The Cesàro operator on weighted Banach spaces of analytic functions

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Joint work with A.A. Albanese and W.J. Ricker

Aim of the lecture

AIM

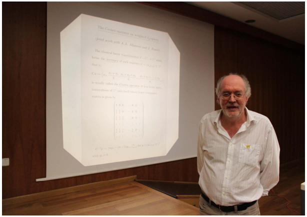
Investigate the behaviour of the Cesàro operator C acting on certain Banach, Fréchet and (LB) spaces of analytic functions on the disc.

We report on joint work in progress with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).

Ernesto Cesàro (1859-1906)

Albanese and Ricker

Angela Albanese



Werner Ricker

The discrete Cesàro operator

The *Cesàro operator* C is defined for a sequence $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ of complex numbers by

$$\mathrm{C}(x) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_n , \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

Proposition.

The operator $C \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself with

$$C^{-1}(y) = (ny_n - (n-1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^{\mathbb{N}}, \tag{1}$$

where we set $y_{-1} := 0$.

Recall that $\mathbb{C}^{\mathbb{N}}$ is a Fréchet space for the topology of coordinatewise convergence.

The Cesàro operator for analytic functions

The Cesàro operator is defined for analytic functions on the disc $\mathbb{D}$ by

$$\mathrm{C}f = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_n \right) z^n, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}).$$

The Cesàro operator acts continuously and has the integral representation

$$\mathrm{C}f(z) = \frac{1}{z} \int_0^z \frac{f(\rho)}{1-\rho} \, d\rho, \quad f \in H(\mathbb{D}), \; z \in \mathbb{D}.$$

Indeed, for $f(z)=\sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, we have

$$\begin{aligned} Cf(z) &= \frac{1}{z}\int_0^z \frac{f(\rho)}{1-\rho}\, d\rho = \frac{1}{z}\int_0^z \left(\sum_{n=0}^{\infty} a_n\rho^n\right)\left(\sum_{m=0}^{\infty}\rho^m\right) \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{1}{z}\int_0^z \rho^{n+m}\, d\rho = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{z^{n+m}}{n+m+1} \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=n} \frac{z^k}{k+1} = \sum_{k=0}^{\infty}\left(\frac{1}{k+1}\sum_{n=0}^{k} a_n\right) z^k. \end{aligned}$$

The discrete Cesàro operator on Banach sequence spaces

Theorem. Hardy. 1920.

Let $1 < p < \infty$. The Cesàro operator maps the Banach space ℓ^p continuously into itself, which we denote by $\mathsf{C}^{(p)} \colon \ell^p \to \ell^p$, and $\|(\mathsf{C}^{(p)})\| = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$, for all $n \in \mathbb{N}$.

In particular, **Hardy's inequality** holds:

$$\|(\mathsf{C}^{(p)})\|_p \leq p' \|x\|_p, \quad x \in \ell^p.$$

Clearly C is not continuous on ℓ_1, since $\mathsf{C}(e_1) = (1, 1/2, 1/3, ...)$.

Proposition.

The Cesàro operators $\mathsf{C}^{(\infty)} \colon \ell^\infty \to \ell^\infty$, $\mathsf{C}^{(c)} \colon c \to c$ and $\mathsf{C}^{(0)} \colon c_0 \to c_0$ are continuous, and $\|\mathsf{C}^{(\infty)}\| = \|\mathsf{C}^{(c)}\| = \|\mathsf{C}^{(0)}\| = 1$.

Moreover, $\lim \mathsf{C}x = \lim x$ for each $x \in c$.

Spectrum and point spectrum

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ (resp. $\mathcal{K}(X)$) is the space of all continuous (resp. compact) linear operators on X.

The **resolvent set** $\rho(T, X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The elements of $\sigma_{pt}(T, X)$ are called eigenvalues of T.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

(i) $\sigma(\mathsf{C};\ell^\infty) = \sigma(\mathsf{C};c_0) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C};\ell^\infty) = \{(1,1,1,...)\}$.

(iii) $\sigma_{pt}(\mathsf{C};c_0) = \emptyset$.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

Let $1 < p < \infty$ and $1/p + 1/p' = 1$.

(i) $\sigma(\mathsf{C}; \ell^p) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C}; \ell^p) = \emptyset$.

