}{1+\log(n^2)}, \tag{3.12}$$

but we can again use Lemma 3.2 with $\tilde m_n = n^4$, $K_n = e^{-n^2}\big(1+\log(n^2)\big)^{-1} = \tilde K_n$, $L_m = (1-n^{-2})^{2m}$ and $\tilde L_m = (1-(n-1)^{-2})^{2m}$, since the calculation (3.8)–(3.11) shows that both the expressions $K_n^2 L_m$ and $\tilde K_{n-1}^2 L_m$ are proportional to

$$\frac{e^{-4n^2}}{\big(1+\log(n^2)\big)^2} \tag{3.13}$$

for all m with $m_n < m \le \tilde m_n$. (To see this, we observe that in comparison with (3.8)–(3.11), $\tilde K_n$ only has the new factor $\big(1+\log(n^2)\big)^{-1} =: g_n$ for which g_n and g_{n-1} are proportional, and that m_n of (3.12) satisfies $n^4 - n^2 < m_n < n^4$ so that m

in (3.13) falls into the interval considered also in (3.8)–(3.11)). Moreover, of course $K_n^2 L_m = \tilde{K}_n^2 L_m$ for all m with $\tilde{m}_n < m \le m_{n+1}$. Thus, we have

$$S(H_v^\infty) = \Big\{(b_m)_{m=0}^\infty : \sup_n \frac{e^{-n^2}}{1+\log(n^2)}\Big(\sum_{m=n^4+1}^{(n+1)^4} |b_m|^2\big(1-n^{-2}\big)^{2m}\Big)^{1/2} < \infty\Big\},$$

$$s(H_v^\infty) = \Big\{(b_m)_{m=0}^\infty : \sup_n \frac{e^{-n^2}}{1+\log(n^2)}\sum_{m=n^4+1}^{(n+1)^4} |b_m|\big(1-n^{-2}\big)^{m} < \infty\Big\}.$$

(iv): $a = b = 1$, $w(r) = \exp(-\log^2(1-r))$. Here we have $w'(r)/w(r) = 2(1-r)^{-1}\log(1-r)$. It is easily seen that (3.2) and (3.3) are satisfied. We obtain $m_n = n^4 - n^2 + 4(n^2-1)\log(n)$ and

$$S(H_v^\infty) = \Big\{(b_m)_{m=0}^\infty : \sup_n \exp(-4\log^2(n) - n^2)\Big(\sum_{\substack{m\in\mathbb{N}\\ m_n < m \le m_{n+1}}} |b_m|^2\big(1-n^{-2}\big)^{2m}\Big)^{1/2} < \infty\Big\},$$

$$s(H_v^\infty) = \Big\{(b_m)_{m=0}^\infty : \sup_n \exp(-4\log^2(n) - n^2)\sum_{\substack{m\in\mathbb{N}\\ m_n < m \le m_{n+1}}} |b_m|\big(1-n^{-2}\big)^{m} < \infty\Big\}.$$

Remark 3.4. Fix $m > 1$ and put

$$f(r) = r^m v(r) = r^m w(r)\exp\Big(-\frac{a}{(1-r)^b}\Big).$$

Due to the continuity of f and the fact that $f(0) = f(1) = 0 \le f(r)$, $r \in]0,1[$, the function f has a global maximum on $]0,1[$. It is easily seen that $r \in]0,1[$ is a zero of f' if and only if

$$m = ab\frac{r}{(1-r)^{b+1}} - r\frac{w'(r)}{w(r)}. \tag{3.14}$$

Since $-rw'(r)/w(r)$ is assumed to be increasing, the right-hand side of (3.14) is strictly increasing in r. Hence (3.14) has exactly one solution, denoted by r_m, which is the unique global maximum of f. In particular, if

$$M = abm^{1+1/b}\Big(1-\Big(\frac{1}{m}\Big)^{1/b}\Big) - \Big(1-\Big(\frac{1}{m}\Big)^{1/b}\Big)\frac{w'\Big(1-\big(\frac{1}{m}\big)^{1/b}\Big)}{w\Big(1-\big(\frac{1}{m}\big)^{1/b}\Big)} \tag{3.15}$$

for some $m > 1$, then

$$r_M = 1 - \Big(\frac{1}{m}\Big)^{1/b}. \tag{3.16}$$

Proof of Theorem 3.1. 1°. We first consider that case $b \ge 1$.

