Largest solid extensions of ℓ_p spaces and operators between them

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On joint work with Angela A. Albanese and Werner J. Ricker

Aim of the lecture

AIM

We present several results about the structure of solid extensions of the spaces ℓ_p and ℓ_{p+} and we compare them with the ℓ_p-type spaces which generate them. The continuity, compactness, spectrum and ergodic properties of the Cesàro operator defined on these spaces will be also investigated.

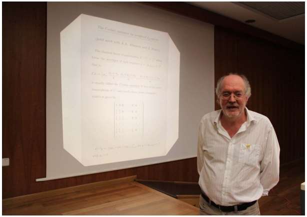
We report on joint work with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).

Lecture dedicated to Manuel López Pellicer

Ernesto Cesàro (1859-1906)

Albanese and Ricker

Angela Albanese



Werner Ricker

The discrete Cesàro operator

The *Cesàro operator* C is defined for a sequence $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ of complex numbers by

$$\mathrm{C}(x) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_n, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

Proposition.

The operator $C \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself with

$$C^{-1}(y) = (ny_n - (n-1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^{\mathbb{N}}, \tag{1}$$

where we set $y_{-1} := 0$.

Recall that $\mathbb{C}^{\mathbb{N}}$ is a Fréchet space for the topology of coordinatewise convergence.

Theorem. Hardy. 1920.

Let $1 < p < \infty$. The Cesàro operator maps the Banach space ℓ^p continuously into itself and $\|\mathrm{C}\| = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$, for all $n \in \mathbb{N}$.

In particular, **Hardy's inequality** holds:

$$\|\mathrm{C}\|_p \leq p' \|x\|_p, \quad x \in \ell^p.$$

Clearly C is not continuous on ℓ_1, since $\mathrm{C}(e_1) = (1, 1/2, 1/3, ...)$.

Proposition.

The Cesàro operators $\mathrm{C}\colon \ell^\infty \to \ell^\infty$, $\mathrm{C}\colon c \to c$ and $\mathrm{C}\colon c_0 \to c_0$ are continuous, and $\|\mathrm{C}\| = 1$ in the three spaces.

Moreover, $\lim \mathrm{C}x = \lim x$ for each $x \in c$.

The space $ces(p), 1 < p < \infty$,

For each element $x = (x_n)_n = (x_1, x_2, \ldots)$ of $\mathbb{C}^{\mathbb{N}}$ let $|x| := (|x_n|)_n$.

The Cesàro operator $C : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$, satisfies $|C(x)| \leq C(|x|)$ for $x \in \mathbb{C}^{\mathbb{N}}$

For each $1 < p < \infty$ define

$$ces(p) := \left\{ x \in \mathbb{C}^{\mathbb{N}} : \|x\|_{ces(p)} := \|(\frac{1}{n}\sum_{k=1}^{n}|x_k|)_n\|_p = \|C(|x|)\|_p < \infty \right\},$$

where $\|\cdot\|_p$ denotes the standard norm in ℓ_p.

An intensive study of the Banach spaces $ces(p), 1 < p < \infty$, was undertaken by **G. Bennett** in 1996.

Properties of $ces(p), 1 < p < \infty$,

- Hardy's inequality implies $\ell_p \subseteq ces(p)$ with a continuous inclusion. The inclusion is proper.
- $C : ces(p) \to \ell_p$ is continuous. Indeed, for $x \in ces(p)$,

$$\|C(x)\|_p = \| \, |C(x)| \, \|_p \leq \|C(|x|)\|_p = \|x\|_{ces(p)}.$$

- **Theorem.** (Bennett, 1996)

$$x \in ces(p) \;\text{ if and only if }\; C(|x|) \in ces(p).$$

Properties of $ces(p), 1 < p < \infty$,

- (**Grosse-Erdmann, 1998**) Description using dyadic decomposition: $x \in \mathbb{C}^{\mathbb{N}}$ belongs to $ces(p)$ if and only if

$$\|x\|_{[p]} := \left(\sum_{j=0}^{\infty} 2^{j(1-p)} \left(\sum_{k=2^j}^{2^{j+1}-1} |x_k| \right)^p \right)^{1/p} < \infty.$$

- (**Curbera, Ricker, 2014**) $ces(p)$ coincides with the largest solid Banach lattice X in $\mathbb{C}^{\mathbb{N}}$ containing ℓ_p such that $C : X \to \ell_p$ acts continuously.