In particular, C is not compact in the spaces $\ell^p, 1 < p \leq \infty$, or in the space c_0.

More about the spectrum and point spectrum

Theorem

The Cesàro operator satisfies

$$(a) \quad \sigma(\mathsf{C}, H(\mathbb{D})) = \sigma_{pt}(\mathsf{C}, H(\mathbb{D})) = \{\frac{1}{m} \colon m \in \mathbb{N}\}.$$

Persson showed in 2008 the following facts:

For every $m \in \mathbb{N}$ the operator $(\mathsf{C} - \frac{1}{m}I) \colon H(\mathbb{D}) \to H(\mathbb{D})$ is not injective because $\mathrm{Ker}(\mathsf{C} - \frac{1}{m}I) = \mathrm{span}\{e_m\}$, where $e_m(z) = z^{m-1}(1-z)^{-m}$, $z \in \mathbb{D}$, and it is not surjective because the function $f_m(z) := z^{m-1}$, $z \in \mathbb{D}$, does not belong to the range of $(\mathsf{C} - \frac{1}{m}I)$.

The growth spaces

For $\gamma > 0$ the growth classes $A^{-\gamma}$ and $A_0^{-\gamma}$ are the Banach spaces defined by

$$A^{-\gamma} = \{f \in H(\mathbb{D}) \colon \|f\|_{-\gamma} := \sup_{z \in \mathbb{D}} (1 - |z|)^{\gamma} |f(z)| < \infty\}.$$

$$A_0^{-\gamma} = \{f \in H(\mathbb{D}) \colon \lim_{|z| \to 1} (1 - |z|)^{\gamma} |f(z)| = 0\}.$$

$A_0^{-\gamma}$ is the closure of the polynomials on $A^{-\gamma}$.

The Cesàro operator acts continuously on $A^{-\gamma}$. Its spectrum on these (and many other spaces of analytic functions on the disc) has been studied by Aleman and Persson 2008-2010.

The result of Aleman and Persson

Theorem. Aleman, Persson.

Let $\gamma > 0$. The Cesàro operator $\mathrm{C}_{\gamma,0} \colon A_0^{-\gamma} \to A_0^{-\gamma}$ has the following properties.

(i) $\sigma_{pt}(\mathrm{C}_{\gamma,0}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\}$.

(ii) $\sigma(\mathrm{C}_{\gamma,0}) = \sigma_{pt}(\mathrm{C}_{\gamma,0}) \cup \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2\gamma}\right| \leq \frac{1}{2\gamma}\right\}$.

(iii) If $\left|\lambda - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma}$ (equivalently $\mathrm{Re}\left(\frac{1}{\lambda}\right) > \gamma$), then $\mathrm{Im}(\lambda I - \mathrm{C}_{\gamma,0})$ is a closed subspace of $A_0^{-\gamma}$ and has codimension 1.

Moreover, the Cesàro operator $\mathrm{C}_{\gamma} : A^{-\gamma} \to A^{-\gamma}$ satisfies

(iv) $\sigma_{pt}(\mathrm{C}_{\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m \leq \gamma\}$, and

(v) $\sigma(\mathrm{C}_{\gamma}) = \sigma(\mathrm{C}_{\gamma,0})$.

The norm

The continuity of C_γ and $\mathsf{C}_{\gamma,0}$ as established by Aleman and Persson gives no quantitative estimate for their operator norm.

Theorem.

(i) Let $\gamma \geq 1$. Then $\|\mathsf{C}_\gamma^n\| = \|\mathsf{C}_{\gamma,0}^n\| = 1$ for all $n \in \mathbb{N}$.

(ii) Let $0 < \gamma < 1$. Then $\|\mathsf{C}_\gamma^n\| = \|\mathsf{C}_{\gamma,0}^n\| = 1/\gamma^n$ for all $n \in \mathbb{N}$.

The optimal domain

The *optimal domain* $[\mathrm{C}, X]$ of the Cesàro operator C when it acts in any Banach space of analytic functions X on $\mathbb{D}$ is defined by

$$[\mathrm{C}, X] := \{f \in H(\mathbb{D}) \;:\; \mathrm{C}(f) \in X\},$$

$$\|f\|_{[\mathrm{C},X]} := \|\mathrm{C}(f)\|_X, \qquad f \in [\mathrm{C}, X].$$

It is a Banach space as of a consequence of $\mathrm{C} : H(\mathbb{D}) \to H(\mathbb{D})$ being a topological Fréchet space isomorphism.