a) Some estimates. If $1 \le x \le y$ and $0 < \beta < 1$ then the mean value theorem yields

$$y^\beta - x^\beta \le \beta x^{\beta-1}(y-x) \quad \text{and} \quad x^\beta - y^\beta \le \beta y^{\beta-1}(x-y). \tag{3.17}$$

Moreover we use

$$1 + x \le e^x \quad \text{for all } x \in \mathbb{R}. \tag{3.18}$$

Now let $1 < m \le k$ and define M as in (3.15) and K in the same way with k replacing m. Then

$$r_M = 1 - \frac{1}{m^{1/b}} \quad \text{and} \quad r_K = 1 - \frac{1}{k^{1/b}}$$

and we can rewrite (3.15) as

$$M = abm^{1+1/b} r_M - r_M \frac{w'(r_M)}{w(r_M)} \ , \quad K = abk^{1+1/b} r_K - r_K \frac{w'(r_K)}{w(r_K)}. \tag{3.19}$$

Then, with (3.17), (3.18) we obtain

$$\begin{aligned} \frac{r_K}{r_M} &= \frac{1 - \big(\frac{1}{k}\big)^{1/b}}{1 - \big(\frac{1}{m}\big)^{1/b}} = 1 + \frac{\big(\frac{1}{m}\big)^{1/b} - \big(\frac{1}{k}\big)^{1/b}}{1 - \big(\frac{1}{m}\big)^{1/b}} \\ &\le \exp\left(\frac{1}{b} \Big(\frac{1}{k}\Big)^{1/b-1} \Big(\frac{1}{m} - \frac{1}{k}\Big) \frac{1}{r_M} \right) = \exp\left(\frac{1}{b} \ \frac{k-m}{k^{1/b} m} \ \frac{1}{r_M} \right) \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \frac{r_M}{r_K} &= \frac{1 - \big(\frac{1}{m}\big)^{1/b}}{1 - \big(\frac{1}{k}\big)^{1/b}} = 1 - \frac{\big(\frac{1}{m}\big)^{1/b} - \big(\frac{1}{k}\big)^{1/b}}{1 - \big(\frac{1}{k}\big)^{1/b}} \\ &\le \exp\left(- \frac{1}{b} \ \frac{k-m}{m^{1+1/b}} \ \frac{1}{r_K} \right). \end{aligned} \tag{3.21}$$

Now we write

$$\Big(\frac{r_K}{r_M}\Big)^K \frac{v(r_K)}{v(r_M)} = \exp\left(K \log\Big(\frac{r_K}{r_M}\Big) \right) \frac{w(r_K)}{w(r_M)} \exp(-a(k-m))$$

which in view of (3.19), (3.20) is bounded by

$$\begin{aligned} &\frac{w(r_K)}{w(r_M)} \ \exp\left(a \frac{k(k-m)}{m} \ \frac{r_K}{r_M} - \frac{1}{b} \ \frac{r_K}{r_M} \ \frac{w'(r_K)}{w(r_M)} \ \frac{k-m}{k^{1/b} m} - a(k-m) \right) \\ &= \frac{w(r_K)}{w(r_M)} \ \exp\left(- a(k-m) + a \Big(\frac{k - k^{1-1/b}}{m - m^{1-1/b}} \Big) (k-m) + c_1(k,m) \right. \\ &= \frac{w(r_K)}{w(r_M)} \ \exp\left(a \frac{(k-m)^2}{m - m^{1-1/b}} \Big(1 - \frac{k^{1-1/b} - m^{1-1/b}}{k-m} \Big) + c_1(k,m) \right) \end{aligned} \tag{3.22}$$

where

$$c_1(k,m) = -\frac{1}{b} \ \frac{r_K}{r_M} \ \frac{w'(r_K)}{w(r_K)} (1 - r_K) \frac{k-m}{m}. \tag{3.23}$$

