Some questions about spaces of Dirichlet series.

José Bonet

Instituto Universitario de Matemática Pura y Aplicada

Universitat Politècnica de València

Eichstaett, June 2017

Dirichlet Series

A **Dirichlet Series** is an expression of the form $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with $s \in \mathbb{C}$.

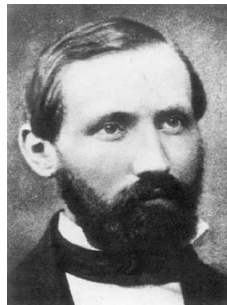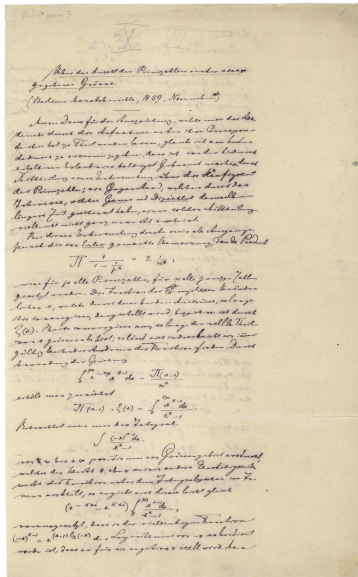**Riemann's zeta function:** If $s \in \mathbb{C}$,

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Euler 1737 considered it for s real:

- He proved $\zeta(2) = \pi^2/6$,

- and he relation with the prime numbers:

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1},$$

the product extended to all the prime numbers p.

Riemann (1826-1866) and his Memoir "Über die Anzahl der Primzahlen unter einer gegebenen Grösse", in which he formulated the "Riemann Hypothesis".

Riemann

- The series $\zeta(s)$ converges if the real part of s is greater than 1, and $\zeta(s)$ has an analytic extension to all the complex plane except $s = 1$, where it has a simple pole.

- Obtained the functional equation.

$$\zeta(s) = 2^s \pi^{s-1} \text{sen}(\pi s/2) \Gamma(1-s) \zeta(1-s), \quad s \in \mathbb{C}.$$

- Proved that there are no zeros of $\zeta(s)$ outside $0 \leq \Re(s) \leq 1$, except the trivial zeros $s = -2, -4, -6, \ldots$

Riemann's Hypothesis

- Riemann **conjectured that all the non-trivial zeros of $\zeta(s)$ are in the line $\Re(s) = 1/2$.**

- His work was essential for **the prime number theorem of Hadamard and de la Vallée Poussin (1896):**

$$\lim_{n \to \infty} \frac{\pi(n) \log n}{n} = 1,$$

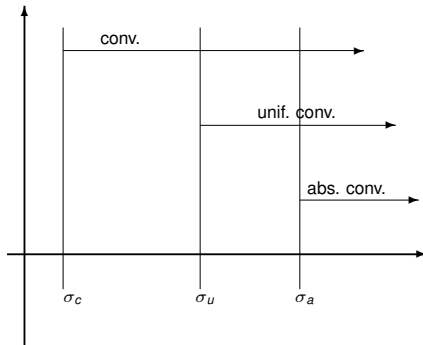$\pi(n) = $ number of primes smaller or equal than $n \in \mathbb{N}$.

Given a Dirichlet Series $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ in $\mathbb{C}$ we define, with $\mathbb{C}_r = [\mathbf{Re}\, s > r]$,

$$\sigma_a(D) := \text{ínf}\{r \in \mathbb{R} \;:\; \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ is abs. conv. in the half plane } \mathbb{C}_r\}$$

$$\sigma_u(D) := \text{ínf}\{r \in \mathbb{R} \;:\; \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ is unif. conv. in the half plane } \mathbb{C}_r\}$$

$$\sigma_c(D) := \text{ínf}\{r \in \mathbb{R} \;:\; \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ is conv. in the half plane } \mathbb{C}_r\}$$

Dirichlet Series

$\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D)$ for each Dirichlet series D.

Moreover $D(s)$ defines a holomorphic function on
$\mathbb{C}_{\sigma_c(D)} = [\textbf{Re } s > \sigma_c(D)]$.

