The mathematical work of Paweł Domański

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Abstract

We report on the mathematical work of Paweł Domański (AMU Poznań).

1 Introduction

Paweł Domański was born on the 5th of June 1959, and died much too early the 4th of August 2016. He studied Mathematics at the Adam Mickiewicz University, Poznań (Poland), and he presented his Ph.D. Thesis at this University in 1987 and his Habilitation in 1991. He was Full professor at this University since 2003. He got several awards and international recognitions: Corresponding Foreign Member of Real Academia de Ciencias Exactas, Físicas y Naturales of Spain since 2009, member of the Committee of Mathematics of the Polish Academy of Science from 2007 till 2015, J. Marcinkiewicz Award of the Polish Mathematical Society for the best mathematical work of a student in Poland in 1987; Banach Award of the Polish Mathematical Society in 1991; Award of 3rd Section of the Academy of Sciences of Poland in 1993; Scientific Award of the Ministry of Education of Poland for joint works with D. Vogt in 2001.

He was executive Editor of Studia Mathematica, editor of Functiones et Approximatio (Poznań) and member of the scientific Committee of Revista RACSAM of the Real Academia

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de Ciencias Exactas Físicas y Naturales, and Main Researcher of the grant MAESTRO of National Center of Science in Poland with budget of around 500.000 euro (2014-2019). He was Alexander von Humboldt fellow at the University of Wuppertal (hosted by Prof. D. Vogt) in 01.09.1991-30.04.1993, 15.09-15.12.2000 and 1.10- 31.12.2006.

Paweł was an excellent mathematician highly estimated in the international mathematical community with an impressively wide interest and knowledge in many related areas of mathematics, especially in Mathematical Analysis: functional analysis, Banach spaces, topological algebras, homological algebras, complex analysis of one and several variables and partial differential operators, among many other topics. Thus he could easily contribute with new ideas, techniques and deep results. He was a very good lecturer, as well as a clear expositor in papers and surveys creating a very active group of young mathematicians in Poznań. Above all, he was a very good friend, always ready to help and very generous. We miss him very much.

In this paper we try to present a selection of results of Paweł, among his many relevant and important contributions to functional analysis and related areas. We hope that our presentation will give the reader an idea of the originality, creativity and deepness of the work of our dear friend Paweł Domański.

2 The early work

s.early

Since the late seventies there was a growing interest in the group around Drewnowski at Poznań in non-locally convex topological vector spaces and F-spaces (i.e. complete metrizable topological vector spaces). Several deep results had been obtained by Kalton, Peck and Roberts. The atmosphere of those times is nicely described in the paper about Drewnowski's work by Domański and Wnuk [D86].

In order to state some of the first results of Domański, we recall that a topological vector space is minimal if there is no strictly weaker vector Hausdorff topology on the space. Minimal locally convex spaces are precisely the products of copies of the scalar field. Drewnowski had defined in 1977 the q-minimal spaces as those for which every Hausdorff quotient is minimal. They played an important role in the theory of F-spaces. The first example of a minimal non-locally convex F-space was obtained in 1995 by Kalton [32] modifying the famous method of Gowers and Maurey [21]. It is still an open question whether there are q-minimal non-locally convex F-spaces.

In the first papers of Domański in this direction [D28] and [D30], he improved an example of Lohman and Stiles and gave an example of a complete non-separable topological vector space embeddable into the product of $2^{\aleph_0}$ copies of the Banach space c_0. He later extended this result by showing that the product of $2^{\aleph_0}$ topological vector spaces contains a nonseparable closed subspace if each factor admits a strongly regular semibasic sequence. This holds on every non-minimal F-space, by a theorem of Kalton and Shapiro. The later result was also extended in [D28] by proving that these semibasic sequences exist in every topological vector space whose completion is not q-minimal.

The second type of problems Domański investigated is concerned with the splitting of twisted sums and the three space problem. A twisted sum of two topological vector spaces Y and Z is a topological vector space X with a subspace Y_1 isomorphic to Y such that the quotient X/Y_1 is isomorphic to Z. The twisted sum is said to split if Y_1 is complemented in X. This can be represented in terms short topologically exact sequences as follows

$$(*) \qquad 0 \to Y \to X \to Z \to 0,$$

where $j : Y \to Z$ and $q : X \to Z$ are open continuous linear maps and $j(Y) = \ker q$. The twisted sum X splits if and only if there is a continuous linear right inverse $T : Z \to X$ of q, that is, $q \circ T$ coincides with the identity on Z. In this case we also say that the topologically exact sequence $(*)$ splits. There are two main questions about twisted sums. The first one is the so-called three-space problem. A property (P) of topological vector spaces is a three-space property if every twisted sum of Y and Z has (P) whenever Y and Z have property (P). Kalton proved in 1978 that being a locally convex space is not a three-space property. The second main question is to characterize those pairs (Y, Z) of topological vector spaces such that every twisted sum of Y and Z splits. In modern terminology, using homological methods, this is expressed by writing $Ext^1(Z, Y) = 0$. Already in the mid eighties there was an extensive literature about these two problems. See the extensive work of S. Dierolf about the three-space problem reported by Frerick and Wengenroth in [19] and the seminal paper by Vogt [61]. In his papers [D29], [D31] and [D32] Paweł proved among many others the following results:

Theorem 2.1 *(1) Every twisted sum of a Banach space Y and a nuclear space Z splits.*

(2) Every twisted sum of a nuclear Fréchet space Y and a normed space Z splits if and only if every twisted sum of Y equal to the scalar field and Z splits. This characterization also holds if Y is a Banach space and Z is a Köthe sequence space.

(3) A locally convex space Z satisfies that every twisted sum of an arbitrary locally convex space Y and Z is locally convex if and only if, for every index set I, every twisted sum of $\ell_\infty(I)$ and Z splits.

Some years later, in 1992, Paweł wrote the paper [D22] with C. Fernández, J.C. Díaz and S. Dierolf on the three-space problem for dual Fréchet spaces. A Fréchet space is called a dual space if it is the strong dual of a barrelled (DF)-space in the sense of Grothendieck. It was proved in the paper that the twisted sum X of two dual Fréchet spaces Y and Z need not be a dual Fréchet space. However, they also proved that this is the case if every bounded set B in Z is contained in the image $q(A)$ of a bounded set A in X by the open surjection $q : X \to Z$.

3 Injective locally convex spaces

s.injective

Paweł Domański dedicated several papers between 1989 and 1993 to injective locally convex spaces and related questions about the structure theory of Fréchet spaces. In this section we briefly report about his work in this direction. A locally convex space is called injective if it is complemented in any locally convex space containing it. Clearly an injective locally convex space can be embedded as a complemented subspace of a product $\prod_{i \in I} \ell_\infty(\Gamma_i)$ of Banach spaces $\ell_\infty(\Gamma_i)$ of bounded families on the set Γ_i. The product of injective Banach space is an injective locally convex space. These considerations led L. Nachbin in 1960 to conjecture that every injective locally convex space should be isomorphic to a product of injective Banach spaces. This seems to be still an open question. The related problem, asked by Domański, Metafune and Moscatelli, whether every complemented subspace of a countable product of Banach spaces is isomorphic to a product of Banach spaces, was solved in the negative by M.I. Ostrovskii in [47] in 1996. Later on in 2001, Chigogidze [11] proved that every injective locally convex space is isomorphic to a product of injective Fréchet spaces.

Motivated by Nachbin's question mentioned above, Paweł contributed many interesting results. In [D33] he proved that a complemented subspace of a arbitrary product of Hilbert spaces is isomorphic to a product of Hilbert spaces. On the other hand, in the joint article [D77] with

Ortyński they presented a systematic study of complemented subspaces of products of Banach spaces which are L_1-predual spaces or $\ell_p(\Gamma), 1 \leq p \leq \infty$. As a consequence of their results they proved that $(\ell_p)^{\mathcal{M}}, 1 \leq p \leq \infty$ is primary for each cardinal number $\mathcal{M}$. A locally convex X is primary if each time one has the topological decomposition $X = Y \oplus Z$, then either Y or Z is isomorphic to X. Extending a result due to Lindenstrauss they also proved that an injective locally convex space is isomorphic to a product of copies of the scalar field or it contains a copy of ℓ_∞. The following result was proved in [D34].

Theorem 3.1 *Let X be a Hausdorff locally compact topological space. If the space $C(X)$ of continuous functions on X endowed with the topology of uniform convergence on compact sets is injective, then it either contains a copy of $\prod_{i\in\mathbb{N}} \ell_\infty(\Gamma_i), \Gamma_i$ uncountable, or $C(X)$ is isomorphic to $\prod_{i\in\mathbb{N}} C(X_i)$, where each X_i is a compact-open subset of X.*

In [D37] Domański solved a problem of Wolfe, and in [D38] he introduced certain estimates of injectivity $i(U)$ of an open subset U in a completely regular Hausdorff topological space X that enabled him to generalize Banach space results due to Amir and to Isbell and Semadeni.

Paweł's article [D35] constitutes an important contribution to the local structure theory of Fréchet spaces. The aim of this paper is to extend the theory of Banach $\mathcal{L}_p$-spaces of Lindenstrauss and Pełzyński to the context of locally convex spaces. The main tool of the paper is the notion of ultrapower of locally convex spaces and in particular an interesting generalization of the Stern theorem which is proved in the article. The definitions of $\mathcal{L}_p$-spaces and $\mathcal{D}\mathcal{L}_p$-spaces, that are the analogues for (DF)-spaces, are technical. They are spaces "full" of subspaces "similar" to products (respectively, direct sums) of finite-dimensional ℓ_p spaces, and similarity is measured in terms of equicontinuity of the corresponding linear isomorphisms. Among many other results, the following ones are presented.

Theorem 3.2 *(1) For $p = 1, 2, \infty$, every complemented subspace of the product (resp. direct sum) of infinitely many $L_p(\mu)$-spaces, or L_1-predual spaces for $p = \infty$, is an $\mathcal{L}_p$-space (resp. $\mathcal{D}\mathcal{L}_p$-space).*

(2) For $1 \leq p \leq \infty$, the strong dual of a Fréchet $\mathcal{L}_p$-space (resp. (LB)-$\mathcal{D}\mathcal{L}_p$-space) is a complemented subspace of a space of type $\oplus_{i\in\mathbb{N}} L_q(\mu_i)$ (resp. $\prod_{i\in\mathbb{N}} L_q(\mu_i)$) with $1/p + 1/q = 1$.

(3) A Fréchet space is injective if and only if it is an $\mathcal{L}_\infty$-space that is complemented in its bidual.

(4) If $p \neq 2$, then there exist Fréchet $\mathcal{L}_p$-spaces (resp. (LB)-$\mathcal{D}\mathcal{L}_p$-spaces) that are not isomorphic to any complemented subspace of any product (resp. direct sum) of Banach spaces.

Every Fréchet $\mathcal{L}_p$-space is a quojection, that is a Fréchet space such that every quotient with a continous norm is a Banach space or equivalently a surjective limit of a sequence of Banach spaces. Moreover, every complete (LB)-$\mathcal{D}\mathcal{L}_p$-space is a strict (LB)-space. In fact, an extension of the principle of local reflexivity for quojections and for operators was necessary in the paper [D35] and it was obtained by Domański in [D36]. Quojections and prequojections (i.e. Fréchet spaces whose bidual is a quojection) play a relevant role in the structure theory of Fréchet spaces in connection with several problems. We refer the interested reader to the survey by Metafune and Moscatelli [46]. In fact, in his paper [D43] Paweł solved a question of Moscatelli and constructed a Fréchet space of continuous functions $C(X)$ on a completely regular Hausdorff space X, that is necessarily a quojection, but it is not isomorphic to a complemented subspace of a countable product of Banach spaces. Domański's research in this direction was complemented in his papers [D39] and [D40].

In 1987 Paweł prepared a re-worked and slightly extended version of his Ph.D. thesis, that had been presented at the University of Poznań under the supervision of Drewnowski. These notes, entitled "Extensions and liftings of linear operators" were never published. Due to the interest in extendable and liftable operators around 2000 by authors like Kalton, and Pełczyński [33], who quoted in fact Domański's notes, Paweł came back to this topic in his paper [D45] about ideals of extendable and liftable operators. The notes "Extensions and liftings of linear operators" connect the extension and lifting of operators with the splitting of short exact sequences. They are mainly concerned with the case of p-Banach spaces. The approach is new and it is based on operator ideals. This research is also naturally related to injective and projective locally convex spaces. Many applications, especially relevant for Banach spaces, are collected in the last chapter. For example $\mathcal{L}_\infty$ spaces are characterized in terms of liftings and $\mathcal{L}_1$ spaces in terms of extensions. Moreover, a Banach space Z is an $\mathcal{L}_1$ space if and only if every short exact sequence $0 \to Y \to X \to Z \to 0$ with Y a dual Banach space splits.