Using (3.21) instead of (3.20) we get with the help of (3.17)

$$\begin{aligned} &\Big(\frac{r_M}{r_K}\Big)^K \frac{v(r_M)}{v(r_K)} \\ &\le \frac{w(r_M)}{w(r_K)} \exp\left(a(k-m) - a(k-m) \left(\frac{k}{m}\right)^{1/b} + c_2(k,m) \right) \\ &= \frac{w(r_M)}{w(r_K)} \exp\left(a(k-m) \Big(\frac{m^{1/b} - k^{1/b}}{m^{1/b}} \Big) + c_2(k,m) \right) \\ &\le \frac{w(r_M)}{w(r_K)} \exp\left(-\frac{a}{b} (k-m)^2 \frac{1}{k^{1-1/b} m^{1/b}} + c_2(k,m) \right) \end{aligned} \tag{3.24}$$

where

$$c_2(k,m) = \frac{1}{b}\left(\frac{1}{m}\right)^{1/b} \frac{w'(r_K)}{w(r_K)}\ \frac{k-m}{k}. \tag{3.25}$$

Similarly, (3.20) implies

$$\begin{aligned}
&\left(\frac{r_K}{r_M}\right)^M \frac{v(r_K)}{v(r_M)} \\
&\le \frac{w(r_K)}{w(r_M)} \exp\left(-a(k-m) + a\Big(\frac{m}{k}\Big)^{1/b}(k-m) + c_3(k,m)\right) \\
&= \frac{w(r_K)}{w(r_M)} \exp\left(-a(k-m)\Big(\frac{k^{1/b}-m^{1/b}}{k^{1/b}}\Big) + c_3(k,m)\right) \\
&\le \frac{w(r_K)}{w(r_M)} \exp\left(-\frac{a}{b}(k-m)^2\frac{1}{k} + c_3(k,m)\right) \qquad (3.26)
\end{aligned}$$

with

$$c_3(k,m) = -\frac{1}{b}\ \frac{w'(r_M)}{w(r_M)}\ \frac{k-m}{m}\ (1-r_K). \tag{3.27}$$

Finally, (3.21) implies

$$\begin{aligned}
&\left(\frac{r_M}{r_K}\right)^M \frac{v(r_M)}{v(r_K)} \\
&= \frac{w(r_M)}{w(r_K)} \exp\left(a(k-m) - a\frac{m}{k}(k-m)\frac{r_M}{r_K} + c_4(k,m)\right) \\
&= \frac{w(r_M)}{w(r_K)} \exp\left(a\frac{(k-m)^2}{k-k^{1-1/b}}\Big(1-\frac{k^{1-1/b}-m^{1-1/b}}{k-m}\Big) + c_4(k,m)\right) \qquad (3.28)
\end{aligned}$$

with

$$c_4(k,m) = \frac{1}{b}\frac{r_M}{r_K}\ \frac{w'(r_M)}{w(r_M)}\ \frac{k-m}{m^{1/b}k}. \tag{3.29}$$

b) The parameters m_n. Now put for every $n \in \mathbb{N}$ large enough

$$j_n = \frac{b}{a}n^2 \tag{3.30}$$

and in the above calculations choose

$$m = j_n \quad , \quad k = j_{n+1}.$$

We denote the numbers in (3.19) by

$$m_n = M \quad , \quad m_{n+1} = K$$

so that the following relations hold, by (3.16):

$$r_{m_n} = 1 - \frac{1}{j_n^{1/b}} = 1 - \left(\frac{a}{bn^2}\right)^{1/b} \quad , \quad r_{m_{n+1}} = 1 - \left(\frac{a}{b(n+1)^2}\right)^{1/b}.$$

c) Final estimates. With (3.23) we obtain

$$c_1(j_{n+1}, j_n) = -\frac{1}{b}\Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)\frac{w'(r_{m_{n+1}})}{w(r_{m_{n+1}})}(1-r_{m_{n+1}})\ \frac{2n+1}{n^2}$$

By assumption (3.2) there is $d > 0$ and $0 < \alpha < 1 + b/2$ with

$$\frac{w'(r_{m_{n+1}})}{w(r_{m_{n+1}})} \le \frac{d}{(1 - r_{m_{n+1}})^{\alpha}}.$$