Proposition

Given a Dirichlet Series $D = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ in $\mathbb{C}$ one has

$$0 \leq \sigma_a(D) - \sigma_c(D) \leq 1,$$

but

$$0 \leq \sigma_a(D) - \sigma_u(D) \leq \frac{1}{2}.$$

The second one is not completely trivial.

Proposition

(i)
$$\sup_{D}(\sigma_a(D) - \sigma_c(D)) = 1.$$

(ii)
$$\sup_{D}(\sigma_u(D) - \sigma_c(D)) = 1.$$

(i) $D = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$ satisfies $\sigma_a(D) = 1$ and $\sigma_c(D) = 0$.

(ii) $(p_n)_n$ prime numbers. $D = \sum_{n=1}^{\infty} \frac{(-1)^n}{(p_n)^s}$ satisfies $\sigma_u(D) = 1$ (the proof requires Kronecker's theorem and the prime number theorem) and $\sigma_c(D) = 0$.

Definition

$$S := \sup \left\{ \sigma_a(D) - \sigma_u(D) \mid D = \sum_n a_n \frac{1}{n^s} \text{ Dirichlet series} \right\}.$$

Problem of absolute convergence of Bohr

$$S = \ ?$$

Theorem, Bohr (1913), Bohnenblust, Hille (1931)
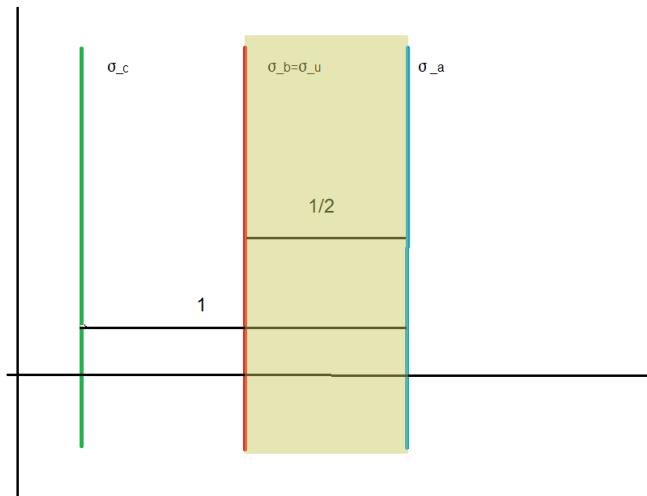
$$S = \frac{1}{2}$$

Bohr's absolute convergence problem

- Given a Dirichlet Series $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ in $\mathbb{C}$ we define the **bounded abscissa** $\sigma_b(D)$ as the infimum of $r \in \mathbb{R}$ such that $D(s)$ defines a bounded holomorphic function in the half plane $\mathbb{C}_r = [\text{Re } s > r]$.

- $\sigma_c(D) \leq \sigma_b(D) \leq \sigma_u(D) \leq \sigma_a(D)$ for each Dirichlet series D.

Theorem. Bohr, 1913

$\sigma_b(D) = \sigma_u(D)$ for each Dirichlet series D.

Bohr's absolute convergence problem

Harald Bohr (1887-1951)

- Younger brother of Niels Bohr, Nobel Prize in Physics 1922.

- Football player. He played for Denmark national team. Silver medal in Olympic Games in London in 1908. They defeated France 17-1. England won the tournament.

- He presented his Ph.D. thesis "Contributions to the theory of Dirichlet series" in 1910. The audience consisted mainly of team mates and soccer fans.

- He had a long fruitful collaboration with Landau.

- Perhaps his most famous contribution is the theory of almost periodic functions.
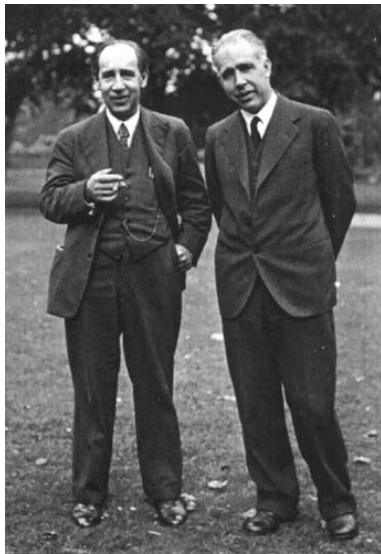
Landau