4 Joint work with Susanne Dierolf

s.dierolf

A celebrated theorem of Davis, Figiel, Johnson, and Pełczyński [15] says that every weakly compact operator between Banach spaces factorizes through a reflexive Banach space. In the realm of locally convex spaces there are two natural candidates for the generalization of (weakly) compact operators: Either mapping a neighbourhood of the origin into a relatively (weakly) compact set or mapping all bounded sets into relatively (weakly) compact ones. The former operators are still called (weakly) compact (and questions reduce quite easily to the Banach case), while the latter ones are usually called Montel (reflexive) operators. Susanne Dierolf and Domański considered in [D25] and [D27] the question whether every Montel operator between Fréchet spaces factorizes through a Fréchet Montel space and, by duality, whether every Montel operator between (LB)-spaces factorizes through a Montel (LB)-space. They proved for example that every Montel operator from a quasinormable Fréchet space into a Fréchet space factors through a Fréchet Schwartz space, and that every Montel operator from a Köthe echelon space of order one into a Köthe echelon space of order zero factors through a Fréchet Montel space. It turned out that this problem was related to a still open question about (LB)-spaces, that is attributed to Grothendieck. An (LB)-space $E = \mathrm{ind}_{n \in \mathbb{N}} E_n$ is called regular if every bounded subset in E is contained and bounded in one of the steps E_n. Every complete (LB)-space is regular. It is unknown if the converse holds. There was much research concerning the completeness of (LB)-spaces and (LF)-spaces in the 1980's and 1990's. See Bierstedt, Bonet [5] and Wengenroth [69]. The following result was obtained in [D25].

Theorem 4.1 *Consider the following conditions.*

(a) Every regular (LB)-space is complete.

(b) Every Montel operator between (LB)-spaces factorizes through a Montel (LB)-space.

(c) For every complete (LB)-space F the space $C(\beta\mathbb{N}, F)$ is bornological.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) holds.

The proof required a deep investigation of the structure of compact sets in complete (LB)-spaces. It is also an open problem if the space $C(K, X)$ is bornological, hence an (LB)-space, for every Hausdorff compact topological space K and every (LB)-space E. This question was explicitly formulated by Bierstedt and Schmets in the 1977. The following interesting, partial positive results were presented in [D26] and [D44].

Theorem 4.2 *(1) If E is a Montel (LB)-space, then $c_0(E)$ is bornological.*

(2) If k_∞ is a Köthe co-echelon space of order infinity, then $c_0(k_\infty)$ is bornological.

(3) If K is a Hausdorff compact topological space, then $C(K, \lambda_p(A)')$ is bornological for every Köthe echelon space $\lambda_p(A), 1 < p < \infty$.

More results about bornological spaces of type $C(K, F)$ are due to Frerick and Wengenroth [18]. In joint work with Bonet and Mujica [D19], Domański obtained related results about the completeness of spaces of vector valued holomorphic germs.

The research about factorization of operators between Fréchet spaces was continued by Paweł in the joint work with Juan Carlos Díaz [D23] in the case of weakly compact and reflexive operators. Among other results, they proved that if a Fréchet space E is either quasinormable or a distinguished Köthe echelon space of order one, then every reflexive operator from E into a Fréchet space F factors through a reflexive Fréchet space. The proof required a careful analysis of weakly compact subsets of Köthe co-echelon spaces of order infinity.

5 Joint work with Lech Drewnowski. Spaces of operators and vector valued continuous functions

s.drew

For Banach spaces X and Y we denote by $K_{w^*}(X', Y)$ the space of all compact weak*-weak continuous linear operators $T : X' \to Y$. It is contained in the space $L_{w^*}(X', Y)$ of all weak*-weak continuous linear operators from X' into Y. Many examples of operators, vector valued measures and vector valued functions can be naturally identified with spaces of this type. Drewnowski, who was the thesis advisor of Domański, investigated the question if copies of c_0 and ℓ_∞ are contained in these spaces. This problem is interesting in itself, but it is also relevant in connection with the question whether the small space is complemented in the biggest one. One of the main motivating problems was whether there exists an infinite dimensional Banach space X such that the space $K(X)$ of compact operators on X is complemented in the space $L(X)$ of all continuous operators on X. This problem was only solved in 2011 by Argyros and Haydon [1]. They constructed a Banach space X_K such that every continuous operator on X_K has the form $\lambda I + K$ for a scalar λ and a compact operator K. This is the first infinite dimensional Banach space in which every continuous operator has a nontrivial invariant subspace and for which the space $L(X_K)$ is separable. The construction combines techniques due to Bourgain and Delbaen as well as more recent tools from the theory of hereditarily indecomposable Banach spaces of Gowers and Maurey [21].

Drewnowski, inspired by an earlier result of Kalton, proved in 1990 in [16] that the Banach space $K_{w^*}(X', Y)$ contains an isomorphic copy of ℓ_∞ if and only if either X or Y contain such a copy. This result implies Kalton's theorem, but it is more powerful, as it yields consequences about tensor products and spaces of vector valued measures and continuous functions. The theorems of Drewnowski triggered a big amount of research on related topics. They were extended to the setting of operators on Fréchet and (DF)-spaces by Domański together with Bonet, Lindström and Ramanujan in [D13] and [D17].

In the early 1990's Domański and Drewnowski studied in [D50], [D51] and [D52] spaces $C(K, X, \tau)$ of vector valued continuous functions $f : K \to X$ from a compact space K into a Banach space X which are continuous from K into the space X endowed with the vector topology τ. If τ is the original topology we write $C(K, X)$. The space $C(K, X, \tau)$ is endowed with the topology of uniform convergence with respect to the original topology of X. The still open conjecture is that the space $C(K,X)$ is complemented in $C(K,X,\tau)$ if and only if both spaces coincide. This research was also related to injective locally convex spaces; see Section 3. Among many others they proved the following results.

Theorem 5.1 *(1) If the Banach space X contains a τ-convergent sequence which is not norm convergent, then $C(K,X)$ is not complemented in $C(K,X,\tau)$ for every infinite compact space K. In particular $C(K,X')$ is not complemented in $C(K,X',w*)$ for all infinite dimensional Banach spaces X.*

(2) $C(K,X)$ is complemented in $C(K,X,w)$ if and only if every weakly null sequence in X is a norm null sequence.

(3) Let E be a completely regular Hausdorff space that contains an infinite compact set and let X be a non-Montel Fréchet space. Then $C(E,X)$ contains a complemented copy of c_0. In particular $C(E,X)$ is not injective.

(4) Let E be a completely regular Hausdorff space that contains an infinite compact set and let X be a locally convex space containing an isomorphic copy of ℓ_1. Then $C(E,X,w)$ contains a complemented copy of ℓ_1, hence it is not injective.

In his note [D42] Paweł proved that the space of Riemann integrable functions is not complemented $L_\infty(0,1)$ since it contains a complemented copy of c_0.

As mentioned in [D86], the collaboration of Domański and Drewnowski in this topic produced a long preprint called "Injective spaces of bounded vector sequences and spaces of operators" which pushes the methods developed by them to its limits. This preprint was never published.

Paweł worked shortly afterwards with Lindström [D73] and with Lindström and Schlüchtermann [D75] on a somewhat related topic. A locally convex space X is called a Grothendieck space if every weak* convergent sequence in X' is weakly convergent. Accordingly, an operator $T : X \to Y$ between Banach spaces is called a Grothendieck operator if its transpose $T' : Y' \to X'$ maps weak* convergent sequences into weakly convergent ones.

Theorem 5.2 *(1) Let X and Y be Fréchet spaces such that Y is Montel and either X''_b or Y has the approximation property. Then the injective tensor product $X\hat{\otimes}_\varepsilon Y$ is a Grothendieck space if and only if X is a Grothendieck space. This had been proved for $X = C(K)$ by Freniche in 1986.*

(2) If $T : X \to Y$ is a Grothendieck operator and $S : X \to Y$ is compact, then the completed tensor product $T\hat{\otimes}_\varepsilon S : X\hat{\otimes}_\varepsilon Y \to X\hat{\otimes}_\varepsilon Y$ is a Grothendieck operator.

J.C. Díaz and Domański studied in [D24] when the complete injective tensor product of two distinguished Fréchet spaces is also distinguished.

6 Composition operators on weighted spaces of holomorphic functions

s.compoH

During his stay in Valencia in the academic year 1996/97 Paweł started to collaborate with Bonet, Linström and Taskinen on composition operators on weighted Banach spaces of holomorphic functions on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. We explain the context and state some results. Let G be an open subset of $\mathbb{C}$, and let $v : G \to \mathbb{R}$ be a continuous and strictly positive weight on G. We define the following *weighted Banach spaces of holomorphic functions* on G

$$Hv(G) := \{f \in H(G);\ ||f||_v := \sup_{z \in G} v(z)|f(z)| < \infty\},$$

$$Hv_0(G) := \{f \in H(G);\ v|f| \text{ vanishes at } \infty \text{ on } G\}.$$

Recall that a function g vanishes at infinity on G if for every $\varepsilon > 0$ there is a compact subset K of G such that $|g(z)| < \varepsilon$ if $z \notin K$.

We assume $Hv(G) \neq \{0\}$, which always happens in the cases considered below. Banach spaces of the type mentioned above appear naturally in the study of growth conditions of analytic functions and have been considered, since the work of Shields and Williams, by many authors like Bierstedt, Meise, Summers Kaballo and Lusky, among others. If v is a (continuous and strictly positive) weight on G, its *associated weight* is defined by $\tilde{v}(z) := 1/||\delta_z||_{Hv(G)'}$. By our assumption above, $\tilde{v}(z)$ is finite for every $z \in G$. Moreover $v \leq \tilde{v}$ on G, $1/\tilde{v}$ is continuous and subharmonic, and the Banach spaces $Hv(G)$ and $H\tilde{v}(G)$ coincide isometrically. A weight v is called *essential* if there is $C \geq 1$ such that $v \leq \tilde{v} \leq Cv$ on G.

Composition operators on various spaces of analytic functions on the unit disc have been studied very thoroughly by a number of authors; cf. the books of Cowen, MacCluer [14] and of Shapiro [55]. Composition operators constitute still now a very active area of research as a search in the databases of Mathematical Reviews or Zentralblatt shows. We now some results from the papers [D18] and [D14]. To do this, we suppose that all the weights v are radial, non-increasing on $\mathbb{D}$ and satisfy that $\lim_{r \to 1-} v(r) = 0$. We denote by $\varphi : \mathbb{D} \to \mathbb{D}$ an analytic map. The composition operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by $C_\varphi(f) := f \circ \varphi$.

Theorem 6.1 *The following conditions are equivalent for the composition operator $C_\varphi : Hv(\mathbb{D}) \to Hw(\mathbb{D})$:*

(1) The operator C_φ is continuous.

(2) $\sup_{z \in \mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))} < \infty$.

(3) $\sup_n \frac{||\varphi(z)^n||_w}{||z^n||_v} < \infty$.

Theorem 6.2 *The following conditions are equivalent for the composition operator $C_\varphi : Hv(\mathbb{D}) \to Hw(\mathbb{D})$:*

(1) The operator C_φ is (weakly) compact.

(2) $\lim_{|z| \to 1} \frac{w(z)}{\tilde{v}(\varphi(z))} = 0$.

(3) $\lim_{n \to \infty} \frac{||\varphi(z)^n||_w}{||z^n||_v} = 0$.

Estimates of the essential norm of a composition operator $C_\varphi : Hv(\mathbb{D}) \to Hw(\mathbb{D})$ were obtained in [D14]. Extensions of these results and consequences for (weighted) composition operators on Bloch spaces were obtained by Montes and by Contreras and Hernández-Díaz. The case of composition operators on weighted spaces of vector valued holomorphic functions was discussed in [D16] where previous work by Liu, Saksman and Tylli was continued.

In the paper [D15] pointwise multiplication operators $M_g : Hv(\mathbb{D}) \to Hv(\mathbb{D})$, $M_g(f) := gf$, for $g \in H(\mathbb{D}), g \neq 0$, were investigated. Pointwise multiplication operators between different spaces of analytic functions have been studied by many authors, like Axler, Luecking, McDonald and Sundberg and Vukotic. In [D15] properties like continuity, isomorphism, being a Fredholm operator or having closed range were studied and in some cases characterized. Cichoń and Seip improved later some of these results.