Hence

$$\begin{aligned}|c_1(j_{n+1}, j_n)| &\le \frac{d}{b}\Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)(1 - r_{m_{n+1}})^{1-\alpha}\Big(\frac{2n+1}{n^2}\Big)\\ &= \frac{d}{b}\Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)\Big(\frac{a}{b}\Big)^{(1-\alpha)/b}(n+1)^{2(\alpha-1)/b}\Big(\frac{2n+1}{n^2}\Big).\end{aligned}$$

Since $2(\alpha - 1)/b < 1$ we obtain

$$\lim_{n\to\infty} c_1(j_{n+1}, j_n) = 0$$

By assumption (3.3) we have $w(r_{m_{n+1}})/w(r_{m_n}) \le e$. So, using (3.22) and (3.30) we see that there is a constant K_0 with

$$\Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \le K_0 \quad \text{for all } n. \tag{3.31}$$

This follows from the fact that, for $k = j_{n+1}$ and $m = j_n$, the expression in (3.22),

$$\frac{(k-m)^2}{m - m^{1/b}}\ \frac{1 - k^{1-1/b} - m^{1-1/b}}{k - m},$$

remains uniformly bounded for all n.

To obtain a lower estimate of

$$\Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)^{m_{n+1}}\ \frac{v(r_{m_{n+1}})}{v(r_{m_n})}$$

consider (3.24) and (3.25). Exactly as before we see that

$$\lim_{n\to\infty} c_2(j_{n+1}, j_n) = 0 \tag{3.32}$$

For $k = j_{n+1}$ and $m = j_n$ we obtain

$$\Big(\frac{a}{b}\Big)\frac{(k-m)^2}{k^{1-1/b}m^{1/b}} = \frac{(2n+1)^2}{(n+1)^{2-2/b}n^{2/b}}$$

which tends to 4 as $n \to \infty$. Together with (3.32) we find n_0 such that

$$-\Big(\frac{a}{b}\Big)\frac{(j_{n+1} - j_n)^2}{j_{n+1}^{1-1/b} j_n^{1/b}} + c_2(j_{n+1}, j_n) \le -2 \quad \text{for } n \ge n_0.$$

Since by assumption $w(r_{m_n})/w(r_{m_{n+1}}) \le e$ the estimate (3.24) implies

$$\Big(\frac{r_{m_n}}{r_{m_{n+1}}}\Big)^{m_{n+1}} \frac{v(r_{m_n})}{v(r_{m_{n+1}})} \le \frac{1}{e},$$

hence

$$2 < e \le \Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \quad \text{for } n \ge n_0. \tag{3.33}$$

Repeating the preceding arguments using (3.26), (3.27), (3.28), (3.29) instead of (3.22), (3.23), (3.24), (3.25) we see that

$$\lim_{n\to\infty} c_3(j_{n+1}, j_n) = \lim_{n\to\infty} c_4(j_{n+1}, j_n) = 0$$

and there are n_1, N_1 with

$$2 < e \leq \Big(\frac{r_{m_n}}{r_{m_{n+1}}}\Big)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})} \leq K_1 \quad \text{for } n \geq n_1. \tag{3.34}$$

Then the assertion of the theorem in the case $b \geq 1$ follows from (3.31), (3.33), (3.34) and [8], Theorem 2.1.

2°. We prove Theorem 3.1 in the case $0 < b < 1$. Here we use, for $\gamma > 1$, $0 \leq x \leq y$,

$$y^\gamma - x^\gamma \leq \gamma y^{\gamma-1}(y-x) \quad \text{and} \quad x^\gamma - y^\gamma \leq \gamma x^{\gamma-1}(x-y).$$