Bohr

Dirichlet series and complex analysis on polydiscs. Bohr's vision

Definition

$p =$ the sequence of prime numbers: $p_1 < p_2 < p_3 < \ldots$

$p^\alpha = p_1^{\alpha_1} \times \cdots \times p_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots) \in \mathbb{N}_0^{(\mathbb{N})}$

The Bohr transform

$$\mathcal{B} : \sum_n a_n \frac{1}{n^s} \in \mathcal{D} \to \sum_\alpha c_\alpha z^\alpha \in \mathcal{P},$$

with

$$a_n = a_{p^\alpha} = c_\alpha.$$

It is a one to one correspondence between the formal Dirichlet series and the formal power series by the fundamental theorem of prime numbers.

The space $\mathcal{H}_\infty$

$\mathcal{H}_\infty$ is the space of all those Dirichlet series D which converge on $\mathbb{C}_0 := [\text{Re} > 0]$ such that the holomorphic function $D(s)$ is bounded on this halfplane.

It is a Banach algebra under the norm $||D||_\infty := \sup_{s \in \mathbb{C}_0} |D(s)|$.

The space $H_\infty(B_{c_0})$

$H_\infty(B_{c_0})$ is the space of all formal series $\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ such that for each N the series, $\sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha$ defines a holomorphic and bounded function f_N in the polydisk $\mathbb{D}^N$ and, moreover,

$$\sup_N \|f_N\|_\infty = \sup_N \sup_{z \in \mathbb{D}^N} \{ \big| \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha \big| \} < \infty.$$

It coincides with the space of bounded holomorphic functions f on the unit ball of the Banach space c_0 of complex null sequences, and $||f||_\infty = \sup_N \|f_N\|_\infty$.

It is also a Banach algebra.

Theorem. Bohr (2013), Hedenmalm, Lindqvist, Seip, Duke Math. J. (1997))

The Bohr transform $\mathcal{B}$ is a Banach algebra isometry between $\mathcal{H}_\infty$ and $H_\infty(B_{c_0})$.

Hedenmalm, Lindqvist, Seip solved a problem of Beurling from 1945.

Theorem. Hedenmalm, Lindqvist, Seip, Duke Math. J. (1997))

The function $\varphi(x) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin(n\pi x) \in L_2(0,1)$ satisfies that $(\varphi(n.))_n$ is a Riesz basis of $L_2(0,1)$ (i.e. orthonormal with respect to an equivalent norm) if and only if $S\varphi(x) := \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}_{\infty}$ and it is bounded way from 0 in $\mathbb{C}_0$.

Important tool: The Hilbert space $\mathcal{H}_2$ of Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.

Solution of Bohr's problem.

Reformulation of Bohr's problem

$S = \sup_D(\sigma_a(D) - \sigma_u(D)) = \sup\{\sigma_a(D) \mid D \in \mathcal{H}_\infty\}.$

An example of Toeplitz (1913) proved $1/4 \leq S \leq 1/2$.

The final solution was obtained by **Bohnenblust and Hille (Annals Math. 1931)**. They presented the following inequality that implies $S = 1/2$.

Bohnenblust and Hille inequality

Theorem.

There is $C > 1$ such that for each N and for each m homogeneous polynomial $P(z) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m} c_\alpha(P) z^\alpha$ the following holds:

$$\Big(\sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m} \big| c_\alpha(P) \big|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq C^m \sup_{z \in B_{c_0}} |P(z)|.$$

The exponent $2m/(m+1)$ is optimal. The constant C can be taken as close to 1 as necessary.

This is an extension of Littlewood's 4/3 inequality for 2-homogeneous polynomials in $\mathbb{C}^N$. It was improved by Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip in 2011, and again by Bayart, Pellegrino and Seoane in 2014.

Solution of Bohr's problem.

- Bohr's transform maps the space $\mathcal{P}_m(c_0)$ of m-homogeneous continuous polynomials on c_0 bijectively onto $\mathcal{H}_\infty^m$, the space of all $\sum_n a_n n^{-s} \in \mathcal{H}_\infty$ such that $a_n \neq 0$ implies that the number of prime divisor of n counting multiplicity is m.