Domański and Lindström [D74] investigated interpolation and sampling in $Hv(\mathbb{D})$. They used ideas and results of Seip. Let v be a continuous and strictly positive weight on $\mathbb{D}$. For a given sequence $\Gamma = (z_n)_n \subset \mathbb{D}$, we define $T : Hv(\mathbb{D}) \to \ell_\infty$ by $T(f) := (f(z_n)v(z_n))_n$. The sequence Γ is called a set of interpolation for v if T is surjective, a set of linear interpolation for v if T has a continuous and linear right inverse and a sampling set for v if T is a monomorphism. Every sampling set is a set of uniqueness for $Hv(\mathbb{D})$. The classical interpolation problem in H^∞ ($Hv(\mathbb{D})$ for $v \equiv 1$) was solved by Carleson in 1952. Seip [53] characterized the sets of interpolation and sampling for $A^{-p} := Hv_p(\mathbb{D})$ if $v_p(z) = (1-|z|^2)^p$, $p > 0$, in terms of certain densities. Domański and Lindström extended some of the results of Seip and characterized (linearly) interpolating and sampling sequences in $\mathbb{D}$ in terms of certain densities related to the weight v. As a consequence of their results they obtained that, if $v(z) = (1-|z|^2)^p \log(e/(1-|z|))^\varepsilon$, then $Hv(\mathbb{D})$ and $Hv_p(\mathbb{D})$ have the same sets of interpolation and sampling, although they do not coincide as Banach spaces.

A related direction of research was pursued by Domański and Bonet in [D3]. For $f \in H(\mathbb{D})$, $p > 0$, and $S \subset \mathbb{D}$, define $\|f\|_{p,S} = \sup_{z\in S}(1-|z|^2)^p|f(z)|$. The Banach space $A^{-p} = \{f \in H(\mathbb{D}) : \|f\|_{p,\mathbb{D}} < \infty\}$ coincides with $Hv(\mathbb{D})$ for $v(z) = (1-|z|^2)^p$. The Korenblum space is $A^{-\infty} = \mathrm{ind}_{p>0} A^{-p} = \cup_{p>0} A^{-p}$. A subset $S \subset \mathbb{D}$ is called (p, q)-sampling ($p \le q$) if there exists $C > 0$ such that $\|f\|_q \le \|f\|_{p,S}$ for all $f \in A^{-q}$. The set S is (p, p)-sampling if and only if S is sampling for A^{-p} in the sense defined above, i.e. if $T : f \longrightarrow ((v_p f)|_S) \in l_\infty(S)$ is an isomorphism into. The article [D3] presents several results and examples concerning (p, q)-sampling sets, as well as a study of the relation of this concept with $A^{-\infty}$-sampling sets in the sense of Horowitz, Korenblum, Pinchuk and with weakly sufficient sets for $A^{-\infty}$ in the sense of Ehrenpreis.

7 Fréchet and (LB)-algebras

Domański was also interested in topological algebras. In fact, together with Bonet they investigated in [D7] the Köthe coechelon spaces $k_p(V), 1 \le p \le \infty$ or $p = 0$, that are locally convex algebras for pointwise multiplication. They characterized when $k_p(V)$ is an algebra for the pointwise multiplication in terms of the matrix V, as well as when this algebra is unital, locally m-convex, a Q-algebra or an inductive limit of a sequence of Banach algebras. These last three conditions are equivalent in this context. Entire functions acting on the algebras $k_p(V)$ are investigated. Maximal regular ideals and multiplicative functional on an algebra are analyzed, too. Finally, it is proved that all ideals in $k_p(V)$ are solid if and only if this algebra is unital.

Around 2010 Paweł started the investigation of the algebra of smooth operators $L(s', s)$, that is the non-commutative Fréchet algebra of all continuous linear operators from the dual s' of the space s of rapidly decreasing sequences into s. His results can be found in “Algebra of smooth operators” (unpublished note available at www.staff.amu.edu.pl/domanski/salgebra1.pdf).

As a Fréchet space $L(s', s)$ is nuclear and isomorphic to the space s. This algebra appears and plays a significant role in K-theory of Fréchet algebras in the work of Bhatt and Inoue, Cuntz, Glöckner and Langkamp, and Phillips, in noncommutative geometry (Blackadar and Cuntz, Connes) and in C^*-dynamical systems (Elliot, Natsume and Nest). Moreover, it was considered by Schmüdgen in the context of algebras of unbounded operators. This algebra serves also as an example of a Fréchet operator space in the sense of Effros and Webster. Finally, it is also present in quantum mechanics, where it is called the space of physical states and its dual is the so-called space of observables. It can be identified canonically with the algebra of rapidly decreasing matrices with the matrix product and matrix complex conjugation.

In his paper Domanski presented some basic spectral properties of $L(s', s)$ collected various representations of it. In particular, he showed that the algebra of smooth operators consists of compact operators of s-type on ℓ_2, i.e., operators with sequence of singular numbers belonging necessarily to s. Moreover, he proved that the spectrum of any element of $L(s', s)$ is equal to its spectrum in the algebra $L(\ell_2)$ and thus the sequence of eigenvalues belongs to s as well. Moreover, the algebra is a Q-algebra, i.e., the set of invertible elements is open. This direction of research was successfully continued by Ciaś and Piszczek, former students of Paweł. They investigated functional calculus on $L(s', s)$ and closed commutative *-subalgebras, automatic continuity of positive functionals and derivations, amenability, closed maximal ideals, Grothendieck's inequality and the multiplier algebra of $L(s', s)$, among other topics. We refer to their work for precise references.

8 Splitting of smooth and distributional complexes

ss.splitting

A large part of Paweł's work was concerned with splitting of exact sequences in several abstract or concrete analytic settings (see also Sections 2 and 10). In this section we will review his joint results with Vogt [D78, D79, D81] and related work on splitting of complexes of smooth functions or distributions. Previously, Palamodov [49] proved that the $\overline{\partial}$ - complex splits for positive dimensions (but in general not at the 0^{th} place which is a result due to Grothendieck) and that the same holds for complexes of matrices of partial differential operators with constant coefficients over convex open sets ([50]). The splitting problem for a single partial differential operator (known as the problem of Laurent Schwartz) has been solved by Meise, Taylor and Vogt in several spaces of (ultra)differentiable functions in a series of papers starting with [43]. In [D78, D79, D81] Domański and Vogt obtained far reaching extensions of these results based on abstract analysis omitting as far as possible any analytical properties of specific concrete operators. Behind this is the $(DN)-(\Omega)$ type splitting theory of Vogt (see [44]) which is extended to cartesian products of spaces of this type. Linear topological invariants like $(DN), (\Omega), (PA), (P\Omega)\dots$ will also be important in the following sections. We will always dispense with presenting the definitions explicitly and refer to the corresponding literature. A common generalization of the $(DN)-(\Omega)$ splitting theorem and Maurey's extension theorem was proved by Defant, Domański and Mastyło [D21]. A complete solution for the splitting problem in Fréchet-Hilbert spaces by a condition of type (S) was obtained by Domański and Mastyło [D76].

We can only sketch two of the main results of [D78, D79, D81] here. The key notion is the category of graded Fréchet spaces, that is, Fréchet spaces E with a fixed (equivalence class of) projective spectra $\mathscr{E} := (E_n, i_n^k)$ of Fréchet spaces E_n and linking maps i_n^k such that $E = \text{proj}\,_n E_n$. Correspondingly, graded subspaces and quotients, graded homomorphisms and graded exact sequences are introduced in [D78]. Also, the existence of exact projective resolutions relative to sequences of Fréchet spaces is needed (see [D57]). A natural grading on $C^\infty(\Omega)$ is defined by $C^\infty(K_n)$ with a compact increasing exhaustion $K_n \subset \Omega$ and restrictions as linking maps. Notice that (systems of) partial differential operators respect this grading. A grading $\mathscr{E}$ is called strict if

$$\forall k \in \mathbb{N}\ \exists \ell \in \mathbb{N}\ \forall m \geq \ell : i_k^m(E_m) = i_k^\ell(E_\ell).$$

The following theorem considerably improves any of the results known previously.

split1 **Theorem 8.1** *([D78, Corollary 5.6]) Let $\Omega_n \subset \mathbb{R}^d$ be open and let $T_0 : C^\infty(\Omega_0)^s \to C^\infty(\Omega_1)^{s_1}$ be a matrix of convolution operators. If the complex*

$$0 \to \ker(T_0) \to C^\infty(\Omega_0)^{s_0} \xrightarrow{T_0} C^\infty(\Omega_1)^{s_1} \xrightarrow{T_1} C^\infty(\Omega_2)^{s_2} \to \dots \qquad (*)$$

is algebraically exact then the complex splits at $C^\infty(\Omega_k)^{s_k}$ for any $k \geq 1$. The complex splits at $C^\infty(\Omega_0)^{s_0}$ iff $\ker(T_0)$ is strict graded.

The analogous question is studied in [D79, D81] for distributions instead of smooth functions. Here the language of PLS-spaces is used and the results are even more general than in Theorem 8.1.

Theorem 8.2 *Let $\Omega_n \subset \mathbb{R}^d$ be open and let T_k be continuous linear operators.*
(a) If the complex

$$0 \to \ker(T_0) \to \big(D'(\Omega_0)\big)^{s_0} \xrightarrow{T_0} \big(D'(\Omega_1)\big)^{s_1} \xrightarrow{T_1} \big(D'(\Omega_2)\big)^{s_2} \to \dots \qquad (*)$$

is algebraically exact then the complex splits at $(D'(\Omega_0))^{s_k}$ for any $k \geq 1$.
(b) Let $T_0 : (D'(\Omega_0))^{s_0} \to (D'(\Omega_1))^{s_1}$ be a matrix of convolution operators. Then the complex splits at $(D'(\Omega_0))^{s_0}$ iff $\ker(T_0)$ is a strict projective limit of LB-spaces.

The corresponding splitting result for short exact sequences has been improved by Wengenroth [68].

9 Spaces of real analytic functions

s.structure

One of the main interests of Domański was in the space $\mathscr{A}(\Omega)$ of real analytic functions, its topological structure and operators acting there. The following sections are devoted to this part of his scientific work. An impressive overview on that subject is contained in the survey article [D49]. Also, one of his outstanding results is concerned with real analytic functions, namely, the basis problem for real analytic functions. The existence of (Schauder) bases is an important subject in analysis. In fact, the question whether every separable Banach space has a basis dates back to the book of Banach [3] and has been solved in the negative in various classes of locally convex spaces including subspaces of $\ell_p, p \neq 2$, (Enflo), nuclear Fréchet spaces (Mitjagin-Henkin) and in $L(\ell^2)$ (Szankowski) the latter being the only "natural" space without basis (for a concrete Fréchet space of smooth functions without basis see Vogt [64]). The basis problem in $\mathscr{A}(\Omega)$ remained open until Domański and Vogt [D80] published their celebrated result which got the Scientific Award of the Ministry of National Education 2001.

basis **Theorem 9.1** *$\mathscr{A}(\Omega)$ has no basis for every open set $\Omega \subset \mathbb{R}^d$.*

The proof is based on a careful study of linear topological invariants for complemented Fréchet subspaces of $\mathscr{A}(\Omega)$. Specifically, such spaces would have properties $(\overline{\overline{\Omega}})$ and $(\underline{DN})$ which implies that they are nuclear Banach spaces, hence finite dimensional. By the following result, $\mathscr{A}(\Omega)$ is an LB-space, which is false since $\mathscr{A}(\Omega)$ has $\omega := \mathbb{C}^{\mathbb{N}}$ as a quotient by interpolation.

altern **Theorem 9.2** *([D80, Theorem 2.2]) If an ultrabornological Köthe PLS-space E does not admit an infinite dimensional complemented Fréchet subspace then E is an LS-space.*

In [D82] more explicit analytical tools are presented for the proof of Theorem 9.1 for $\mathscr{A}(\mathbb{R})$. Also, the above structural results are applied to obtain the existence of right inverses for convolution operators on $\mathscr{A}(\mathbb{R})$ (cf. Langenbruch [40]).

Using composition operators the subspace structure of $\mathscr{A}(\Omega)$ was clarified in [D58]. Namely, for $\Omega \subset \mathbb{R}^d$ the (LB)-subspaces of $\mathscr{A}(\Omega)$ are isomorphic to a subspace of $H(\overline{\mathbb{D}^d})$ (where $\mathbb{D}$ is the unit disc) while the Fréchet subspaces of $\mathscr{A}(\Omega)$ are isomorphic to subspaces of $H(\mathbb{D}^d)^r$ (where r is the number of components of Ω). In continuation of [D83], Domański, Frerick and Vogt [D53] determined the Fréchet quotients E of $\mathscr{A}(\Omega)$ by the fact that E has $(\overline{\overline{\Omega}})$ and is a quotient of $\mathscr{H}(\mathbb{D})$.

Complemented ideals in $\mathscr{A}(\mathbb{R}^d)$ were studied in [D85] extending and using previous results of Vogt [62, 63]. Specifically, if the vanishing ideal $J_V(\mathbb{R}^d)$, V a complex analytic variety in a neighborhood of $\mathbb{R}^d$, is complemented in $\mathscr{A}(\mathbb{R}^d)$ then V satisfies the local Phragmen-Lindelöf condition of Hörmander at every real point of V (see [D85, Theorem 2.4]).