If C acts in X, then $X \subseteq [\mathrm{C}, X]$ and the natural inclusion map is continuous. Moreover, $[\mathrm{C}, X]$ is the *largest* of all Banach spaces of analytic functions Y on $\mathbb{D}$ that C maps continuously into X

The optimal domain

Theorem

Let $\gamma > 0$ and $\varphi(z) := 1/(1-z)$ for $z \in \mathbb{D}$.
The optimal domain $[C, A^{-\gamma}]$ of $C_\gamma : A^{-\gamma} \to A^{-\gamma}$ is isometrically isomorphic to $A^{-\gamma}$ and is given by

$$[C, A^{-\gamma}] = \{f \in H(\mathbb{D}) \ : \ f\varphi \in A^{-(\gamma+1)}\}. \tag{2}$$

Moreover, the norm $\|.\|_{[C, A^{-\gamma}]}$ is equivalent to the norm $f \to \|f\varphi\|_{-(\gamma+1)}$ and the containment $A^{-\gamma} \subseteq [C, A^{-\gamma}]$ is proper.

A similar result holds for the optimal domain $[C, A_0^{-\gamma}]$ of $C_{\gamma,0} : A_0^{-\gamma} \to A_0^{-\gamma}$.

Definition of the *-spectrum.

X **is a Hausdorff locally convex space (lcs).**

- $\rho^*(T)$ consists of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that each $\mu \in B(\lambda, \delta) := \{z \in \mathbb{C} \colon |z - \lambda| < \delta\}$ belongs to $\rho(T)$ and the set $\{R(\mu, T) \colon \ \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X)$.
- $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$.
- $\sigma^*(T)$ is a closed set containing $\sigma(T)$. If $T \in \mathcal{L}(X)$ with X a Banach space, then $\sigma(T) = \sigma^*(T)$. There exist continuous linear operators T on a Fréchet space X such that $\overline{\sigma(T)} \subset \sigma^*(T)$ properly.

More about the spectrum

Notation:

$\Sigma := \{\frac{1}{m} : m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$.

Proposition.

(i) $\sigma(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \sigma_{pt}(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \Sigma$.

(ii) Fix $m \in \mathbb{N}$. Let $x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^{\mathbb{N}}$ where $x_n^{(m)} := 0$ for $n \in \{1, \ldots, m-1\}$, $x_m^{(m)} := 1$ and $x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!}$ for $n > m$. Then the eigenspace

$$\mathrm{Ker}\left(\frac{1}{m}I - \mathsf{C}\right) = \mathrm{span}\{x^{(m)}\} \subseteq \mathbb{C}^{\mathbb{N}}$$

is 1-dimensional.

(iii) $\sigma^*(\mathsf{C}, H(\mathbb{D})) = \Sigma \cup \{0\} = \sigma^*(\mathsf{C}; \mathbb{C}^{\mathbb{N}})$.

The Fréchet growth spaces

Let $\gamma \geq 0$.

$$A_+^{-\gamma} := \cap_{\mu>\gamma} A^{-\mu} = \cap_{\mu>\gamma} A_0^{-\mu}.$$

The space $A_+^{-\gamma}$ is Fréchet when it is endowed with the lc-topology generated by the fundamental sequence of seminorms

$$\|f\|_k := \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma+\frac{1}{k}} |f(z)|.$$

It is a Fréchet-Schwartz space because the inclusion $A^{-\mu_1} \hookrightarrow A^{-\mu_2}$ is compact for all $0<\mu_1<\mu_2$. In particular, every bounded subset of $A_+^{-\gamma}$ is relatively compact, i.e. the space is Montel

The (LB) growth spaces

Let $0 < \gamma \leq \infty$

$$A_{-}^{-\gamma} := \cup_{\mu<\gamma} A^{-\mu} = \cup_{\mu<\gamma} A_0^{-\mu},$$

and it is endowed with the finest locally convex topology such that all the inclusions $A^{-\mu} \hookrightarrow A^{-\gamma},\ \mu < \gamma$, are continuous.
In particular, $A_{-}^{-\gamma}$ is the (DFS)-space

$$A_{-}^{-\gamma} := \operatorname*{ind}_{k} A^{-(\gamma-\frac{1}{k})} = \operatorname*{ind}_{k} A_0^{-(\gamma-\frac{1}{k})}.$$

As a consequence $A_{-}^{-\gamma}$ is a Montel space, too.