We obtain, instead of (3.20) and (3.20),

$$\frac{1-\big(\frac{1}{k}\big)^{1/b}}{1-\Big(\frac{1}{m}\Big)^{1/b}} = 1 + \frac{\big(\frac{1}{m}\big)^{1/b} - \big(\frac{1}{k}\big)^{1/b}}{1-\big(\frac{1}{m}\big)^{1/b}} \leq \exp\left(\frac{1}{b}\Big(\frac{1}{m}\Big)^{1/b} \frac{k-m}{\Big(1-\big(\frac{1}{m}\big)^{1/b}\Big)k}\right)$$

and

$$\frac{1-\big(\frac{1}{m}\big)^{1/b}}{1-\big(\frac{1}{k}\big)^{1/b}} = 1 - \frac{\big(\frac{1}{m}\big)^{1/b} - \big(\frac{1}{k}\big)^{1/b}}{1-\big(\frac{1}{k}\big)^{1/b}} \leq \exp\left(-\frac{1}{b}\Big(\frac{1}{k}\Big)^{1/b} \frac{k-m}{\Big(1-\big(\frac{1}{k}\big)^{1/b}\Big)m}\right).$$

Then the theorem follows by repeating the same arguments as in the preceding section. □

4. A weight of the form $v_2(z) = \exp(-a/(1-r^2)^b)$.

As a consequence of the preceding discussion we consider here the weight

$$v_2(z) = \exp\Big(\frac{-a}{(1-r^2)^b}\Big)$$

for given constants $a, b > 0$. We compare v_2 with the weight $v_1(z) = \exp(-a/(1-r)^b)$ of the preceding section (with $w \equiv 1$).

Put

$$A = \Big\{f \in H^\infty_{v_2} : f(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k} \text{ for some } a_{2k}\Big\}$$

and

$$B = z \cdot A = \Big\{g \in H^\infty_{v_2} : g(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1} \text{ for some } a_{2k+1}\Big\}.$$

Moreover, let $T_1, T_2 : H^\infty_{v_1} \to H^\infty_{v_2}$ be the maps with

$$(T_1 h)(z) = h(z^2) \quad \text{and} \quad (T_2 h)(z) = z h(z^2), \quad h \in H^\infty_{v_1}, z \in \mathbb{D}.$$

Proposition 4.1. *The operator T_1 maps $H^\infty_{v_1}$ isometrically onto A. The map T_2 is a contractive operator from $H^\infty_{v_1}$ onto B. Moreover, we have*

$$H^\infty_{v_2} = A \oplus B.$$

Proof. The map T_1 is certainly an isometry into A and T_2 is a contractive operator into B. To show the surjectivity, let $f \in B$, say

$$f(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}. \tag{4.1}$$

Then put $h(z) = \sum_{k=0}^{\infty} a_{2k+1} z^k$. In view of (4.1), since the series representing $f(z)/z$ converges uniformly on compact subsets of $\mathbb{D}$, we have

$$c := \sup_{|z| \le 1/2} \frac{|f(z)|}{|z|} < \infty.$$

Hence,

$$\|h\|_{v_1} = \sup_{|z|<1} |h(z)| v_1(z) = \sup_{|z|<1} |h(z^2)| v_1(z^2) = \sup_{|z|<1} \frac{|f(z)|}{|z|} v_2(z)$$

$$= \max\Big(\sup_{|z|\le 1/2} \frac{|f(z)|}{|z|} v_2(z), \sup_{1/2<|z|<1} \frac{|f(z)|}{|z|} v_2(z) \Big) \tag{4.2}$$

We obtain $h \in H^\infty_{v_1}$ and clearly $T_2 h = f$. This shows that T_2 maps $H^\infty_{v_1}$ onto B. Similarly we see that T_1 maps $H^\infty_{v_1}$ onto A.

Now consider the operator P with $(Pf)(z) = (f(z) + f(-z))/2$ for $f \in H^\infty_{v_2}$ and $z \in \mathbb{D}$. P is a contractive projection from $H^\infty_{v_2}$ onto A. We clearly get $(id - P)(H^\infty_{v_2}) = B$. Hence $H^\infty_{v_2} = A \oplus B$. $\Box$

Now we take the numbers m_n of Theorem 3.1 for $w \equiv 1$, i.e.

$$m_n = \frac{b^{1+1/b}}{a^{1/b}}\, n^{2+2/b} - bn^2.$$

Let $[s]$ denote the largest integer which is smaller than or equal to s.