10 Parameter dependence of solutions of partial differential equations and splitting of short exact sequences of $PLS-$ spaces

s.parameter

The classical problem of parameter dependence of solutions of linear equations can be formulated as follows: we are given a locally convex space $F(\Omega)$ of scalar (generalized) functions on an open set $\Omega \subset \mathbb{R}^d$ (like real analytic functions, (ultra)differentiable functions or (ultra)distributions), a continuous linear operator operator $T : F(\Omega) \to F(\Omega)$ and an $E-$valued function $f \in F(\Omega, E)$ where E is a locally convex space of smooth (holomorphic, real analytic or generalized, respectively) functions. Can we find an $E-$valued function $g \in F(\Omega, E)$ solving $Tg = f$? Classical results for partial differential operators are due to Trèves [57, 58] and Browder [8], holomorphic parameter dependance was solved by Mantlik [41, 42]. The question is closely related to a tensor product representation of $F(\Omega, E)$, and, correspondingly, to the surjectivity of the tensorized map $T \otimes id_E$ (and to the splitting of certain short exact sequences, see below) and has been extensively been studied by Bonet and Domański (see [D2, D4, D6, D47, D48]). To define $E-$valued real analytic functions f on an open set $\Omega \subset \mathbb{R}^d$ we in principle have two choices here, namely, that f locally is an $E-$valued power series (i.e. $f \in \mathscr{A}_t(\Omega, E)$) or that f is a (weakly) real analytic function (i.e. $f \in \mathscr{A}(\Omega, E)$, that is, $u \circ f$ is real analytic for any $u \in E'$), the latter one being the proper choice as it turns out, since $\mathscr{A}(\Omega, E) = \mathscr{A}(\Omega)\varepsilon E := L(\mathscr{A}(\Omega)'_b, E)$ if E is sequentially complete. If E is a sequential complete $DF-$space then $\mathscr{A}(V, E) = \mathscr{A}_t(V, E)$ [D1]. For Fréchet spaces however both classes in general do not coincide, in fact the following characterization was proved in [D1].

Theorem 10.1 *Let E be a Fréchet space. Then $\mathscr{A}(\Omega, E) = \mathscr{A}_t(\Omega, E)$ iff E has the property (DN).*

Notice that the space $\mathscr{H}(V)$ of holomorphic functions on a Stein manifold V has property (DN) iff V has the Liouville property.

The results of [D1, D2] on real analytic parameter dependance can be summarized as follows.

Theorem 10.2 *Let $\Omega_k \subset \mathbb{R}^{d_k}$ be open sets and let $T : \mathscr{A}(\Omega_1) \to \mathscr{A}(\Omega_2)$ be a surjective continuous linear map. Then $T \otimes id_E : \mathscr{A}(\Omega_1, E) \to \mathscr{A}(\Omega_2, E)$ is surjective in the following cases.*
(a) E is a Fréchet quojection.
(b) E is a Fréchet space with property (DN).
(c) E is a complete LB-space such that E'_b has property $(\overline{\overline{\Omega}})$.

In their fundamental papers [D4, D6] Bonet and Domański treated the question of parameter dependance in the context of splitting of short exact sequences of $PLS-$spaces and solved the splitting problem using the connection of functors $Proj^1$ and Ext^1 . Recall that $PLS-$spaces are countable projective limits of $LS-$spaces. Many of the standard spaces used in analysis are of that type (e.g. any Fréchet Schwartz space, the spaces of distributions, (ultra)differentiable functions, (ultra)distributions). An informative overview on $PLS-$spaces is given in [D46]. An exact sequence of $PLS-$spaces X, Y, Z with continuous linear mappings j and q

$$0 \to X \xrightarrow{j} Y \xrightarrow{q} Z \to 0 \qquad (*)$$

is called topologically exact if j is a topological isomorphism onto $\ker(q)$ and q is open. We say that $\mathrm{Ext}^1_{PLS}(Z, X) = 0$ if any topologically exact sequence (*) splits (for any $PLS-$ space Y), i.e. if q has a continuous linear right inverse. The following result is an important extension of the $(DN) - (\Omega)$ splitting theory of Vogt (see [D4, Theorem 5.5] and [D6, Corollary 6.4]).

splittingp **Theorem 10.3** *Let X be a $PLS-$space. Then $\mathrm{Ext}^1_{PLS}(F, X) = 0$ in the following cases.*
(a) F is a nuclear Fréchet space and (i) or (ii) holds where
(i) F has (DN) and X has $(P\Omega)$.
(ii) F has $(\underline{DN})$ and X has $(P\overline{\overline{\Omega}})$.
(b) F is an $LN-$space and and (i) or (ii) holds where
(i) F' has $(\overline{\overline{\Omega}})$ and X has $(P\underline{A})$
(ii) F' has (Ω) and X has (PA)

For the classical spaces of analysis and sequence spaces F it is well known which of the above linear topological invariants hold or not (see e.g. [D4, Sections 5 and 6], in case of quasianalytic Roumieu type classes this is studied in [D5]). To get results concerning $E-$valued equations we have to apply Theorem 10.3 to $X := \ker(T)$ (and $F := E'_b$). Also for $\ker(T)$ corresponding results are known for several classes of operators. We mention only one special case [D6, Theorem 5.5].

Corollary 10.4 *Let $\Omega \subset \mathbb{R}^d$ be open and convex and let E be a nuclear Fréchet space. Then $P(D) : \mathscr{D}'(\Omega, E) \to \mathscr{D}'(\Omega, E)$ is surjective if E has property (Ω).*

In [D47, D48] Domański pushed the study of the functor $\mathrm{Ext}^1_{PLS}(F, X)$ still a step further allowing F and X to be nuclear $PLS-$spaces. Instead of formulating the general result here we point out that in this way distributional solutions for linear equations depending on a real analytic parameter could be treated obtaining (see [D47, Corollary 6.4])

Theorem 10.5 *Let U be a real analytic non compact connected manifold and let $\Omega \subset \mathbb{R}^d$ be open. Let $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be a surjective linear operator. Then*

$$T \otimes id_{\mathscr{A}(U)} : \mathscr{D}'(\Omega)\varepsilon\mathscr{A}(U) \to \mathscr{D}'(\Omega)\varepsilon\mathscr{A}(U)$$

is surjective iff $\ker(T)$ *satisfies the dual interpolation estimate for small θ.*

The condition above can be evaluated for constant coefficient partial differential operators $P(D)$ on convex sets Ω using Phragmen-Lindelöff type conditions valid on the characteristic variety of P (see [D47, Section 7]). This implies that $P(D) : \mathscr{D}'(\mathbb{R}^d, \mathscr{A}(U)) \to \mathscr{D}'(\mathbb{R}^d, \mathscr{A}(U))$ is never surjective if P is hypoelliptic and if Ω is convex. Also, the following surprising inheritance result is obtained in [D47].

Theorem 10.6 *Let $\Omega \subset \mathbb{R}^d$ be open and convex and let $P(D) : \mathscr{D}'(\Omega, \mathscr{A}(U)) \to \mathscr{D}'(\Omega, \mathscr{A}(U))$ be surjective. Then*

(a) $P(D) : \mathscr{D}'(\mathbb{R}^d, \mathscr{A}(U)) \to \mathscr{D}'(\mathbb{R}^d, \mathscr{A}(U))$ is surjective.

(b) $P(D) : \mathscr{D}'(H, \mathscr{A}(U)) \to \mathscr{D}'(H, \mathscr{A}(U))$ is surjective for any halfspace H such that ∂H is parallel to a tangent hyperplane of $\partial\Omega$ (in a point of smoothness).

(c) The principal part $P_m(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ has a right inverse.

As mentioned already, the right inverse problem for partial differential operators has been studied intensively by Meise, Taylor and Vogt. The methods of [D47] have been transferred to operators on Roumieu type ultradifferentiable classes $\mathscr{E}(\Omega)$ (see [D48]). Distributional equations depending on a distributional parameter can be viewed as augmented operators on distributions. The surjectivity of these operators has been studied by Kalmes in a series of papers [29, 30, 31].

The $E-$valued interpolation problem for real analytic functions on domains $\omega \subset \mathbb{R}^d$ was solved in [D20] for sequentially complete $(DF)-$spaces E. In fact, this problem always has a solution iff E has property $(\underline{A})$. The scalar case is related to Eidelheit sequences on $\mathscr{A}(\omega)$ which were studied in detail in [D84] including a characterization of Eidelheit sequences on $\mathscr{A}(\omega)$ (see [D84, Theorem 2.2]).

11 Composition operators on real analytic functions

s.compoB

11.1 The range of composition operators

ss.RangecompoB

Composition operators and topological properties of its range have been intensively studied on various spaces of holomorphic functions (see Section 6 and the books [54, 13]) and on spaces of smooth functions, respectively (see [56, 20, 7, 6]). Domański, Langenbruch and Goliński [D58, D59, D60, D54] initiated the study of these operators on real analytic functions. Let M, N be real analytic manifolds and let $\varphi : M \to N$ be real analytic. The corresponding composition operator C_φ is defined by

$$C_\varphi : \mathscr{A}(N) \to \mathscr{A}(M), f \to f \circ \varphi, \text{ for } f \in \mathscr{A}(N).$$

The main problem is to characterize when C_φ has closed range, is open onto its range, and is a topological embedding, respectively. Notice that in contrast to the smooth case the first two questions are not equivalent for real analytic functions.

To state some of the main results, some notation is needed: Let M, N and φ as above and let $S \subset N$. φ is called semiproper if for any compact set $K \subset N$ there is a compact set $L \subset M$ such that $\varphi(L) = \varphi(M) \cap K$. S has the global extension property if every real analytic function on S extends to a real analytic functions on N. S has the semiglobal extension property if for every relatively compact set $\Omega \Subset N$ there is an open set Δ with $\Omega \subset \Delta \Subset N$ such that

$$\forall f \in \mathscr{A}(S \cap \Delta) \exists g \in \mathscr{A}(\Omega) : f\big|_{\Omega\cap S} = g\big|_{\Omega\cap S}.$$

S is called $\mathbb{C}-$analytic if there is $f \in \mathscr{A}(N)$ such that $S = \{a \in N \mid f(a) = 0\}$.

Theorem 11.1 *Let $C_\varphi : \mathscr{A}(N) \to \mathscr{A}(M)$ as above.*

(a) C_φ is a topological embedding iff φ is a real analytic semiproper surjection ([D58, Theorem 3.1]).

(b) C_φ has closed range and is open onto its range iff φ is semiproper, $\varphi(M)$ is $\mathbb{C}-$analytic with global and semiglobal extension property ([D60, Corollary 2.5])

(c) C_φ is open onto its range iff φ is semiproper and $\varphi(M)$ has the semiglobal extension property ([D54, Theorem 3.2]).

From (c) we obtain for compact manifolds M that C_φ has closed range iff C_φ is open onto its range iff φ is semiproper and $\varphi(M)$ has the global extension property ([D54, Corollary 3.5]).

It is easily seen (see [D58]) that algebra homomorphisms between $\mathscr{A}(\Omega_1)$ and $\mathscr{A}(\Omega_2)$ are exactly given by composition operators C_φ with $\varphi : \Omega_2 \to \Omega_1$ real analytic. It follows that $\mathscr{A}(\Omega_1)$ can be topologically embedded as an algebra in $\mathscr{A}(\Omega_2)$ iff the dimension of Ω_2 is at most equal to the dimension of Ω_1, and if Ω_2 has at least as many components as Ω_1. Hence $\mathscr{A}(\Omega_1)$ and $\mathscr{A}(\Omega_2)$ are isomorphic as topological algebras iff Ω_1 and Ω_2 are real analytic diffeomorphic (see [D58, Corollary 2.3]).

ss.DyncompoB 11.2 Dynamics and spectrum of composition operators

A few years after Paweł's joint work with Langenbruch [D58] about composition operators on the space of real analytic functions began, Domański and Bonet started to investigate the dynamical behaviour of those operators. We will now include some results taken from the papers [D8], [D9], [D10] and [D11]. To do this, we recall that a continuous linear operator $T : E \to E$ on a Hausdorff locally convex space E is called power bounded if the sequence of iterates $(T^n)_{n\in\mathbb{N}}$ is equicontinuous in the space $L(E)$ of linear operators from E to E. The operator T on E is called mean ergodic if the limits $Px := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n x, \quad x \in E$, exist in E. If the convergence is uniform on bounded sets we call T *uniformly mean ergodic*. There is a classical theory of mean ergodic operators which goes back to fundamental work of Yosida and Hille especially in the Banach case; cf. the book by Krengl [37].

compAerg **Theorem 11.2** *Let Ω be a real analytic manifold (compact or non-compact) and let $\varphi : \Omega \to \Omega$ be a real analytic map. The following assertions are equivalent:*

(a) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is power bounded.

(b) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is uniformly mean ergodic.

(c) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is mean ergodic.