- Bohnenblust and Hille inequality implies that $S_m := \sup\{\sigma_a(D) \mid D \in \mathcal{H}_\infty^m\} \leq \frac{m-1}{2m}$.

- The proof that $S_m \geq \frac{m-1}{2m}$ requires the construction of certain continuous m-homogeneous polynomials on c_0.

- Since $S_m = \frac{m-1}{2m} \leq S \leq 1/2$ for each m, one concludes $S = 1/2$.

Dirichlet series and complex analysis on polydiscs

Bohr's power series theorem, Math. Ann. 1914

- For all $z \in \mathbb{D}$, $|z| < 1/3$ and for all $f(z) = \sum_n c_n(f)z^n \in H_\infty(\mathbb{D})$ we have
$$\sum_n |c_n(f)z^n| \le \|f\|_\infty.$$

- Moreover, $\dfrac{1}{3}$ is optimal.

Lemma

For every $f(z) = \sum_n c_n(f)z^n \in H_\infty(\mathbb{D})$ with $\|f\|_\infty \le 1$ we have $|c_N| \le 1 - |c_0|^2$ for every $n \in \mathbb{N}$.

Dirichlet series and complex analysis on polydiscs

Definition – Nth Bohr radius

$$K_N := \sup \left\{ r \leq 1 \mid \forall f \in H_\infty(\mathbb{D}^N) : \sup_{z \in r \cdot \mathbb{D}^N} \sum |c_\alpha(f) z^\alpha| \leq \|f\|_\infty \right\}$$

Bohr's power series theorem

$$K_1 = \frac{1}{3}$$

Problem

$$K_N = \mathbf{?}$$

Dirichlet series and complex analysis on polydiscs

> **Theorem, Boas-Khavinson, PAMS 1989**
>
> $$\frac{1}{3}\frac{1}{\sqrt{N}} \leq K_N < 2\sqrt{\log N}\frac{1}{\sqrt{N}}\,,$$

> Aizenberg, Boas, Khavinson, Dineen, Timoney, . . .

> **Theorem Defant-Frerick (2006 Israel J. Math.)**
>
> $$\sqrt{\frac{\log N}{N \log \log N}} \prec K_N \prec \sqrt{\frac{\log N}{N}}$$

Dirichlet series and complex analysis on polydiscs

Theorem Defant-Frerick-OtegaCerdà-Ounaïes-Seip (2011, Annals of Math.)

$$K_N \asymp \sqrt{\frac{\log N}{N}}$$

Theorem Bayart-Pellegrino-Seoane (2014, Advances Math.)

$$K_N \sim \sqrt{\frac{\log N}{N}}$$

Dirichlet Series. The result behind some of these improvements

Theorem, Defant, Frerick, Ortega-Cerdá, Ounaïes, Seip, Annals of Mathematics, 2011

The series

$$\sum_{n=1}^{\infty} |a_n| n^{-1/2} \exp(c(\log n \log \log n)^{1/2})$$

converges for each Dirichlet Series $D = \sum_n a_n \frac{1}{n^s}$ bounded in the half plane $[\mathbf{Re}\, s > 0]$ si $c < 1/2$ and can be divergent if $c > 1/2$.

Dirichlet Series. Further results

The study of (spaces of) Dirichlet series and operators between them continues in many related directions.

This is a field in which complex analysis, Fourier analysis, functional analysis, operator theory, Banach spaces, infinite holomorphy and number theory interplay.

Some Topics: Multipliers (for example $\mathcal{H}_\infty$ is the space of multipliers of $\mathcal{H}_2$), Hardy spaces $\mathcal{H}_p, 1 < p < \infty$, of Dirichlet series, boundary limits, Fatou theorems, zeros and interpolation of Dirichlet series, composition operators, approximation numbers, integral operators, improved inequalities (Lorentz spaces), Sidon constants for homogeneous polynomials, Bohr radii, harmonic analysis in the infinite dimensional torus, Riesz projection, embedding problems, **vector valued Dirichlet series**,...

Authors: Bailleul, Bayart, Brevig, Carando, Defant, García, Hedenmalm, Maestre, Mastylo, Ortega-Cerdá, Pellegrino, Queffélec, Saksman, Seip, Seoane, Sevilla, amongst many others.