The (LB) growth spaces. The Korenblum space

The **Korenblum space** $A_{-}^{-\infty}$ was introduced by Korenblum in 1975 is usually denoted by

$$A^{-\infty} = \cup_{0<\gamma<\infty} A^{-\gamma} = \cup_{n\in\mathbb{N}} A^{-n}.$$

All these spaces play an important role in the study of interpolation and sampling of holomorphic functions on the disc.

Proposition.

(i) The Korenblum space $A^{-\infty}$ is nuclear.

(ii) Each Fréchet space $A_{+}^{-\gamma}$ for $0 \leq \gamma < \infty$, and each (LB)-space $A_{-}^{-\gamma}$, for $0 < \gamma < \infty$, is a Schwartz space but, fails to be nuclear.

Part (i) was well-known. The proof non-nuclearity in (ii) relies on a result of T. Domenig (1999) about absolutely summing composition operators.

Non-nuclearity. Another approach.

Let $e_n(z) = z^n$, $z \in \mathbb{D}$, for $n = 0, 1, 2, \ldots$ and $\Lambda = \{e_n : n = 0, 1, 2, \ldots\}$. It is well-known that Λ is not a Schauder basis for any $A_0^{-\mu}$. This for example is of a consequence of a more general result due to Lusky (2000).

Theorem (Bonet, Lusky, Taskinen, 2017).

(i) Λ is a Schauder basis of $A_+^{-\gamma}$ for any $\gamma \geq 0$.

(ii) Λ is a Schauder basis of $A_-^{-\gamma}$ for any $\gamma > 0$.

The non nuclearity of these spaces now follows from Grothendieck Pietsch criterion for Fréchet of (LB)-spaces with basis, and the fact that $||e_n||_\mu = n^n \mu^\mu / (n + \mu)^{n+\mu}$ for each $n \in \mathbb{N}$ and each $\mu > 0$.

The operator C in the growth spaces

The Cesàro operators

$C\colon A_-^{-\gamma} \to A_-^{-\gamma}$ and $C\colon A_+^{-\gamma} \to A_+^{-\gamma}$

are continuous because C acts continuously in every step.

Theorem

(1) Let $\gamma \in]0,\infty[$.

(a) $\sigma_{pt}(\mathsf{C}, A_+^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m \leq \gamma\}$.

(b) $\sigma(\mathsf{C}, A_+^{-\gamma}) = \{0\} \cup \{\frac{1}{m} \colon m,\ m \leq \gamma\} \cup \{\lambda \in \mathbb{C} \colon |\lambda - \frac{1}{2\gamma}| < \frac{1}{2\gamma}\}$.

(c) $\sigma^*(\mathsf{C}, A_+^{-\gamma}) = \overline{\sigma(\mathsf{C}, A_+^{-\gamma})}$.

(2) Let $\gamma = 0$.

(a) $\sigma_{pt}(\mathsf{C}, A_+^{-0}) = \emptyset$.

(b) $\sigma(\mathsf{C}, A_+^{-0}) = \{0\} \cup \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > 0\}$.

(c) $\sigma^*(\mathsf{C}, A_+^{-0}) = \overline{\sigma(\mathsf{C}, A_+^{-0})}$.

Theorem

(1) Let $\gamma \in]0, \infty[$.

(a) $\sigma_{pt}(\mathsf{C}, A_-^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\}$.

(b) $\sigma(\mathsf{C}, A_-^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\} \cup \{\lambda \in \mathbb{C} \colon |\lambda - \frac{1}{2\gamma}| \leq \frac{1}{2\gamma}\}$.

(c) $\sigma^*(\mathsf{C}, A_-^{-\gamma}) = \sigma(\mathsf{C}, A_-^{-\gamma})$.