(d) For every complex neighbourhood U of Ω there is a complex open neighbourhood $V \subseteq U$ of Ω such that φ extends as a holomorphic function to V, $\varphi(V) \subseteq V$, and φ satisfies that for every compact subset K of U there is a compact subset $L \Subset U$ such that $\varphi^n(K) \subset L$ for $n \in \mathbb{N}$.

The equivalent conditions of the theorem were evaluated further in the case $\Omega = \mathbb{R}$. The proof of Theorem 11.2 required a study of the behaviour of orbits of composition operators $C_\varphi(f) := f \circ \varphi$, φ a holomorphic self map, on spaces $H(U)$ of holomorphic functions defined on an open connected subset U of $\mathbb{C}^d$ or, more generally, of a Stein manifold Ω. This was done in [D8].

An operator $T : X \to X$, X a locally convex space, is called topologically transitive whenever for each pair of non-empty open sets U, V in X there is $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. A vector $x \in X$ is called hypercyclic (or sequentially hypercyclic) if the x-orbit $\{T^n x : n \in \mathbb{N}\}$ of T is dense (or sequentially dense, respectively) in X. Every sequentially hypercyclic operator on a locally convex space X is hypercylic, and hypercyclic operators are topologically transitive. There is a huge literature about the dynamical behavior of various linear continuous operators on Banach, Fréchet and more general locally convex spaces; see the books by Bayart and Matheron [4] and by Grosse-Erdmann and Peris [22].

A map $\varphi : \Omega \to \Omega$ is said to run away on Ω if for every compact set $K \Subset \Omega$ there is $n \in \mathbb{N}$ such that $\varphi^n(K) \cap K = \emptyset$. The next two statements are take from [D10].

Theorem 11.3 *Let $\varphi : \Omega \to \Omega$ be an analytic map on an open subset Ω of $\mathbb{R}^d$. The composition operator $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is topologically transitive if and only if φ is injective, φ' is never* *singular on Ω and φ runs away on Ω.*

hyper

Theorem 11.4 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, $\varphi((-1,1)) \subset (-1,1)$. Then the following assertions are equivalent:*

(a) $C_\varphi : \mathscr{A}(-1,1) \to \mathscr{A}(-1,1)$ is (sequentially) hypercyclic;

(b) $C_\varphi : \mathscr{A}(-1,1) \to \mathscr{A}(-1,1)$ is topologically transitive;

(d) φ runs away on $(-1,1)$ and φ' does not vanish on $(-1,1)$.

The article [D11] gives a full description of eigenvalues and eigenvectors of composition operators $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ for a real analytic self map $\varphi : \mathbb{R} \to \mathbb{R}$ as well as an isomorphic description of corresponding eigenspaces. It also completely characterizes those φ for which Abel's equation $f \circ \varphi = f + 1$ has a real analytic solution on the real line. This research was continued in the paper [D12] that includes results about the spectrum of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.

Domański and Jasiczak have very recently described in [D56] Toeplitz continuous operators on the space of real analytic functions on the real line (i.e., operators for which the associated matrix is Toeplitz). They also proved a necessary and sufficient condition for such operators to be Fredholm operators. Their results show strong similarity to the classical theory of Toeplitz operators on Hardy spaces.

12 Vector valued hyperfunctions

yperfunctions

12.1 The general theory

ctionsgeneral

The sheaf of hyperfunctions was introduced by Sato [52] as a useful tool to study partial differential equations and their singularities. Hyperfunctions may be defined as (formal) boundary values of holomorphic or harmonic functions, as certain relative cohomology groups or as the sheaf generated by analytic functionals with compact support. Vector valued hyperfunctions are used e.g. in theoretical physics or in the study of the abstract Cauchy problem (discussed in Section 12.2), respectively. One technical problem in the vector valued case is the fact that hyperfunctions do not have a useful topology. Nevertheless, Ion and Kawai [25] succeeded in defining Fréchet space valued hyperfunctions. In [D61] Domański and Langenbruch studied $E-$valued hyperfunctions for $PLS-$spaces E and also clarified the limitations of such a theory. In fact, the existence of $E-$valued hyperfunctions is closely related to the solvability of the $E-$valued Laplace equation, and therefore to the circle of problems considered by Bonet and Domanski [D6] (see Section 10). A reasonable theory of $E-$valued hyperfunctions should produce a flabby sheaf $\mathcal{F}$ on $\mathbb{R}^d$ such that the space $\mathcal{F}_0(K)$ of sections supported in a compact set K coincides with the space $L(\mathscr{A}(K), E)$ of $E-$valued analytic functionals. The following characterization is proved in [D61, Theorem 8.9].

Theorem 12.1 *Let E be an ultrabornological PLS-space. The following are equivalent:*

(a) For any $1 \leq d < \infty$ (equivalently, for some $1 \leq d < \infty$) there is a flabby sheaf $\mathcal{F}$ on $\mathbb{R}^d$ such that $\mathcal{F}_0(K) = L(\mathscr{A}(K), E)$ for any compact set $K \subset \mathbb{R}^d$.

(b) E has (PA).

Moreover, any of the above mentioned methods then define the same sheaf of $E-$valued hyperfunctions.

A long list of well known ultrabornological $(PLS)-$spaces having (PA) or failing (PA) is given in [D61, Corollaries 4.8 and 4.9]. Specifically, the spaces of distributions or tempered distributions have (PA) while the spaces of real analytic functions or distributions with compact support do not have (PA).

The ansatz of [D61] has been extended by Kruse [38] to the case of vector valued Fourier hyperfunctions.

12.2 The abstract Cauchy problem

ss.ACP

Hyperfunctions can be used to discuss the abstract Cauchy problem (ACP) under minimal regularity assumptions. Specifically, let $C : F := D(C) \subset E \to E$ be a closed operator with domain $D(C)$ in a locally convex space E. Then

$$(ACP) \qquad\qquad x'(t) = Cx(t); x(0) = x_0.$$

There is an abundance of literature on how to give a precise meaning to the exponential ansatz to solve the ACP, i.e. to study semigroups with decreasing regularity (C_0-, integrated, distributional and even hyperfunction semigroups, see e.g. [45, 17, 39, 28, 48, 34, 35, 67]). For operators in Banach spaces, Komatsu [36] introduced operator valued Laplace hyperfunctions and a corresponding Laplace transform to find a fundamental solution for (ACP), that is, an $L(E, F)-$valued Laplace hyperfunction T such that

$$\left(\frac{d}{dt} - C\right) \circ T = \mathrm{id}_{\,E} \otimes \delta_0 \quad \text{and} \quad T \circ \left(\frac{d}{dt} - C\right) = \mathrm{id}_{\,F} \otimes \delta_0$$

where δ_0 is Dirac's distribution. Komatsu showed that the existence of a fundamental solution for the ACP is equivalent to the fact that C has resolvents satisfying certain exponential growth conditions on an open set $\Omega \subset \mathbb{C}$ containing any cone $\{z \in \mathbb{C} \mid \mathrm{Re}\,(z) > |\mathrm{Im}(z)|/C\}$ near ∞. Elementary examples show that an extension of these results is impossible even for operators on Fréchet spaces if the standard notions of Laplace transform and resolvent are used. To overcome this difficulty, spectral valued holomorphic functions and resolvents, and a spectral valued Laplace transform were introduced in [D62, D64]. We shortly recall these key notions. Let X be a locally convex space given by a projective spectrum $\mathscr{X} := (X_\gamma)_{\gamma\in\Gamma}$ with connecting mappings κ^γ_ν. Let $\mathscr{G} := (G_\gamma)_{\gamma\in\Gamma}$ be directed family of open sets $G_\gamma \subset \mathbb{C}$. A family $\mathscr{S} = (S_\gamma)_{\gamma\in\Gamma}$ is called a spectral-valued (or $\mathscr{X}$-valued) holomorphic function (denoted by $\mathscr{S} : \mathscr{G} \to \mathscr{X}$) if

(i) $S_\gamma : G_\gamma \to X_\gamma$ is holomorphic; (ii) (compatibility) $\forall\, \gamma \geq \nu : \quad \kappa^\gamma_\nu \circ S_\gamma = S_\nu\big|_{G_\gamma}$.

Specifically, this is needed in the operator valued case, i.e. where $X = L_b(E, F)$ with the projective spectrum defined as follows: Let $\mathcal{B}^E$ be the system of bounded absolutely convex subsets of E (and corresponding normed spaces $E_B, B \in \mathcal{B}^E$) and let $\{\| \cdot \|_\alpha, \alpha \in A\}$ be a system of seminorms defining the topology of F (with local Banach spaces $F_\alpha, \alpha \in A$). Set $\mathscr{X} := \mathscr{L}(E, F) := (L(E_B, F_\alpha))_{(B,\alpha)\in(\mathcal{B}^E,A)}$.

Let $\mathscr{G} := (G_{B,\alpha})_{(B,\alpha)\in(\mathcal{B}^E,A)}$ be a directed family of domains. A holomorphic $\mathscr{L}(E, F)$-valued function $\mathscr{R} : \mathscr{G} \to \mathscr{L}(E, F)$ is called a spectral-valued resolvent for a closed operator $C : F := D(C) \subset E \to E$ (F is endowed with the graph topology) if the following compatibility conditions are satisfied

(i) $\forall \alpha \in A \exists \widetilde{\alpha} \in A \forall B \in \mathcal{B}^E \forall \lambda \in G_{(B,\widetilde{\alpha})} : \quad (\lambda - C)^{\widetilde{\alpha}}_\alpha \circ R_{(B,\widetilde{\alpha})}(\lambda) = i^E_{(B,\alpha)}$

(ii) $\forall B \in \mathcal{B}^F \exists \widetilde{B} \in \mathcal{B}^E \forall \alpha \in A \forall \lambda \in G_{(\widetilde{B},\alpha)} : \quad R_{(\widetilde{B},\alpha)}(\lambda) \circ (\lambda - C)^B_{\widetilde{B}} = i^F_{(B,\alpha)}$.

This notion considerably extends the notion of resolvents for operators in Fréchet spaces given by Arikan, Runov and Zahariuta [2], and it can be simplified if E, F both are (FS)-spaces (and (DFS)-spaces, respectively). Notice that our notion is also compatible with duality. Many examples of generalized resolvents for concrete operators are calculated in [D64]. Moreover, a new general operator valued Laplace hyperfunction and a corresponding Laplace transform (producing a spectral valued holomorphic function) are introduced in [D62]. The main result in [D64] now reads as follows:

Theorem 12.2 *Let E be a complete bornological space and let $C : F := D(C) \subseteq E \to E$ be a closed operator. Then the (ACP) has a fundamental solution in the sense of Laplace hyperfunctions if and only if C admits a spectral valued resolvent $\mathscr{R} : \mathscr{G} \to \mathscr{L}(E, F)$ such that*

$$\forall\ B \in \mathcal{B}^E, \alpha \in A, K > 1\ \exists\ k = k(B, \alpha, K) : G_{(B,\alpha)} \supseteq V_{K,k} := \{z \in \mathbb{C} :\ \mathrm{Re}\ z > k + |\,\mathrm{Im}\, z|/K\}$$

$$\textit{and} \sup_{\lambda \in V_{K,k}} \|R_{(B,\alpha)}(\lambda)\|_{L(E_B, F_\alpha)} \exp\left(-\,\mathrm{Re}\ \lambda/K\right) < \infty.$$

13 Hadamard and Euler type operators

s.HadEul

Let $E(\Omega)$ be a locally convex space of generalized functions on an open set $\Omega \subset \mathbb{R}^d$ containing the space of polynomials as a dense subspace. Then a linear and continuous operator $H : E(\Omega) \to E(\Omega)$ is called a Hadamard type operator (or Hadamard multiplier) if all monomials are eigenvectors, that is, if $H(\xi^\alpha)(x) = m_\alpha x^\alpha$ for some sequence $(m_\alpha)_{\alpha \in \mathbb{N}^d}$ called the multiplier sequence of H. Hadamard operators go back to the work of Hadamard [23], and have been intensively studied on holomorphic functions (see [9, 10] and the survey paper [51]) and on hyperfunctions [26, 27] and, recently, on smooth functions and distributions [66, 65, 59, 60]. Though Hadamard operators are uniquely determined by their multiplier sequence they are not just diagonal operators since the monomials in general are not a basis in $E(\Omega)$. In a series of papers [D63, D65, D66, D72, D68, D67, D71, D70] Domański, Langenbruch and Vogt developed a theory of Hadamard operators on real analytic functions and smooth functions considering three basic problems: Firstly, find a representation theorem giving a general formula generating any Hadamard operator. Secondly, characterize multiplier sequences and, thirdly, characterize surjective Hadamard operators.