Dirichlet Series in Banach spaces

The behaviour of $S(X) := \sup\{\sigma_a(D) - \sigma_u(D)\}$, when the Dirichlet series $D = \sum_n a_n \frac{1}{n^s}$ has its coefficients a_n in a Banach space X, depends on the geometry of the X. It is related to the cotype of X. Define $cot(X) := \inf\{2 \leq p \leq \infty \mid X \text{ has cotype } p\}$.

Theorem, Defant, García, Maestre, Pérez-García, Math. Ann. 2008

1. $S(X) = 1 - (1/cot(X))$.
2. If H is a Hilbert space, then $S(H) = 1/2$.
3.
$$S(\ell_p) = \begin{cases} \frac{1}{2}, & 1 \leq p \leq 2 \\ \\ 1 - \frac{1}{p}, & p \geq 2 \end{cases}$$

These results motivated my first work in this direction.

Some questions about Dirichlet Series in locally convex spaces

- X is a Hausdorff (sequentially complete if necessary) locally convex space.
  Examples: $C^\infty(\Omega)$, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{E}'(\omega)$, $H(\Omega)$, $\mathcal{A}(\Omega)$, $L_{loc}^p(\omega)$, the Schwartz spaces S and S', sequence spaces,...

- X' is the topological dual of X.

- For a Dirichlet series $D = \sum_n a_n \frac{1}{n^s}$ with coefficients $a_n \in X$, we denote by $\sigma_i(D), i = c, u, a$ the abscissas defined in a natural way.

- $\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D) \leq \sigma_c(D) + 1$.

Some questions about Dirichlet Series in locally convex spaces

Given $x' \in X'$ and the Dirichlet series D in X, consider for the scalar Dirichlet series $x'(D) = \sum_n x'(a_n) \frac{1}{n^s}$ the abscissas of convergence $\sigma_i(x'(D)), i = c, u, a$.

Definition

$\sigma_i^w(D) := \sup_{x' \in X'} \sigma_i(x'(D))$ are called **abscissas of weak convergence** of the Dirichlet series D in X.

$\sigma_i^w(D) \leq \sigma_i(D), i = c, u, a$; for each Dirichlet series D in X.

AIM

Compare the behaviour of $\sigma_i^w(D)$ and $\sigma_i(D)$ for $i = c, u, a$ for all Dirichlet vector valued series in X, and relate this behaviour with the topological structure of the space X.

Some questions about Dirichlet Series in locally convex spaces

Theorem

Let $D = \sum_n a_n \dfrac{1}{n^s}$ be a Dirichlet series in a sequentially complete locally convex space X. Then

(1) $\sigma_c^w(D) := \sup_{x' \in X'} \sigma_c(x'(D)) = \sigma_c(D)$.

(2) $\sigma_u^w(D) := \sup_{x' \in X'} \sigma_u(x'(D)) = \sigma_u(D)$.

The proof of (2) depends on the behaviour of the abscissa of bounded convergence and Bohr's fundamental theorem $\sigma_u(x'(D)) = \sigma_b(x'(D))$.

Some questions about Dirichlet Series in locally convex spaces

It might happen that $\sigma_c(D) = +\infty$ and $\sigma_c(x'(D)) < +\infty$ for each $x' \in X'$, although the supremum of these values as x' runs in X' must be infinity:

Example

Let $X = \mathbb{C}^{\mathbb{N}}$ the Fréchet space of all the complex sequences endowed with its natural Fréchet topology of pointwise convergence.

Take $a_n = (1, n, n^2, n^3, ...)$ and $D := \sum_n a_n \dfrac{1}{n^s}$.

Then $\sigma_c(D) = +\infty$ and $\sigma_c(x'(D)) < +\infty$ for each $x' \in X'$.

Some questions about Dirichlet Series in locally convex spaces

This phenomenon **cannot** happen for Banach spaces.

Theorem

Let X be a sequentially complete (DF)-space space, for example a Banach space. If a Dirichlet series $D = \sum_n a_n \dfrac{1}{n^s}$ satisfies $\sigma_c(x'(D)) < +\infty$ for all $x' \in X'$, then $\sigma_c(D) < +\infty$.