(2) For the Korenblum space $A^{-\infty}$ (i.e. $\gamma = \infty$) we have:

(a) $\sigma(\mathsf{C}, A^{-\infty}) = \sigma_{pt}(\mathsf{C}, A^{-\infty}) = \{\frac{1}{m} \colon m \in \mathbb{N}\}$.

(b) $\sigma^*(\mathsf{C}, A^{-\infty}) = \{\frac{1}{m} \colon m \in \mathbb{N}\} \cup \{0\}$.

Abstract results.

Theorem.

Let $X = \cap_{n\in\mathbb{N}} X_n$ be a Fréchet space given by the intersection of a sequence of Banach spaces $((X_n, \|\cdot\|_n))_{n\in\mathbb{N}}$ satisfying $X_{n+1} \subset X_n$ with $\|x\|_n \le \|x\|_{n+1}$, for each $n \in \mathbb{N}$ and $x \in X_{n+1}$. Let $T \in \mathcal{L}(X)$ satisfy the following condition:

(a) For each $n \in \mathbb{N}$ there exists $T_n \in \mathcal{L}(X_n)$ such that the restriction of T_n to X (resp. of T_n to X_{n+1}) coincides with T (resp. with T_{n+1}).

Then $\sigma(T, X) \subseteq \cup_{n\in\mathbb{N}} \sigma(T_n, X_n)$ and $R(\lambda, T)$ coincides with the restriction of $R(\lambda, T_n)$ to X for each $n \in \mathbb{N}$ and each $\lambda \in \cap_{n\in\mathbb{N}} \rho(T_n, X_n)$. Moreover, if $\cup_{n\in\mathbb{N}} \sigma(T_n, X_n) \subseteq \overline{\sigma(T, X)}$, then

$$\sigma^*(T, X) = \overline{\sigma(T, X)}.$$

Abstract results.

Theorem.

Let $E = \operatorname{ind}_n(E_n, \| \ \|_n)$ be a Hausdorff inductive limit of Banach spaces. Let $T \in \mathcal{L}(E)$ satisfy the following condition:

(A) For each $n \in \mathbb{N}$ the restriction T_n of T to E_n maps E_n into itself and belongs to $\mathcal{L}(E_n)$.

Then the following properties are satisfied.

(i) $\sigma_{pt}(T, E) = \cup_{n \in \mathbb{N}} \sigma_{pt}(T_n, E_n)$.

(ii) $\sigma(T, E) \subseteq \cap_{m \in \mathbb{N}} \cup_{n=m}^{\infty} \sigma(T_n, E_n)$. Moreover, if $\lambda \in \cap_{n=m}^{\infty} \rho(T_n, E_n)$ for some $m \in \mathbb{N}$, then $R(\lambda, T_n)$ coincides with the restriction of $R(\lambda, T)$ to E_n for each $n \geq m$.

(iii) If $\cup_{n=m}^{\infty} \sigma(T_n, E_n) \subseteq \overline{\sigma(T, E)}$ for some $m \in \mathbb{N}$, then $\sigma^*(T, E) = \overline{\sigma(T, E)}$.

A bit of linear dynamics.

Mean ergodic properties. Definitions

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Banach space, an operator T is power bounded if and only if $\sup_n ||T^n|| < \infty$.

If X is a barrelled space, an operator T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded. This is a consequence of the uniform boundedness principle.

For $T \in \mathcal{L}(X)$**, we set** $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$.

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} T^m x, \quad x \in X, \tag{3}$$

exist in X.

If T is mean ergodic, then one then has the direct decomposition

$$X = \mathrm{Ker}(I - T) \oplus \overline{(I - T)(X)}.$$

Mean ergodic properties. Definitions

Uniformly mean ergodic operators

If $\{T_{[n]}\}_{n=1}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$ to $P \in \mathcal{L}(X)$, then T is called *uniformly mean ergodic*.

Theorem. Lin. 1974.

Let T a (continuous) operator on a Banach space X which satisfies $\lim_{n\to\infty} ||T^n/n|| = 0$. The following conditions are equivalent:

(1) T is uniformly mean ergodic.

(4) $(I - T)(X)$ is closed.

Ergodic properties of C on classical sequence spaces

Proposition.