Properties of $ces(p), 1 < p < \infty$,

- (**Curbera, Ricker, 2014**) $ces(p)$ is a reflexive, p-concave Banach lattice for the order induced by $\mathbb{C}^{\mathbb{N}}$, and the canonical vectors $e_k := (\delta_{nk})_n$, for $k \in \mathbb{N}$, form an unconditional basis.
- (**Bennett, 1996**) Let $1 < p, q < \infty$ with $p \neq q$. Then $ces(p)$ is not Banach space isomorphic to ℓ_q.
- (**Albanese, Bonet , Ricker**) Let $1 < p, q < \infty$ with $p \neq q$. Then $ces(p)$ is not isomorphic to $ces(q)$

Properties of $ces(p), 1 < p < \infty$,

An isometric description of the dual of $ces(p)$ was obtained by **Jagers in 1974**.

A more convenient isomorphic description is due to **Bennett**: Let $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. The map $\Phi\colon (ces(q))' \to d(q')$, $\Phi(f) := (\langle e_j, f\rangle)_{j\in\mathbb{N}}$, is a linear isomorphism of the dual Banach space $(ces(q))'$ onto the Banach space

$$d(q') := \left\{ x \in \mathbb{C}^{\mathbb{N}} \colon \|x\|_{d(q')} := \left(\sum_{n=1}^{\infty} \sup_{k\geq n} (|x_k|^{q'}) \right)^{1/q'} < \infty \right\}.$$

Spectrum and point spectrum

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ is the space of all continuous linear operators on X.

The **resolvent set** $\rho(T,X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T,X) := \mathbb{C} \setminus \rho(T,X)$. The **point spectrum** is the set $\sigma_{pt}(T,X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The elements of $\sigma_{pt}(T,X)$ are called eigenvalues of T.

Spectrum and point spectrum of the Cesàro operator

Notation:

$\Sigma := \{\frac{1}{m} \colon m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$.

Proposition.

(i) $\sigma(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \sigma_{pt}(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \Sigma$.

(ii) Fix $m \in \mathbb{N}$. Let $x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^{\mathbb{N}}$ where $x_n^{(m)} := 0$ for $n \in \{1, \ldots, m-1\}$, $x_m^{(m)} := 1$ and $x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!}$ for $n > m$. Then the eigenspace

$$\mathrm{Ker}\left(\frac{1}{m} I - \mathsf{C}\right) = \mathrm{span}\{x^{(m)}\} \subseteq \mathbb{C}^{\mathbb{N}}$$

is 1-dimensional.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

(i) $\sigma(\mathsf{C};\ell^\infty)=\sigma(\mathsf{C};c_0)=\{\lambda\in\mathbb{C}\mid \left|\lambda-\frac{1}{2}\right|\leq\frac{1}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C};\ell^\infty)=\{(1,1,1,...)\}$.

(iii) $\sigma_{pt}(\mathsf{C};c_0)=\emptyset$.

Theorem. Leibowitz. 1972.

Let $1 < p < \infty$ and $1/p + 1/p' = 1$.

(i) $\sigma(\mathsf{C}; \ell^p) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C}; \ell^p) = \emptyset$.

In particular, C is not compact in the spaces $\ell^p, 1 < p \leq \infty$.

Spectrum and point spectrum of the Cesàro operator in $ces(p)$

Theorem. Curbera, Ricker. 2014.

Let $1 < p < \infty$ and $1/p + 1/p' = 1$.

(i) We have

$$\sigma(\mathrm{C}; ces(p)) = \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\} \quad \text{and} \quad \sigma_{pt}(\mathrm{C}; ces(p)) = \emptyset.$$

(ii) If $|\lambda - \frac{p'}{2}| < \frac{p'}{2}$, then $\overline{\mathrm{Im}(\lambda \mathrm{I} - \mathrm{C})} \neq ces(p)$.

(iii) In addition,

$$\sigma_{pt}(\mathrm{C}'; (ces(p))') = \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\}.$$

The Fréchet space $\ell_{p+}, 1 \leq p < \infty$,

- For each $1 \leq p < \infty$ define

$$\ell_{p+} := \bigcap_{q>p} \ell_q.$$

- It is a Fréchet space (and a lattice) with respect to the increasing sequence of lattice norms

$$x \longmapsto \|x\|_{p_k}, \quad x \in \ell_{p+}, \quad k \in \mathbb{N},$$

for any sequence $p < p_{k+1} < p_k$ with $p_k \downarrow p$.

Properties of ℓ_{p+}

- (**J.C. Díaz, Metafune, Moscatelli**) ℓ_{p+} is a reflexive, quasinormable, non-Montel, countably normed Fréchet space which is solid in $\mathbb{C}^{\mathbb{N}}$ and contains no isomorphic copy of any infinite dimensional Banach space.
- $\ell_p \subseteq \ell_{p+}$ continuously and with a proper inclusion.
- The Cesàro operator $\mathsf{C} : \ell_{p+} \to \ell_{p+}$ is continuous.