Some questions about Dirichlet Series in locally convex spaces

Theorem

A Fréchet space X satisfies every continuous linear operator from the Schwartz space S of rapidly decreasing functions into X is bounded, i.e. maps a neighbourhood of S into a bounded set of X, if and only if every Dirichlet series $D = \sum_n a_n \dfrac{1}{n^s}$ in X such that $\sigma_c(x'(D)) < +\infty$ for all $x' \in X'$ must also satisfy $\sigma_c(D) < +\infty$.

These Fréchet spaces X were characterized in terms of a topological invariant by Vogt in 1983.

Some questions about Dirichlet Series in locally convex spaces

Example

Let $D = \sum_n \frac{e_n}{n^s}$, where $(e_n)_n$ is the canonical basis of $X = \ell_p, 1 \le p < +\infty$ or $X = c_0$. Then $\sigma_a(D) = 1$ in all cases, but $\sigma_a^w(D) = 1/p, 1 \le p < +\infty$, and $\sigma_a^w(D) = 0$ if $X = c_0$.

Definition

The **gap for absolute convergence of Dirichlet series in** X is defined by $G_a(X) := \sup_D(\sigma_a(D) - \sigma_a^w(D))$, where the supremum is taken over all the Dirichlet series D with coefficients in X such that $\sigma_c(D) < +\infty$.

$0 \le G_a(X) \le 1$ for every space X. If X is finite dimensional, then $G_a(X) = 0$.

Some questions about Dirichlet Series in locally convex spaces

Theorem

Let X be an infinite dimensional Banach space.

- $G_a(X) \geq 1/2$. This is a consequence of Dvoretzky-Rogers Theorem.

- Let X have cotype $p \geq 2$, then $G_a(X) = 1 - 1/Cot(X)$, where $Cot(X) := \inf\{2 \leq p \leq \infty \,|\, X \text{ has cotype } p\}$.

- The following conditions are equivalent for X: (1) $G_a(X) = 1$. (2) X does not have finite cotype. (3) ℓ_∞ is finite representable in X.

Some questions about Dirichlet Series in locally convex spaces

Theorem

(a) If X is a nuclear sequentially complete locally convex space, then $G_a(X) = 0$.

(b) The following conditions are equivalent for a Fréchet space X:

(i) X is nuclear.

(ii) $G_a(X) = 0$.

(iii) $G_a(X) < 1/2$.

Some questions about Dirichlet Series in locally convex spaces

Examples

- $G_a(\ell_p) = 1/2$ if $1 \leq p \leq 2$ and $G_a(\ell_p) = 1 - 1/p$ if $2 \leq p \leq \infty$.

- For every $t \in [1/2, 1]$ there are Banach spaces X_t and non-normable Fréchet spaces Y_t such that $G_a(X_t) = G_a(Y_t) = t$.

The Fréchet algebra of uniformly convergent Dirichlet series

AIM

We study the Fréchet space $\mathcal{H}_+^\infty$ of uniformly convergent Dirichlet series in the half-plane $\mathbb{C}_0 := \{s \in \mathbb{C} \mid \mathbf{Re}\, s > 0\}$.

Motivation

- (Bohr, 1913) The set of Dirichlet series $f(s) = \sum_n a_n n^{-s}$ such that $\sigma_u(f) \leq 0$ coincides with the set of holomorphic functions on $\mathbb{C}_0$ that are bounded on $\mathbb{C}_\varepsilon$ for each $\varepsilon > 0$ and that can be represented as a convergent Dirichlet series in $\mathbb{C}_0$.

- (Improved Montel principle of Bayart, 2002) If $(f_k)_k$ is a bounded sequence in $\mathcal{H}^\infty$, then there are $f \in \mathcal{H}^\infty$ and a subsequence $(f_{k(j)})_j$ of the original sequence that converges uniformly to f on $\mathbb{C}_\varepsilon$ for each $\varepsilon > 0$.