To begin with, Hadamard operators are closely connected to *multiplicative* convolution (see Theorem 13.1 below). Let $xy := (x_1y_1, \ldots, x_dy_d)$ denote the coordinatewise multiplication on $\mathbb{R}^d$ and let $V(\Omega) := \{x \in \mathbb{R}^d \mid x\Omega \subset \Omega\}$ denote the dilation set where $\Omega \subset \mathbb{R}^d$ always is an open set in this section. Let $E(V(\Omega))'$ denote the functionals on $E(\mathbb{R}^d)$ with support in $V(\Omega)$. The following Representation Theorem for the space $M_E(\Omega)$ of Hadamard operators on $E(\Omega)$ is proved in [D65] (for $d = 1$) and in [D72] in the analytic case (and in [66] in the smooth case).

repr **Theorem 13.1** *Let $E(\Omega) = \mathscr{A}(\Omega)$ or $E(\Omega) = C^\infty(\Omega)$. The map*

$$\mathscr{B} : E(V(\Omega))'_b \to M_E(\Omega) \subset L_b(E(\Omega)), \quad \mathscr{B}(T)(g)(y) := \langle g(y\cdot)\,,\, T\rangle, T \in E(V(\Omega))', g \in E(\Omega),$$

is a bijective continuous linear linear map and the multiplier sequence $(m_\alpha)_{\alpha \in \mathbb{N}^d}$ of $\mathscr{B}(T)$ is equal to the sequence of moments of the functional T on $E(\Omega)$, i.e.

$$m_\alpha = \langle x^\alpha, T\rangle \textit{ for any } \alpha \in \mathbb{N}^d. \qquad (*)$$

The Represetation Theorem means that Hadamard operators are just multiplicative convolution operators with a functional supported in $V(\Omega)$. Specifically, the following classes of continuous linear operators are Hadamard operators:

(a) Euler operators: $P(\theta) := \sum_{|\beta|\le m} a_\beta \theta^\beta$ where $\theta = (\theta_1, \ldots, \theta_d)$ and $\theta_j = x_j\partial_j$;

(b) integral operators: $M(g)(x) := \int_{[0,1]^d} g(xy)dy$ (for $d = 1$ this is Hadamard's operator [23]);

(c) dilation operators: $M_a(g)(x) := g(ax)$ for $a > \mathbf{0}$;

Using the Cauchy transform (and hyperfunction theory) a representation of Hadamard multipliers by an algebra of holomorphic functions can be given with Hadamard multiplication of the Laurent coefficients (see [D72, Section 3] and [66]). Besides the strong topology τ_b there are two other natural topologies on $\mathscr{A}(V(\Omega))'$, and [D72] contains a detailed discussion of the question if and when these topologies coincide with the one induced from $L_b(\mathscr{A}(\Omega))$ via $\mathscr{B}$. Specifically, $B \subset \mathscr{A}(V(\Omega))'_b$ is bounded iff $\mathscr{B}(B) \subset M_{\mathscr{A}}(\Omega)$ is bounded.

13.1 Surjectivity of Hadamard operators on real analytic functions

ss.Hadanalytic

By (*) in Theorem 13.1 multiplier sequences for Hadamard operators on $\mathscr{A}(\Omega)$ are moment sequences of analytic functionals, and these can be characterized by holomorphic interpolation (see Theorem 13.2 below). An open set $\omega \subset \mathbb{C}$ is called an *asymptotic halfspace* if $0 \in \omega$ and if $\omega = \bigcup_n(\kappa_n + \omega_{K_n})$ where $K_n \to \infty$ and $\omega_K := \{z \in \mathbb{C} \mid |y| < Kx\}$. Asymptotic halfspaces in $\mathbb{C}^d$ are the cartesian product ω^d of an asymptotic halfspace $\omega \subset \mathbb{C}$. For $a \in \mathbb{R}^d$ let $Q_a = \prod_{k=1}^d [-e^{a_k}, e^{a_k}]$ and

$$\mathscr{H}_a(\omega^d) := \{f \in \mathcal{H}(\omega^d) \mid \forall 1 \le j \in \mathbb{N} : \sup_{z\in\Gamma_j} |f(z)|e^{-\langle a+\mathbf{1}/j, \mathrm{Re}\,(z)\rangle} < \infty\}$$

where Γ_j is an exhaustion of ω^d by closures of asymptotic half spaces. We now have the following characterization of moment sequences of analytic functionals by interpolation (see [D68, Th. 6.1, Cor. 6.4]).

surjgen **Theorem 13.2** *There is a surjective continuous linear mapping $I : \mathscr{H}_a(\omega^d)^{2^d} \to \mathscr{A}(Q_a)'_b$ such that*

$$\langle I((f_\sigma)_{\sigma\in\{0,1\}^d}), x^\alpha\rangle = f_\sigma(\alpha) \qquad \text{for } \alpha \in \mathbb{N}^d_\sigma.$$

The mapping I is not injective. However, the kernel of I can be described rather precisely since I is the tensor product of the corresponding mappings in one variable (see [D68]). This leads to large sets where the functions $(f_\sigma)_\sigma \in \ker(I)$ are small. As a consequence we have the following description of multiplier projections, i.e. Hadamard multipliers which are projections (see [D66] (for $d = 1$) and [D69]).

Theorem 13.3 *The following assertions are equivalent.*

(a) There is a Hadamard multiplier projection $M : \mathscr{A}(\mathbb{R}^d) \to \mathscr{A}(\mathbb{R}^d)$ with multiplier sequence $(m_\alpha)_{\alpha\in\mathbb{N}^d}$ and $I := \{\alpha \in \mathbb{N}^d \mid m_\alpha = 1\}$.

(b) For any real analytic function $f \in \mathscr{A}(\mathbb{R}^d)$ with Taylor expansion $f(z) = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha$ at $\mathbf{0}$, also the function f_I with Taylor expansion $f_I(z) = \sum_{\alpha\in I} f_\alpha z^\alpha$ at $\mathbf{0}$, extends to a real analytic function on $\mathbb{R}^d$.

(c) The set I belongs to the set algebra in $\mathbb{N}^d$ generated by the products of sets which are either finite subsets of $\mathbb{N}$ or the set of even numbers $2\mathbb{N}$.

Using the above Interpolation Theorem and the description of the kernel of I, the surjectivity of Euler operators $P(\theta)$ has been discussed in detail in [D68] (including a perturbation result for surjectivity). It turns out that surjectivity is closely related to the so called half plane property. We can only mention one typical result here.

Theorem 13.4 *The following are equivalent for a polynomial P.*
(a) $P_m(\theta)$ is invertible on $\mathscr{A}_0(\mathbb{R}^d) := \{f \in \mathscr{A}(\mathbb{R}^d) \mid f(0) = 0\}$.
(b) P_m satisfies the so-called closed halfplane property, i.e.,

$$P_m(z) \neq 0 \text{ if } 0 \neq x \text{ and } x \geq 0.$$

There is an extensive literature on the halfplane property partly motivated by image processing (see the survey paper [12]).

The above results can be applied to operators built from operators conjugate to θ_j. This class of operators (far more general than Euler type partial differential operators) has been characterized in [D67].

ss.Hadasmooth

13.2 Surjectivity of Hadamard operators on smooth functions

In [D70] and [D71] Domański and Langenbruch studied surjectivity of Hadamard operators on smooth functions. It turns out that the methods as well as the results are completely different from those for real analytic functions sketched in the previous section. First notice that range of $P(\theta)$ is contained in
$C^\infty_{I(P)}(\Omega) := \{f \in C^\infty(\Omega) \mid \forall \emptyset \neq J \subset D \; \forall \alpha \in \mathbb{N}^J :$

$$P(\alpha, \xi_{D\setminus J}) \equiv 0 \Rightarrow f^{(\alpha)}(0_J, x_{D\setminus J}) = 0 \text{ if } (0_J, x_{D\setminus J}) \in \Omega\}.$$

The main results are the following:

Theorem 13.5 *Any Euler operator $0 \neq P(\theta) : C^\infty(\mathbb{R}^d) \to C^\infty_{I(P)}(\mathbb{R}^d)$ is surjective.*

This holds for $C^\infty(\Omega)$ instead of $C^\infty(\mathbb{R}^d)$ if and only if Ω is a so called m-convex set.

The proof relies on a reduction and induction procedure using Euler operators on certain Whitney jets and reducing the problem to the surjectivity of $P(\theta)$ on the space

$$\mathscr{E}([0,\infty[^d) := \{f \in C^\infty(\mathbb{R}^d) \mid \operatorname{supp}(f) \subset [0,\infty[^d\}.$$

Notice that this makes no sense in the case of real analytic functions. A suitably defined Mellin transform identifies $\mathscr{E}([0,\infty[^d)'_b$ with a corresponding space of holomorphic germs $\mathscr{H}_{\mathscr{M}}$ transforming the action of $P(\theta)$ to the multiplication by $P(z)$.

surjective **Theorem 13.6** *The Euler operator $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ is surjective iff Ω is $P(\theta)$-convex (for supports), i.e.*
$\forall K \subset\subset \Omega \; \forall k \in \mathbb{N} \; \exists \widetilde{K} \subset\subset \Omega \; \forall T \in C^{\mathbf{k}}_{I(P)}(\Omega)' :$

$$\operatorname{supp}_{I(P)}(T) \subset \widetilde{K} \text{ if } \operatorname{supp}({}^tP(\theta)T) \subset K.$$

Here $\operatorname{supp}_{I(P)}(T)$ is the support of T in the sense of $C^\infty_{I(P)}(\Omega)'$ which can be defined since $C^\infty_{I(P)}(\Omega)$ is a module over an algebra containing sufficiently many resolutions of the identity.

Though Theorem 13.6 resembles very much the constant coefficient case (see [24]) the consequences differ very much: if $P(\theta) : C^\infty(\Omega_i) \to C^\infty_{I(P)}(\Omega_i), i \in I$, is surjective for any $i \in I$ then $P(\theta) : C^\infty(\Omega) \to C^\infty_{I(P)}(\Omega)$ need not be surjective for $\Omega := (\cap_{i\in I}\Omega_i)^\circ$ or $\Omega := (\liminf_{i\in I}\Omega_i)^\circ$, moreover, in general there is no minimal open set $\widetilde{\Omega} \supset \Omega$ such that $P(\theta) : C^\infty(\widetilde{\Omega}) \to C^\infty_{I(P)}(\widetilde{\Omega})$ is surjective. Also, surjectivity depends on lower order terms for operators in two variables.

Finally, the kernel of Euler type operators is studied in [D71, Section 11] showing that there are many operators with trivial kernel.

Theorem 13.7 *The Euler operator $P(\theta) : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ is bijective iff there is $k \in \mathbb{N}$ such that*

$$P(z) \neq 0 \text{ if } z \in (\mathbb{C}_{\geq k})^d$$

where $\mathbb{C}_{\geq k} := \{0, \ldots, k-1\} \cup \{z \in \mathbb{C} \mid \mathrm{Re}\,(z) \geq k\} \supset \mathbb{N}$.

For general Hadamard operators on $C^\infty(\mathbb{R}^d)$ surjectivity is connected to some new slowly decreasing conditions (see [D70]).

14 List of Paweł PhD. students

Here is the list of PhD. students of Paweł Domański with the year and the university where the defence took place:

1. Daren Kunkle (2001), University of Wuppertal.
2. Kinga Cichon (2002), Adam Mickiewicz University in Poznan.
3. Krzysztof Piszczek (2007), Adam Mickiewicz University in Poznan.
4. Piotr Ziolo (2011), Adam Mickiewicz University in Poznan.
5. Michal Golinski (2013), Adam Mickiewicz University in Poznan.
6. Tomasz Cias (2014), Adam Mickiewicz University in Poznan.
7. Adam Przestacki (2015), Adam Mickiewicz University in Poznan.
8. Anna Golinska - she is still a PhD student, now her supervisor is Dr. Michal Jasiczak.

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Publications of Paweł Domański

Sec:1

MR1633255 [D1] J. Bonet and P. Domański. Real analytic curves in Fréchet spaces and their duals. *Monatsh. Math.*, 126(1):13–36, 1998.

MR1800237 [D2] J. Bonet and P. Domański. Parameter dependence of solutions of partial differential equations in spaces of real analytic functions. *Proc. Amer. Math. Soc.*, 129(2):495–503 (electronic), 2001.

MR1961252 [D3] J. Bonet and P. Domański. Sampling sets and sufficient sets for $A^{-\infty}$. *J. Math. Anal. Appl.*, 277(2):651–669, 2003.

MR2186216 [D4] J. Bonet and P. Domański. Parameter dependence of solutions of differential equations on spaces of distributions and the splitting of short exact sequences. *J. Funct. Anal.*, 230(2):329–381, 2006.

MR2363694 [D5] J. Bonet and P. Domański. The structure of spaces of quasianalytic functions of Roumieu type. *Arch. Math. (Basel)*, 89(5):430–441, 2007.

MR2370276 [D6] J. Bonet and P. Domański. The splitting of exact sequences of PLS-spaces and smooth dependence of solutions of linear partial differential equations. *Adv. Math.*, 217(2):561–585, 2008.