- The Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded and uniformly mean ergodic.
- The Cesàro operator $\mathrm{C}^{(p)}\colon \ell^p \to \ell^p$, $1 < p < \infty$, is not power bounded and not mean ergodic.
- The Cesàro operator $\mathrm{C}^{(0)}\colon c_0 \to c_0$ is power bounded, not mean ergodic.

Hypercyclicity. Definitions

Hypercyclic operator

$T \in \mathcal{L}(X)$, with X separable, is called **hypercyclic** if there exists $x \in X$ such that the orbit $\{T^n x\colon n \in \mathbb{N}_0\}$ is dense in X.

Supercyclic operator

If, for some $z \in X$, the projective orbit $\{\lambda T^n z\colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*.

Clearly, hypercyclicity always implies supercyclicity.

Dynamics of C on classical sequence spaces

Proposition.

- The Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded, uniformly mean ergodic and not supercyclic.

- The Cesàro operator $\mathsf{C}^{(p)}\colon \ell^p \to \ell^p$, $1 < p < \infty$, is not power bounded, not mean ergodic and not supercyclic.

- The Cesàro operator $\mathsf{C}^{(0)}\colon c_0 \to c_0$ is power bounded, not mean ergodic and not supercyclic.

Ergodicity of C on growth Banach spaces

Theorem

(i) Let $0 < \gamma < 1$. Both of the operators C_γ and $\mathsf{C}_{\gamma,0}$ fail to be power bounded and are not mean ergodic. Moreover,

$$\mathrm{Ker}(I - \mathsf{C}_\gamma) = \mathrm{Ker}(I - \mathsf{C}_{\gamma,0}) = \{0\},$$

and $\mathrm{Im}(I - \mathsf{C}_\gamma)$ (resp. $\mathrm{Im}(I - \mathsf{C}_{\gamma,0})$) is a proper closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$).

(ii) Both of the operators C_1 and $\mathsf{C}_{1,0}$ are power bounded but not mean ergodic. Moreover, $\mathrm{Im}(I - \mathsf{C}_1)$ (resp. $\mathrm{Im}(I - \mathsf{C}_{1,0})$) is not a closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$).

Theorem continued

(iii) Let $\gamma > 1$. Both of the operators C_γ and $\mathrm{C}_{\gamma,0}$ are power bounded and uniformly mean ergodic. Moreover, $\mathrm{Im}(I - \mathrm{C}_\gamma)$ (resp. $\mathrm{Im}(I - \mathrm{C}_{\gamma,0})$) is a proper closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$). In addition,

$$\mathrm{Im}(I - \mathrm{C}_\gamma) = \{h \in A^{-\gamma} : h(0) = 0\}. \tag{4}$$

Moreover, with $\varphi(z) := 1/(1-z)$, for $z \in \mathbb{D}$, the linear projection operator $P_\gamma : A^{-\gamma} \to A^{-\gamma}$ given by

$$P_\gamma(f) := f(0)\varphi, \qquad f \in A^{-\gamma},$$

is continuous and satisfies $\lim_{n\to\infty}(\mathrm{C}_\gamma)_{[n]} = P_\gamma$ in the operator norm.

Theorem

The Cesàro operator C acting on $H(\mathbb{D})$ is power bounded, uniformly mean ergodic and not supercyclic, hence not hypercyclic.

As a consequence, C is not supercyclic on the spaces $A^{-\gamma}, \ \gamma \geq 0$, and $A_0^{-\gamma}, 0 < \gamma \leq \infty$.

Ergodicity of C on the Fréchet growth spaces

Proposition

Let $\gamma \in [0, \infty[$.

The following conditions are equivalent:

(a) C is power bounded on $A_+^{-\gamma}$.

(b) C is (uniformly) mean ergodic on $A_+^{-\gamma}$.

(c) $1 \leq \gamma < \infty$.

Ergodicity of C on the (LB) growth spaces

Proposition

Let $\gamma \in]0, \infty]$.

The following conditions are equivalent:

(a) C is power bounded on $A_{-}^{-\gamma}$.

(b) C is (uniformly) mean ergodic on $A_{-}^{-\gamma}$.

(c) $1 < \gamma \leq \infty$.

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