The Fréchet algebra of uniformly convergent Dirichlet series

- $\mathcal{H}_+^\infty$ is the space of all analytic functions on the half-plane $\mathbb{C}_0$ which are bounded on $\mathbb{C}_\varepsilon$ for each $\varepsilon > 0$ and that can be represented as a convergent Dirichlet series $f(s) = \sum_n a_n n^{-s}$ in $\mathbb{C}_0$. Seminorms:

$$P_\varepsilon(f) := \sup_{s \in \mathbb{C}_\varepsilon} |f(s)|, \qquad f \in \mathcal{H}_+^\infty.$$

- $\mathcal{H}_+^\infty$ is a Fréchet space, that can be written as the projective limit $\mathcal{H}_+^\infty = \mathrm{proj}_k \mathcal{H}_k^\infty$.

- $\mathcal{H}_k^\infty$ is the Banach space of all bounded analytic functions on $\mathbb{C}_{1/k}$ that can be represented as a convergent Dirichlet series in $\mathbb{C}_{1/k}$, endowed with the norm $||g||_k = \sup_{s \in \mathbb{C}_{1/k}} |g(s)|$.

The Fréchet algebra of uniformly convergent Dirichlet series

- The space $\mathcal{H}^\infty$ of bounded Dirichlet series in $\mathbb{C}_0$ is contained in $\mathcal{H}_+^\infty$.

- $\mathcal{H}_+^\infty$ coincides with the space of all Dirichlet series $f(s) = \sum_n a_n n^{-s}$ such that $\sigma_u(f) \leq 0$.

- $\mathcal{H}_+^\infty$ is continuously included in the Fréchet space $H(\mathbb{C}_0)$.

The Fréchet algebra of uniformly convergent Dirichlet series

Examples

- $f(s) = (s-1)/(s+1)$ and $f(a) = a^{-s}$, $a > 1$ not a positive integer, belong to $H(\mathbb{C}_0)$, they are bounded in modulus by 1, but they are not a convergent Dirichlet series, hence they do not belong to $\mathcal{H}^\infty_+$.

- $f(s) = (1 - 2^{-s})\zeta(s) = \sum_n (-1)^{n-1} n^{-s-1}$ does not belong to $\mathcal{H}^\infty$. However, $f \in \mathcal{H}^\infty_+$ since $\sigma_a(f) = 0$.

The Fréchet algebra of uniformly convergent Dirichlet series

Theorem

The Fréchet space $\mathcal{H}_+^\infty$ is Schwartz, non-nuclear and the Dirichlet monomials $e_n(s) = n^{-s}$, $n \in \mathbb{N}$ are a Schauder basis of the space. Moreover, for each $m \in \mathbb{N}$, $\mathcal{H}_+^\infty$ contains an isomorphic copy of the space $H(\mathbb{D}^m)$ of holomorphic functions on the m-dimensional polydisc $\mathbb{D}^m$.

$\mathcal{H}_+^\infty$ contains a complemented subspace isomorphic to a non-nuclear Köthe echelon space of order one. This is a consequence of **Bohr's inequality:** $\sum_{p \in \mathcal{P}} |a_p| \leq \sup_{s \in \mathbb{C}_0} |f(s)|$ for each $f = \sum_n a_n n^{-s} \in \mathcal{H}^\infty$.

The map $\Delta : H(\mathbb{D}^m) \to \mathcal{H}_+^\infty$ given by $\Delta(g)(s) := g(2^{-s}, 3^{-s}, ..., (p_m)^{-s})$ defines a topological isomorphism into its image by Kronecker's theorem.

The Fréchet algebra of uniformly convergent Dirichlet series

Theorem

(1) The Fréchet space $\mathcal{H}_+^\infty$ is a multiplicatively convex Fréchet algebra for the pointwise product.

(2) An element $f \in \mathcal{H}_+^\infty$ is invertible if and only if $\inf_{s \in \mathbb{C}_\varepsilon} |f(s)| > 0$ for each $\varepsilon > 0$.

(3) The space $\mathcal{H}_+^\infty$ is not a Q-algebra. This means that the set of invertible elements is not open.

Statement (3) depends on an approximation result due to Bayart.

The Fréchet algebra of uniformly convergent Dirichlet series

Examples

- Every invertible element of $\mathcal{H}^\infty$ is also invertible in $\mathcal{H}^\infty_+$.