MR2678904 [D7] J. Bonet and P. Domański. Köthe coechelon spaces as locally convex algebras. *Studia Math.*, 199(3):241–265, 2010.

MR2826716 [D8] J. Bonet and P. Domański. A note on mean ergodic composition operators on spaces of holomorphic functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 105(2):389–396, 2011.

MR2772328 [D9] J. Bonet and P. Domański. Power bounded composition operators on spaces of analytic functions. *Collect. Math.*, 62(1):69–83, 2011.

MR2990627 [D10] J. Bonet and P. Domański. Hypercyclic composition operators on spaces of real analytic functions. *Math. Proc. Cambridge Philos. Soc.*, 153(3):489–503, 2012.

MR3323563 [D11] J. Bonet and P. Domański. Abel's functional equation and eigenvalues of composition operators on spaces of real analytic functions. *Integral Equations Operator Theory*, 81(4):455–482, 2015.

MR3595980 [D12] J. Bonet and P. Domański. A Note on the Spectrum of Composition Operators on Spaces of Real Analytic Functions. *Complex Anal. Oper. Theory*, 11(1):161–174, 2017.

MR1388616 [D13] J. Bonet, P. Domański, and M. Lindström. Cotype and complemented copies of c_0 in spaces of operators. *Czechoslovak Math. J.*, 46(121)(2):271–289, 1996.

MR1692002 [D14] J. Bonet, P. Domański, and M. Lindström. Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions. *Canad. Math. Bull.*, 42(2):139–148, 1999.

MR1734396 [D15] J. Bonet, P. Domański, and M. Lindström. Pointwise multiplication operators on weighted Banach spaces of analytic functions. *Studia Math.*, 137(2):177–194, 1999.

MR1816570 [D16] J. Bonet, P. Domański, and M. Lindström. Weakly compact composition operators on analytic vector-valued function spaces. *Ann. Acad. Sci. Fenn. Math.*, 26(1):233–248, 2001.

MR1356892 [D17] J. Bonet, P. Domański, M. Lindström, and M. S. Ramanujan. Operator spaces containing c_0 or l_∞. *Results Math.*, 28(3-4):250–269, 1995.

MR1490150 [D18] J. Bonet, P. Domański, M. Lindström, and J. Taskinen. Composition operators between weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A*, 64(1):101–118, 1998.

MR1308945 [D19] J. Bonet, P. Domański, and J. Mujica. Complete spaces of vector-valued holomorphic germs. *Math. Scand.*, 75(1):150–160, 1994.

MR1920411 [D20] J. Bonet, P. Domański, and D. Vogt. Interpolation of vector-valued real analytic functions. *J. London Math. Soc. (2)*, 66(2):407–420, 2002.

MR1740463 [D21] A. Defant, P. Domański, and M. Mastyło. Extension and splitting theorems for Fréchet spaces of type 2. *Rev. Mat. Complut.*, 12(2):325–337, 1999.

MR1401875 [D22] J. C. Diaz, S. Dierolf, P. Domański, and C. Fernandez. On the three-space problem for dual Fréchet spaces. *Bull. Polish Acad. Sci. Math.*, 40(3):221–224, 1992.

MR1651403 [D23] J. C. Díaz and P. Domański. Reflexive operators with domain in Köthe spaces. *Manuscripta Math.*, 97(2):189–204, 1998.

MR1670440 [D24] J. C. Díaz and P. Domański. On the injective tensor product of distinguished Fréchet spaces. *Math. Nachr.*, 198:41–50, 1999.

MR1239423 [D25] S. Dierolf and P. Domański. Factorization of Montel operators. *Studia Math.*, 107(1):15–32, 1993.

MR1336223 [D26] S. Dierolf and P. Domański. Bornological spaces of null sequences. *Arch. Math. (Basel)*, 65(1):46–52, 1995.

MR1439173 [D27] S. Dierolf and P. Domański. Null sequences in coechelon spaces. *Math. Nachr.*, 184:167–176, 1997.

MR721060 [D28] P. Domański. Nonseparable closed subspaces in separable products of topological vector spaces, and q-minimality. *Arch. Math. (Basel)*, 41(3):270–275, 1983.

MR781936 [D29] P. Domański. Local convexity of twisted sums. In *Proceedings of the 12th winter school on abstract analysis (Srní, 1984)*, number Suppl. 5, pages 13–31, 1984.

MR817350 [D30] P. Domański. On the separable topological vector spaces. *Funct. Approx. Comment. Math.*, 14:117–122, 1984.

MR823973 [D31] P. Domański. On the splitting of twisted sums, and the three-space problem for local convexity. *Studia Math.*, 82(2):155–189, 1985.

MR835872 [D32] P. Domański. Twisted sums of Banach and nuclear spaces. *Proc. Amer. Math. Soc.*, 97(2):237–243, 1986.

MR1000152 [D33] P. Domański. Complemented subspaces of products of Hilbert spaces. *Proc. Amer. Math. Soc.*, 110(1):187–196, 1990.

MR1078419 [D34] P. Domański. Fréchet injective spaces of continuous functions. *Results Math.*, 18(3-4):209–230, 1990.

MR1067609 [D35] P. Domański. L_p-spaces and injective locally convex spaces. *Dissertationes Math. (Rozprawy Mat.)*, 298:76, 1990.

MR1052978 [D36] P. Domański. Principle of local reflexivity for operators and quojections. *Arch. Math. (Basel)*, 54(6):567–575, 1990.

MR1135223 [D37] P. Domański. A note on injective $C(T)$-spaces and the Amir boundary. *Israel J. Math.*, 73(3):357–359, 1991.

MR1149682 [D38] P. Domański. On the injective $C(T)$-spaces. *Indag. Math. (N.S.)*, 2(3):289–299, 1991.

MR1296989 [D39] P. Domański. A characterization of preduals of Fréchet injective spaces. *Funct. Approx. Comment. Math.*, 21:37–40, 1992.

MR1258562 [D40] P. Domański. On the projective LB-spaces. *Note Mat.*, 12:43–48, 1992. Dedicated to the memory of Professor Gottfried Köthe.

MR1213517 [D41] P. Domański. A note on projective limits of LB-spaces. *Arch. Math. (Basel)*, 60(5):464–472, 1993.

MR1401887 [D42] P. Domański. On the space of Riemann integrable functions. *Bull. Polish Acad. Sci. Math.*, 41(1):61–64, 1993.

MR1205758 [D43] P. Domański. Twisted Fréchet spaces of continuous functions. *Results Math.*, 23(1-2):45–48, 1993.

MR1721846 [D44] P. Domański. On spaces of continuous functions with values in coechelon spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.)*, 92(1):61–66, 1998.

MR2068176 [D45] P. Domański. Ideals of extendable and liftable operators. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 97(2):229–241, 2003.

MR2099459 [D46] P. Domański. Classical PLS-spaces: spaces of distributions, real analytic functions and their relatives. In *Orlicz centenary volume*, volume 64 of *Banach Center Publ.*, pages 51–70. Polish Acad. Sci., Warsaw, 2004.

MR2666313 [D47] P. Domański. Real analytic parameter dependence of solutions of differential equations. *Rev. Mat. Iberoam.*, 26(1):175–238, 2010.

MR2807900 [D48] P. Domański. Real analytic parameter dependence of solutions of differential equations over Roumieu classes. *Funct. Approx. Comment. Math.*, 44(part 1):79–109, 2011.

MR2905538 [D49] P. Domański. Notes on real analytic functions and classical operators. In *Topics in complex analysis and operator theory*, volume 561 of *Contemp. Math.*, pages 3–47. Amer. Math. Soc., Providence, RI, 2012.

MR1100690 [D50] P. Domański and L. Drewnowski. Uncomplementability of the spaces of norm continuous functions in some spaces of "weakly" continuous functions. *Studia Math.*, 97(3):245–251, 1991.

MR1170555 [D51] P. Domański and L. Drewnowski. Fréchet spaces of continuous vector-valued functions: complementability in dual Fréchet spaces and injectivity. *Studia Math.*, 102(3):257–267, 1992.

MR1241691 [D52] P. Domański and L. Drewnowski. Complementability and injectivity of Fréchet spaces of continuous vector-valued functions. In *Functional analysis (Essen, 1991)*, volume 150 of *Lecture Notes in Pure and Appl. Math.*, pages 405–413. Dekker, New York, 1994.

MR2052220 [D53] P. Domański, L. Frerick, and D. Vogt. Fréchet quotients of spaces of real-analytic functions. *Studia Math.*, 159(2):229–245, 2003. Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday (Polish).

MR2855302 [D54] P. Domański, M. Goliński, and M. Langenbruch. A note on composition operators on spaces of real analytic functions. *Ann. Polon. Math.*, 103(2):209–216, 2012.

MR1992857 [D55] P. Domański and B. Jakubczyk. Linear continuous division for exterior and interior products. *Proc. Amer. Math. Soc.*, 131(10):3163–3175 (electronic), 2003.

DJ* [D56] P. Domański and M. Jasiczak. Toeplitz operators on the space of real analytic functions: The Fredholm property. *Banach J. Math. Anal.*, 2016.

MR1441538 [D57] P. Domański, J. Krone, and D. Vogt. Standard exact projective resolutions relative to a countable class of Fréchet spaces. *Studia Math.*, 123(3):275–290, 1997.

MR1983956 [D58] P. Domański and M. Langenbruch. Composition operators on spaces of real analytic functions. *Math. Nachr.*, 254/255:68–86, 2003.

MR2139710 [D59] P. Domański and M. Langenbruch. Coherent analytic sets and composition of real analytic functions. *J. Reine Angew. Math.*, 582:41–59, 2005.

MR2250756 [D60] P. Domański and M. Langenbruch. Composition operators with closed image on spaces of real analytic functions. *Bull. London Math. Soc.*, 38(4):636–646, 2006.

MR2477906 [D61] P. Domański and M. Langenbruch. Vector valued hyperfunctions and boundary values of vector valued harmonic and holomorphic functions. *Publ. Res. Inst. Math. Sci.*, 44(4):1097–1142, 2008.

MR2767167 [D62] P. Domański and M. Langenbruch. On the Laplace transform for vector valued hyperfunctions. *Funct. Approx. Comment. Math.*, 43(part 2):129–159, 2010.

MR3008439 [D63] P. Domański and M. Langenbruch. Algebra of multipliers on the space of real analytic functions of one variable. *Studia Math.*, 212(2):155–171, 2012.

MR2978913 [D64] P. Domański and M. Langenbruch. On the abstract Cauchy problem for operators in locally convex spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 106(2):247–273, 2012.

MR3043719 [D65] P. Domański and M. Langenbruch. Representation of multipliers on spaces of real analytic functions. *Analysis (Munich)*, 32(2):137–162, 2012.

MR3046319 [D66] P. Domański and M. Langenbruch. Hadamard multipliers on spaces of real analytic functions. *Adv. Math.*, 240:575–612, 2013.

MR3514312 [D67] P. Domański and M. Langenbruch. Euler type partial differential operators on real analytic functions. *J. Math. Anal. Appl.*, 443(2):652–674, 2016.

MR3474335 [D68] P. Domański and M. Langenbruch. Interpolation of holomorphic functions and surjectivity of Taylor coefficient multipliers. *Adv. Math.*, 293:782–855, 2016.

DL16b [D69] P. Domański and M. Langenbruch. Multiplier projections on spaces of real analytic functions in several variables. *Compl. Var. Ell. Equ.*, 2016. to appear.

DL16d [D70] P. Domański and M. Langenbruch. Surjectivity of Hadamard type operators on spaces of smooth functions. 2016. manuscript.

DL16c [D71] P. Domański and M. Langenbruch. Surjectivity of Euler type operators on spaces of smooth functions. *AMS Transactions*, 2017. to appear.

MR3418073 [D72] P. Domański, M. Langenbruch, and D. Vogt. Hadamard type operators on spaces of real analytic functions in several variables. *J. Funct. Anal.*, 269(12):3868–3913, 2015.

MR1405494 [D73] P. Domański and M. Lindström. Grothendieck spaces and duals of injective tensor products. *Bull. London Math. Soc.*, 28(6):617–626, 1996.

MR1957801 [D74] P. Domański and M. Lindström. Sets of interpolation and sampling for weighted Banach spaces of holomorphic functions. *Ann. Polon. Math.*, 79(3):233–264, 2002.

MR1372028 [D75] P. Domański, M. Lindström, and G. Schlüchtermann. Grothendieck operators on tensor products. *Proc. Amer. Math. Soc.*, 125(8):2285–2291, 1997.

MR2324722 [D76] P. Domański and M. Mastyło. Characterization of splitting for Fréchet-Hilbert spaces via interpolation. *Math. Ann.*, 339(2):317–340, 2007.