Theorem 4.2. *The solid hull of $H^\infty_{v_2}$ is equal to*

$$\Big\{ (b_m) : \sup_n e^{-bn^2} \Big(\sum_{\substack{m\in\mathbb{N} \\ 2[m_n]+1<m\le 2[m_{n+1}]+1}} |b_m|^2 \Big(1 - \Big(\frac{a}{bn^2}\Big)^{1/b}\Big)^{2[m/2]} \Big)^{1/2} < \infty \Big\}.$$

Moreover, the solid core of $H^\infty_{v_2}$ is equal to

$$\Big\{ (b_m) : \sup_n e^{-bn^2} \sum_{\substack{m\in\mathbb{N} \\ 2[m_n]+1<m\le 2[m_{n+1}]+1}} |b_m| \Big(1 - \Big(\frac{a}{bn^2}\Big)^{1/b}\Big)^{[m/2]} < \infty \Big\}.$$

Proof. Using Proposition 4.1 and Theorem 3.1 we see that $(b_m) \in S(H^\infty_{v_2})$ if and only if

$$\sup_n e^{-bn^2} \Big(\sum_{\substack{m\in\mathbb{N} \\ m_n<m\le m_{n+1}}} |b_{2m}|^2 \Big(1 - \Big(\frac{a}{bn^2}\Big)^{1/b}\Big)^{2m} \Big)^{1/2} < \infty \tag{4.3}$$

and

$$\sup_n e^{-bn^2} \Big(\sum_{\substack{m\in\mathbb{N} \\ m_n<m\le m_{n+1}}} |b_{2m+1}|^2 \Big(1 - \Big(\frac{a}{bn^2}\Big)^{1/b}\Big)^{2m} \Big)^{1/2} < \infty. \tag{4.4}$$

If $m \in \mathbb{N}$ and $m_n < m \le m_{n+1}$ then $[m_n]+1 \le m \le [m_{n+1}]$. We obtain $2[m_n]+2 \le 2m \le 2[m_{n+1}]$ and $2[m_n]+3 \le 2m+1 \le 2[m_{n+1}]+1$. Hence (4.3) and (4.4) are equivalent to

$$\sup_n e^{-bn^2}\Big(\sum_{\substack{m\in\mathbb{N}\\ 2[m_n]+1<m\le 2[m_{n+1}]+1}} |b_m|^2\Big(1-\Big(\frac{a}{bn^2}\Big)^{1/b}\Big)^{2[m/2]}\Big)^{1/2} < \infty.$$

The proof for the solid core is the same. □

5. Maximal solid cores and minimal solid hulls.

In this section we prove some general results on the relations of Schauder bases and solid hulls and cores for H_v^∞-spaces. We refer the reader to [14] for terminology about bases in Banach spaces. If the monomials form a Schauder basic sequence in H_v^∞, then obviously the condition of being solid is related with the property of $\{z^m\}$ being an unconditional basis. Another, related fact will be proven in Theorem 5.2. We also show the unexpected fact that for some special weights, H_v^∞ is solid.

Proposition 5.1. *We have $S(H_v^\infty) = H_v^\infty$ if and only if $s(H_v^\infty) = H_v^\infty$.*

Proof. If $S(H_v^\infty) = H_v^\infty$ then $h_f \in H_v^\infty$ (see (2.4)) for all $f \in H_v^\infty$. Hence $s(H_v^\infty) = H_v^\infty$.

Now assume $s(H_v^\infty) = H_v^\infty$ and take $g \in S(H_v^\infty)$ with $g(z) = \sum_{k=0}^\infty b_k z^k$. There is $f \in H_v^\infty$ with $f(z) = \sum_{k=0}^\infty a_k z^k$ and $|b_k| \le |a_k|$ for all k. Since by assumption $h_f \in H_v^\infty$ we obtain

$$\|g\|_v \le \sup_r v(r)\sum_{k=0}^\infty |a_k| r^k = \|h_f\|_v < \infty$$

which implies $g \in H_v^\infty$. Hence $S(H_v^\infty) = H_v^\infty$. □

Example. Consider the weight $v(r) = \exp(-\log^2(r))$ on the complex plane $\mathbb{C}$. According to [15], Theorem 2.5., there is a constant $d > 0$ such that for every $f \in H_v^\infty$ with $f(z) = \sum_{k=0}^\infty a_k z^k$ we have