Proposition. Albanese, Bonet, Ricker.

For every $1 \leq p, q < \infty, p \neq q$, the spaces ℓ_{p+} and ℓ_{q+} are not isomorphic.

Idea: Take $p < q$. Suppose that there exists an isomorphism $T : \ell_{q+} \to \ell_{p+}$. Then $T|_{\ell_q} : \ell_q \to \ell_r$ is continuous for each $r \in (p, q)$. By Pitt's theorem, $T : \ell_q \to \ell_r$ is compact. Therefore $\{T(e_j)\}_{j=1}^{\infty}$ is a relatively compact subset of ℓ_r. Consequently $\{T(e_j)\}_{j=1}^{\infty}$ is also a relatively compact subset of ℓ_{p+}. There exists $y \in \ell_{p+}$ and a subsequence $\{T(e_{j(k)})\}_{k=1}^{\infty}$ of $\{T(e_j)\}_{j=1}^{\infty}$ such that $T(e_{j(k)}) \to y$ in ℓ_{p+}. By continuity of the inverse operator $T^{-1} : \ell_{p+} \to \ell_{q+}$ it follows that $e_{j(k)} \to T^{-1}(y)$ in ℓ_{q+}. Choose any $s > q$, in which case $\ell_{q+} \subseteq \ell_s$ continuously, then also $e_{j(k)} \to T^{-1}(y)$ in the Banach space ℓ_s. This is impossible as $\|e_{j(k)} - e_{j(l)}\|_{\ell_s} = 2^{1/s} \geq 1$ for all $k \neq l$.

The Fréchet space $ces(p+), 1 \leq p < \infty$,

Define

$$ces(p+) := \bigcap_{q>p} ces(q), 1 \leq p < \infty,$$

which is Fréchet space equipped with the lattice norms

$$x \longmapsto \|x\|_{ces(p_k)}, \quad x \in ces(p+), \quad k \in \mathbb{N},$$

for any sequence $p < p_{k+1} < p_k$ satisfying $\lim_{k\to\infty} p_k = p$.

Operators on $ces(p+)$

- The Cesàro operator $C_{p+} : ces(p+) \to ces(p+)$ is a positive continuous operator.
- Given $\varphi = (\varphi_i)_i \in \mathbb{C}^{\mathbb{N}}$, the multiplier operator $M_\varphi : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is defined by $M_\varphi(x) := (\varphi_i x_i)_i$ for each $x = (x_i)_i \in \mathbb{C}^{\mathbb{N}}$.
- Continuous multipliers M_φ from $ces(p)$ to $ces(q)$ were characterized by Bennet. Compactness and spectrum have been studied by Albanese, Bonet and Ricker.
- The case of multipliers M_φ from $ces(p+)$ to $ces(q+)$ has been also studied by us.

The Fréchet space $ces(p+), 1 \leq p < \infty$,

WARNING

$ces(p+)$ behaves VERY different from ℓ_{p+}.

Properties of $ces(p+)$

- The space $ces(p+)$ is a solid Fréchet lattice subspace of $\mathbb{C}^{\mathbb{N}}$ and $\ell_{p+} \subseteq ces(p+)$ with a continuous and proper inclusion.
- For $x \in \mathbb{C}^{\mathbb{N}}$ we have $x \in ces(p+)$ if and only if $\mathsf{C}(|x|) \in ces(p+)$.
- The space $ces(p+)$ is the largest solid Fréchet lattice X in $\mathbb{C}^{\mathbb{N}}$ which contains ℓ_{p+} such that $\mathsf{C}(X) \subset \ell_{p+}$.
- For $1 \leq p < q < \infty$ $ces(p+) \subseteq ces(q+)$ with proper inclusion. .

Main results about $ces(p+)$

A power series space of finite type $r \in \mathbb{R}$ and order 1 is defined, for any given strictly increasing sequence $\alpha = (\alpha_k)_k \subseteq (0, \infty)$ satisfying $\lim_{k\to\infty} \alpha_k = \infty$, by

$$\Lambda_r(\alpha) := \{x \in \mathbb{C}^{\mathbb{N}} : ||x||_t := \sum_{k=1}^{\infty} |x_k| e^{t\alpha_k} < \infty, \quad \forall t < r\}.$$

Theorem.

The space $ces(p+)$, $1 \leq p < \infty$, coincides algebraically and topologically with the power series space $\Lambda_{-1/p'}(\alpha)$, $\frac{1}{p} + \frac{1}{p'} = 1$, of finite type $-1/p'$ and order 1, where $\alpha = (\log k)_k$.