- The Dirichlet polynomial $f(s) = 1 - 2^{-s}$ belongs to $\mathcal{H}^\infty$, it is invertible in $\mathcal{H}^\infty_+$, but not in $\mathcal{H}^\infty$ since $\inf_{s \in \mathbb{C}_0} |f(s)| = 0$.

- The Dirichlet monomial $g(s) = 2^{-s}$ is an invertible element in the space $H(\mathbb{C}_0)$, but it is not invertible in $\mathcal{H}^\infty_+$.

Composition operators on the Fréchet algebra of uniformly convergent Dirichlet series

- Composition operators on the Hilbert space $\mathcal{H}^2$ of Dirichlet series were characterized by **Gordon and Hedenmalm** in 1999.

- **Bayart** proved in 2002 that an analytic map $\phi : \mathbb{C}_0 \to \mathbb{C}_0$ defines a continuous composition operator $C_\phi(f) := f \circ \phi$ on the Banach space $\mathcal{H}^\infty$ if and only if $\phi(s) = c_0 s + \sum_n c_n n^{-s}$ with c_0 a non-negative integer and $\varphi(s) := \sum_n c_n n^{-s}$ a Dirichlet series convergent in some half-plane.

- **Bayart** also proved that C_ϕ is compact on $\mathcal{H}^\infty$ if and only if $\phi(\mathbb{C}_0) \subset \mathbb{C}_\varepsilon$ for some $\varepsilon > 0$.

- An operator $T : E \to E$ on a locally convex space E is called **bounded** if there is a 0-neighbourhood U in E such that $T(U)$ is bounded in E. Every bounded operator is continuous.

Composition operators on the Fréchet algebra of uniformly convergent Dirichlet series

Theorem

Let $\phi : \mathbb{C}_0 \to \mathbb{C}_0$ be analytic.

(1) The composition operator C_ϕ is continuous on the space $\mathcal{H}_+^\infty$ if and only if $\phi(s) = c_0 s + \sum_n c_n n^{-s}$ with c_0 a non-negative integer and $\varphi(s) := \sum_n c_n n^{-s}$ a Dirichlet series convergent in some half-plane.

(2) Suppose that the composition operator C_ϕ is continuous on the space $\mathcal{H}_+^\infty$. Then C_ϕ is bounded on $\mathcal{H}_+^\infty$ if and only if there is $\varepsilon > 0$ such that $\phi(\mathbb{C}_0) \subset \mathbb{C}_\varepsilon$.

Composition operators on the Fréchet algebra of uniformly convergent Dirichlet series

Examples

- If $\phi(s) = c_0 s + i\tau$, $s \in \mathbb{C}_0$, with c_0 a positive integer and $\tau \in \mathbb{R}$, then the operator C_ϕ is continuous but not bounded in $\mathcal{H}_+^\infty$.

- If $\phi(s) = c_0 s + c_1 + c_2 2^{-s}$, with c_0 a positive integer and $c_2 \neq 0$, then C_ϕ is bounded in $\mathcal{H}_+^\infty$ if and only if $\mathbf{Re}\, c_1 > |c_2|$. This is a consequence of a more general result due to Bayart (2003).

1. **F. Bayart**, Hardy spaces of Dirichlet series and their composition operators, Monatsh. Math. 136 (2002), 203–236.

2. **H.P. Boas**, The football player and the infinite series, Notices of the Amer. Math. Soc. 44 (1997) 1430–1435.

3. **H. Bohr**, Über die gleichmässige Konvergenz Dirichletscher Reihen, J. reine angew. Math. 143 (1913), 203–211.

4. **J. Bonet**, Abscissas of weak convergence of vector valued Dirichlet series, J. Funct. Anal. 269 (2015), 3914–3927

5. **J. Bonet**, The Fréchet Schwartz algebra of uniformly convergent Dirichlet series, Proc. Edinburgh Math. Soc. (to appear).

6. **A. Defant, D. García, M. Maestre, D. Pérez-García**, Bohr's strip for vector valued Dirichlet series, Math. Ann. 342 (2008), no. 3, 533–555.

7. **A. Defant, L. Frerick, J. Ortega-Cerdá, M. Ounaïes, K. Seip**, The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive, Annals Math. 174 (2011), 485-497.