$$\sup_k(|a_k|\exp(k^2/4)) \le \|f\|_v \le d\sup_k(|a_k|\exp(k^2/4)).$$

Then clearly $h_f \in H_v^\infty$. Indeed, let $(h_f)_n$ be the partial sums of h_f, i.e. $(h_f)_n(z) = \sum_{k=0}^n |a_k| z^k$. Then $(h_f)_n \in H_v^\infty$, $(h_f)_n \to h_f$ pointwise on $\mathbb{C}$ and

$$\|h_f\|_v \le \sup_n \|(h_f)_n\|_v \le d\sup_k(|a_k|\exp(k^2/4)) \le d\|f\|_v < \infty.$$

Hence $S(H_v^\infty) = H_v^\infty = s(H_v^\infty)$.

Recall that we denote by H_v^0 the closure of the polynomials in H_v^∞. We put $\Lambda = \{z^k : k = 0, 1, 2, \ldots\}$.

Theorem 5.2. *If $S(H_v^\infty) = H_v^\infty$ then Λ is a Schauder basis of H_v^0.*

We prove Theorem 5.2 at the end of this section. At first we state

Corollary 5.3. *In the case of analytic functions on the disc $\mathbb{D}$, one always has $S(H_v^\infty(\mathbb{D})) \ne H_v^\infty(\mathbb{D})$ and $s(H_v^\infty(\mathbb{D})) \ne H_v^\infty(\mathbb{D})$.*

Proof. According to [15], Theorem 2.2., Λ is never a basis for $H_v^0(\mathbb{D})$. This proves Corollary 5.3 in view of Theorem 5.2 $\square$

For the proof of Theorem 5.2 we need two lemmas.

Lemma 5.4. *(i) Fix $m \in \mathbb{N}$ and $\epsilon > 0$. Then there is $r_0 < R$ such that, for every f with $f(z) = \sum_{k=0}^m a_k z^k$, we have*

$$\sup_{r_0 \le |z| < R} |f(z)|v(z) \le \epsilon \|f\|_v.$$

(ii) Fix $0 \le r_1 < R$ and $\epsilon > 0$. Then there is $n \in \mathbb{N}$ such that, for any $g \in H_v^\infty$ with $g(z) = \sum_{k=n}^\infty a_k z^k$, we have

$$\sup_{0 \le |z| \le r_1} |g(z)|v(z) \le \epsilon \|g\|_v.$$

Proof. (i) Fix $r < R$. Then we clearly have

$$|a_k| \le \frac{\|f\|_v}{r^k v(r)} \quad \text{for all } k = 0, 1, \ldots, m.$$

We obtain

$$|f(z)|v(z) \le \sum_{k=0}^m \Big(\frac{|z|}{r}\Big)^k \frac{v(|z|)}{v(r)} \|f\|_v.$$

Find $r_0 > 0$ such that

$$\Big(\frac{|z|}{r}\Big)^k \frac{v(|z|)}{v(r)} \le \frac{\epsilon}{m+1}$$

whenever $|z| > r_0$. This is possible since, by assumption, $\lim_{|z| \to R} |z|^k v(|z|) = 0$ for all k. This implies (i).

(ii) Fix $r > r_1$ and consider $g(z) = \sum_{k=n}^\infty a_k z^k$. We have $|a_k| \le \|g\|_v / (r^k v(r))$ for all k. This implies, if $|z| \le r_1$,

$$|g(z)|v(z) \le \sum_{k=n}^\infty \frac{|z|^k v(|z|)}{r^k v(r)} \|g\|_v \le \sum_{k=n}^\infty \Big(\frac{r_1}{r}\Big)^k \frac{v(0)}{v(r)} \|g\|_v.$$

We find n so large that

$$\sum_{k=n}^\infty \Big(\frac{r_1}{r}\Big)^k \frac{v(0)}{v(r)} \le \epsilon$$

which proves the lemma. $\square$

Lemma 5.5. *Let $f = \sum_{j=0}^\infty a_j z^j$ be an analytic function on the disc, let $m_1 < m_2 < \ldots$ be indices and $f_n(z) = \sum_{j=m_n+1}^{m_{n+1}} a_j z^j$. Then there is a subsequence $(f_{n_k})_{k=0}^\infty$ such that*

$$\sup_{k \in \mathbb{N}} \|f_{n_k}\|_v \le 2 \Big\| \sum_{k=0}^\infty f_{n_k} \Big\|_v. \tag{5.1}$$