Idea of the proof of the Theorem

For $1 < p < q < \infty$ ($\frac{1}{q} + \frac{1}{q'} = 1$). If $x \in ces(q)$

$$||x||_{ces(q)} \leq \sum_{j=1}^{\infty} |x_j| \cdot ||e_j||_{ces(q)} \leq B_q \sum_{j=1}^{\infty} |x_j| j^{-\frac{1}{q'}},$$

B_q only depends on q and $-\frac{1}{q'} < -\frac{1}{p'}$.
Given $p < s < q$ by Hölder's inequality we get

$$q' \sum_{j=1}^{\infty} |x_j| j^{-\frac{1}{q'}} \leq \sum_{j=1}^{\infty} |x_j| \sum_{n=j}^{\infty} \frac{1}{n} n^{-\frac{1}{q'}} =$$

$$= \sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{j=1}^{n} |x_j|\Big) n^{-\frac{1}{q'}} \leq ||y||_{s'} ||x||_{ces(s)}.$$

Main results about $ces(p+)$

Corollary.

Each of the Fréchet spaces $ces(p+)$, for $1 \leq p < \infty$, is isomorphic to the power series space $\Lambda_0(\alpha)$ of finite type 0 and order 1, where $\alpha = (\log k)_k$.

Corollary.

(i) The Fréchet space $ces(p+)$ is a Köthe echelon space of order 1 and the canonical vectors $(e_j)_{j\in\mathbb{N}}$ form an unconditional basis of $ces(p+)$.

(ii) $ces(p+)$ is a Fréchet-Schwartz space but, it is not nuclear.

(iii) $ces(p+)$ is not isomorphic to ℓ_{q+} for each $1 \leq p, q < \infty$.

Definition of the *-spectrum of Waelbroeck.

X **is a Hausdorff locally convex space (lcs).**

- $\rho^*(T)$ consists of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that each $\mu \in B(\lambda, \delta) := \{z \in \mathbb{C} \colon |z - \lambda| < \delta\}$ belongs to $\rho(T)$ and the set $\{R(\mu, T) \colon \ \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X)$.
- $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$.
- $\sigma^*(T)$ is a closed set containing $\sigma(T)$. If $T \in \mathcal{L}(X)$ with X a Banach space, then $\sigma(T) = \sigma^*(T)$. There exist continuous linear operators T on a Fréchet space X such that $\overline{\sigma(T)} \subset \sigma^*(T)$ properly.

Theorem.

(i) Let $1 < p < \infty$. The following statements are valid.

(a1) $\sigma_{pt}(\mathsf{C}; ces(p+)) = \sigma_{pt}(\mathsf{C}; \ell_{p+}) = \emptyset$.

(a2) $\sigma(\mathsf{C}; ces(p+)) = \sigma(\mathsf{C}; \ell_{p+}) = \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| < \frac{p'}{2}\} \cup \{0\}$.

(a3) $\sigma^*(\mathsf{C}; ces(p+)) = \overline{\sigma(\mathsf{C}; ces(p+))} = \{\lambda \in \mathsf{C} : |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\}$.

(a4) $\sigma^*(\mathsf{C}; \ell_{p+}) = \sigma^*(\mathsf{C}; ces(p+))$.

Theorem continued.

(ii) For $p = 1$ the following statements are valid.

(b1) $\sigma_{pt}(\mathsf{C}; ces(1+)) = \sigma_{pt}(\mathsf{C}; \ell_{1+}) = \emptyset$.

(b2) $\sigma(\mathsf{C}; ces(1+)) = \sigma(\mathsf{C}; \ell_{1+}) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\} \cup \{0\}$.

(b3) $\sigma^*(\mathsf{C}; ces(1+)) = \overline{\sigma(\mathsf{C}; ces(1+))} = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq 0\}$.

(b4) $\sigma^*(\mathsf{C}; \ell_{1+}) = \sigma^*(\mathsf{C}; ces(1+))$.

Ergodic properties of the Cesàro operator on $ces(p+)$ and ℓ_{p+}

An operator $T \in \mathcal{L}(X)$, is called **power bounded** if $\{T^n\}_{n=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$. Given $T \in \mathcal{L}(X)$, the averages are $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$, $n \in \mathbb{N}$. The operator T is said to be **mean ergodic** if $\{T_{[n]}x\}_{n=1}^{\infty}$ is a convergent sequence in X for every $x \in X$.

An operator $T \in \mathcal{L}(X)$ is called **hypercyclic** if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X. If, for some $z \in X$, the projective orbit $\{\lambda T^n z \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*.

Proposition.

The Cesàro operator C on $ces(p+)$ and on ℓ_{p+}, for $1 \leq p < \infty$, is not power bounded, not mean ergodic and not supercyclic.

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