MR937243 [D77] P. Domański and A. Ortyński. Complemented subspaces of products of Banach spaces. *Trans. Amer. Math. Soc.*, 316(1):215–231, 1989.

MR1614582 [D78] P. Domański and D. Vogt. A splitting theorem for the space of smooth functions. *J. Funct. Anal.*, 153(2):203–248, 1998.

MR1758575 [D79] P. Domański and D. Vogt. Distributional complexes split for positive dimensions. *J. Reine Angew. Math.*, 522:63–79, 2000.

MR1792604 [D80] P. Domański and D. Vogt. The space of real-analytic functions has no basis. *Studia Math.*, 142(2):187–200, 2000.

MR1763882 [D81] P. Domański and D. Vogt. A splitting theory for the space of distributions. *Studia Math.*, 140(1):57–77, 2000.

MR1861751 [D82] P. Domański and D. Vogt. Linear topological properties of the space of analytic functions on the real line. In *Recent progress in functional analysis (Valencia, 2000)*, volume 189 of *North-Holland Math. Stud.*, pages 113–132. North-Holland, Amsterdam, 2001.

MR2009563 [D83] P. Domański and D. Vogt. A non-trivial Fréchet quotient of the space of real analytic functions. *Arch. Math. (Basel)*, 81(2):208–214, 2003.

MR2084083 [D84] P. Domański and D. Vogt. Infinite systems of linear equations for real analytic functions. *Proc. Amer. Math. Soc.*, 132(12):3607–3614 (electronic), 2004.

MR2606942 [D85] P. Domański and D. Vogt. The coherence of complemented ideals in the space of real analytic functions. *Math. Ann.*, 347(2):395–409, 2010.

MR3189501 [D86] P. Domański and W. Wnuk. On the work of Lech Drewnowski. *Funct. Approx. Comment. Math.*, 50(1, [2013 on table of contents]):7–54, 2014.

Further references

Sec:2

MR2784662 [1] S. A. Argyros and R. G. Haydon. A hereditarily indecomposable $\mathscr{L}_\infty$-space that solves the scalar-plus-compact problem. *Acta Math.*, 206(1):1–54, 2011.

ARZ [2] R.-L. Arikan, H. and V. Zahariuta. Holomorphic functional calculus for operators on a locally convex space. *Results Math.*, 43:23–36, 2003.

B31 [3] S. Banach. *Theorja operacyj, tom I. Operacje liniowe.* Kasa Mianowskiego, Warszawa, 1931. in polish.

MR2533318 [4] F. Bayart and É. Matheron. *Dynamics of linear operators*, volume 179 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2009.

MR2068172 [5] K. D. Bierstedt and J. Bonet. Some aspects of the modern theory of Fréchet spaces. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 97(2):159–188, 2003.

BM82 [6] E. Bierstone and P. Milman. Composite differentiable functions. *Math. Ann.*, 116:541–558, 1982.

BMP [7] M.-P. Bierstone, E. and W. Pawłucki. Composite differentiable functions. *Duke Math.*, 83:607–620, 1996.

Br62 [8] F. Browder. Analyticity and partial differential equations. i. *Amer. J. Math.*, 84:666–710, 1962.

BM92 [9] R. Brück and J. Müller. Invertible elements in a convolution algebra of holomorphic functions. *Math. Ann.*, 94:421–438, 1992.

BR00 [10] R. Brück and H. Render. Invertibility of holomorphic functions with respect to the Hadamard product. *Complex Variables*, 42:207–223, 2000.

MR1840100 [11] A. Chigogidze. Complemented subspaces of products of Banach spaces. *Proc. Amer. Math. Soc.*, 129(10):2959–2963 (electronic), 2001.

COSW [12] O.-J. G. S. A. D. Choe, Y. and D. G. Wagner. Homogeneous multivariate polynomials with the half-plane property. *Advances in Appl. Math.*, 32:88–187, 2004.

CM [13] C. Cowen and B. MacCluer. *Composition operators on spaces of analytic functions.* CRC Press, Boca Raton, 1995.

MR1397026 [14] C. C. Cowen and B. D. MacCluer. *Composition operators on spaces of analytic functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

MR0355536 [15] W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński. Factoring weakly compact operators. *J. Functional Analysis*, 17:311–327, 1974.

MR1068453 [16] L. Drewnowski. Copies of l_∞ in an operator space. *Math. Proc. Cambridge Philos. Soc.*, 108(3):523–526, 1990.

EN00 [17] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Mathematics*. Springer, New York, 2000.

MR2418806 [18] L. Frerick and J. Wengenroth. (LB)-spaces of vector-valued continuous functions. *Bull. Lond. Math. Soc.*, 40(3):505–515, 2008.

MR2807896 [19] L. Frerick and J. Wengenroth. The mathematical work of Susanne Dierolf. *Funct. Approx. Comment. Math.*, 44(part 1):7–31, 2011.

Gl [20] G. Glaeser. Fonctions composées différentiables. *Math. Ann.*, 77:193–209, 1963.

MR1201238 [21] W. T. Gowers and B. Maurey. The unconditional basic sequence problem. *J. Amer. Math. Soc.*, 6(4):851–874, 1993.

MR2919812 [22] K.-G. Grosse-Erdmann and A. Peris Manguillot. *Linear chaos.* Universitext. Springer, London, 2011.

H92 [23] J. Hadamard. Essai sur l'etude des fonctions données pour leurs développements de Taylor. *J. de Mathematique*, 8(4):101–186, 1982.

Hoe [24] L. Hörmander. *The analysis of linear partial differential operators I + II*, volume 256+257 of *Grundlehren.* Springer, Berlin, 1983.

IK75 [25] P. Ion and T. Kawai. Theory of vector-valued hyperfunctions. *Publ. Res. Inst. Math. Sci.*, 11(1):1–19, 1975/76.

Is85 [26] R. Ishimura. Existence locale de solutions holomorphes pour les équations différentielles d'ordre infini. *Ann. Inst. Fourier*, 35:49–57, 1985.

Is90 [27] R. Ishimura. Sur les équations différentielles d'ordre infini d'Euler. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, 44((1)):1–10, 1990.

It821 [28] Y. Ito. Fourier hyperfuction semigroups. *J. Math. Tokushima Univ.*, 16:33–53, 1982.

Ka15 [29] T. Kalmes. Some results on P-convexity and the problem of parameter dependence for solutions of linear partial differential equations. arXiv-Preprint: 1408.4356.

Ka12b [30] T. Kalmes. The augmented operator of a surjective partial differential operator with constant coefficients need not be surjective. *Bull. London Math. Soc.*, 44:610–614, 2012.

Ka12a [31] T. Kalmes. Some results on surjectivity of augmented differential operators. *J. Math. Anal. Appl.*, 386:125–134, 2012.

MR1354137 [32] N. J. Kalton. The basic sequence problem. *Studia Math.*, 116(2):167–187, 1995.

MR1467651 [33] N. J. Kalton and A. Pełczyński. Kernels of surjections from $\mathcal{L}_1$-spaces with an application to Sidon sets. *Math. Ann.*, 309(1):135–158, 1997.

Ki02 [34] J. Kisyński. Distribution semigroups and one parameter semigroups. *Bull. Pol. Acad. Sci. Math.*, 50:189–216, 2002.

Ki07 [35] J. Kisyński. On Fourier transforms of distribution semigroups. *J. Funct. Anal.*, 242:400–441, 2007.

Ko64 [36] H. Komatsu. Semigroups of operators in locally convex spaces. *J. Math. Soc. Japan*, 16:230–262, 1964.

MR797411 [37] U. Krengel. *Ergodic theorems*, volume 6 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.

K14 [38] K. Kruse. Vector valued Fourier hyperfunctions. 2014. Dissertation Oldenburg.

Ku [39] P. C. Kunstmann. Distribution semigroups and abstract Cauchy problems. *Trans. Amer. Math. Soc.*, 351:837–856, 1999.

La94 [40] M. Langenbruch. Continuous linear right inverses for convolution operators in spaces of real analytic functions. *Studia Math.*, 110:65–82, 1994.

Ma91 [41] F. Mantlik. Partial differential operators depending analytically on a parameter. *Ann. Inst. Fourier (Grenoble)*, 41:577–599, 1991.

Ma92 [42] F. Mantlik. Fundamental solutions for hypoelliptic differential operators depending analytically on a parameter. *Trans. Amer. Math. Soc.*, 334:245–257, 1992.

MTV2 [43] R. Meise, B. Taylor, and D. Vogt. Characterization of the linear partial differential operators with constant coefficients admitting a continuous linear right inverse. *Ann. Inst. Fourier (Grenoble)*, 40:619–655, 1990.

MR1483073 [44] R. Meise and D. Vogt. *Introduction to functional analysis*, volume 2 of *Oxford Graduate Texts in Mathematics*. The Clarendon Press, Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.

MF [45] I. Melnikova and A. Filinkov. *Abstract Cauchy problems: three approaches*, volume 120 of *Monographs and Surveys in Pure Appl. Math.* Chapman & Hall, Boca Raton, 2001.

MR1083567 [46] G. Metafune and V. B. Moscatelli. Quojection and prequojections. In *Advances in the theory of Fréchet spaces (Istanbul, 1988)*, volume 287 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 235–254. Kluwer Acad. Publ., Dordrecht, 1989.

MR1328368 [47] M. I. Ostrovskiĭ. On complemented subspaces of sums and products of Banach spaces. *Proc. Amer. Math. Soc.*, 124(7):2005–2012, 1996.

Ou2 [48] S. Ōuchi. Semi-groups of operators in locally convex spaces. *J. Math. Soc. Japan*, 29:265–276, 1973.

Pa72 [49] V. Palamodov. On a Stein manifold the Dolbeault complex splits in positive dimensions. *Math.USSR Sbornik*, 17:289–315, 1972.

Pa72a [50] V. Palamodov. A criterion for splitness of differential complexes with constant coefficients. In *Geometrical and aalgebraical aspects in several variables*, C.A.Berenstein and D. Struppa, Eds., pages 265–291. EdtEl, 1991.

Re97 [51] H. Render. Hadamard's multiplication theorem — recent developments. *Colloquium Math.*, 74:79–92, 1997.

S1 [52] M. Sato. Theory of hyperfunctions. *J. Fac. Sci. Univ. Tokyo, Sec. I*, 8:139–193, 1959/60.

MR1223222 [53] K. Seip. Beurling type density theorems in the unit disk. *Invent. Math.*, 113(1):21–39, 1993.

SA [54] J. Shapiro. *Composition operators and classical function theory.* Springer, New York, 1993.

MR1237406 [55] J. H. Shapiro. *Composition operators and classical function theory.* Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

To [56] J. Tougeron. Fonctions composées différentiables. *Ann. Inst. Fourier (Grenoble)*, 30(4):51–74, 1980.

Tr62a [57] F. Trèves. Fundamental solutions of linear partial differential equations with constant coefficients depending on parameters. *Bull. Soc. Math. France*, 90:473–486, 1962.

Tr62 [58] F. Trèves. Un théorème sur les équations aux dérivées partielles à coefficients constants dépendant de paramètres. *Amer. J. Math.*, 84:561–577, 1962.

Vo15 [59] D. Vogt. Hadamard operators on $\mathscr{D}'(\mathbb{R}^d)$. *Studia Math.* Online First Version, DOI:10.4064/sm8481-1-2017.

Vo16 [60] D. Vogt. Hadamard operators on $\mathscr{D}'(\omega)$. *Math. Nachr.* to appear.

MR887320 [61] D. Vogt. On the functors $\mathrm{Ext}^1(E,F)$ for Fréchet spaces. *Studia Math.*, 85(2):163–197, 1987.

Vo07b [62] D. Vogt. A division theorem for real analytic functions. *Bull. Lond. Math. Soc.*, 39(5):818–828, 2007.

Vo07a [63] D. Vogt. Extension operators for real analytic functions on compact subvarieties on $\mathbb{R}^d$. *J. Reine Angew. Math.*, 606:217–233, 2007.

Vo09 [64] D. Vogt. A nuclear Fréchet space consisting of $C^\infty-$functions and failing the bounded approximation property. *Proc. Amer. Math. Soc.*, 138(4):1421–1423, 2010.

Vo13b [65] D. Vogt. $\mathscr{E}$ as an algebra by multiplicative convolution. *arXiv*, 2015. 1509.05759.

Vo13a [66] D. Vogt. Operators of Hadamard type on spaces of smooth functions. *Math. Nachr.*, 288:353–361, 2015.

WW [67] S. Wang Wang. Quasi-distribution semigroups and integrated semigroups. *J. Funct. Anal.*, 146:352–381, 1997.

We01 [68] J. Wengenroth. A splitting theorem for subspaces and quotients of $\mathscr{D}'$. *Bull. Pol. Acad. Sci., Math.*, 49(4):349–354, 2001.

MR1977923 [69] J. Wengenroth. *Derived functors in functional analysis*, volume 1810 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.