We remark that for any subsequence $(f_{n_k})_{k=0}^\infty$, the sum $\sum_k f_{n_k}$ on the right-hand side of (5.1) is the Taylor series of an analytic function on the disc, so the sum converges at least uniformly on compact subsets of $\mathbb{D}$; if the sum does not belong to H_v^∞, its norm is infinity and the inequality (5.1) becomes a triviality.

Proof. Use Lemma 5.4 and induction to find a subsequence (f_{n_k}) and radii $r_1 < r_2 < \dots$ such that $|f_{n_k}(z)|v(|z|) \le 3^{-k}\|f_{n_k}\|_v$ whenever $|z| \le r_k$ or $|z| \ge r_{k+1}$. Hence

$$\|f_{n_k}\|_v = \sup_{r_k \le |z| \le r_{k+1}} |f_{n_k}(z)|v(|z|). \tag{5.2}$$

By the remark above, we may assume that $\sum_k f_{n_k} \in H_v^\infty$. Fix j. If $r_j \le |z| \le r_{j+1}$ we obtain

$$\begin{aligned}
\|\sum_k f_{n_k}\|_v &\ge |\sum_k f_{n_k}(z)|v(z) \\
&\ge |f_{n_j}(z)|v(z) - \sum_{k \ne j} |f_{n_k}(z)|v(z) \\
&\ge |f_{n_j}(z)|v(z) - \sum_{k \ne j} \frac{1}{3^k}\|f_{n_k}\|_v \\
&\ge |f_{n_j}(z)|v(z) - \frac{1}{2}\sup_k \|f_{n_k}\|_v.
\end{aligned}$$

In view of (5.2) this implies

$$\|\sum_k f_{n_k}\|_v \ge \|f_{n_j}\|_v - \frac{1}{2}\sup_k \|f_{n_k}\|_v \quad \text{for all } j$$

and hence

$$\|\sum_k f_{n_k}\|_v \ge \frac{1}{2}\sup_k \|f_{n_k}\|_v$$

which proves the lemma. □

Proof of Theorem 5.2. For any subset N of $\mathbb{N}$, let T_N be the operator with $T_N(\sum_{k=0}^\infty a_k z^k) = \sum_{k\in N} a_k z^k$. If $S(H_v^\infty) = H_v^\infty = s(H_v^\infty)$ then $T_N(H_v^\infty) \subset H_v^\infty$. The closed graph theorem implies that T_N is bounded.

Now let P_n be the Dirichlet projections, i.e. $P_n(\sum_{k=0}^\infty a_k z^k) = \sum_{k=0}^n a_k z^k$. Assume that Λ is not a basis for H_v^0. Then the P_n are not uniformly bounded. By the uniform boundedness theorem we obtain a function $f \in H_v^0$ such that $\sup_n \|P_n(f)\|_v = \infty$. Hence we can find a subsequence P_{n_m} with $\lim_{m\to\infty} \|(P_{n_{m+1}} - P_{n_m})(f)\|_v = \infty$. Put $f_m = (P_{n_{m+1}} - P_{n_m})(f)$. Then, $\sum_m f_m \in H_v^\infty$, since this sum is of the form $T_N f$ for some subset N of $\mathbb{N}$. We apply Lemma 5.5 to find a subsequence f_{m_k} such that

$$\sup_k \|f_{m_k}\|_v \le 2\|\sum_k f_{m_k}\|_v.$$

The left hand side of this inequality is infinite while the function on the right-hand side is again of the form $T_{\tilde{N}} f$ for some $\tilde{N} \subset \mathbb{N}$ and thus has finite norm as an element H_v^∞. So we arrive at a contradiction. Therefore Λ is a basis of H_v^0. □

Acknowledgements. The research of Bonet was partially supported by the project MTM2016-76647-P. The research of Taskinen was partially supported by the Väisälä Foundation of the Finnish Academy of Sciences and Letters.

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