ASSOCIATED WEIGHTS FOR SPACES OF p-INTEGRABLE ENTIRE FUNCTIONS

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Abstract. In analogy to the notion of associated weights for weighted spaces of analytic functions with sup-norms, p-associated weights are introduced for spaces of entire p-integrable functions, $1 \leq p < \infty$. As an application, necessary conditions for the boundedness of composition operators acting between general Fock type spaces are proved.

Introduction

Weighted L^p-spaces of entire functions, $1 \leq p < \infty$, arise in several different settings, such as partial differential equation, Fourier analysis, and spectral theory. In particular, the Fock L^2-space (also known as Bargmann space or Segal-Bargmann space) plays an important role in quantum physics (see [2, 21], see also [15, Chapter I, Sec. 6] for an historical introduction) .

Properties of composition operators acting between these spaces have been extensively studied by several authors in the last two decades (see [1, 10, 17, 24, 25, 26, 27, 28, 29]).

There are essentially two approaches to the analysis of weighted L^p-spaces of entire functions and of composition operators. The first approach has been developed for $p = \infty$ and relies on the study of associated growth conditions, namely weights that are intrisically defined for the space of entire functions under consideration (see, e.g. the seminal paper [4]). The other approach has been initially applied in the case $p = 2$ and then extended to $1 \leq p < \infty$, and involves the use of complex and functional analysis techniques, such as representing kernels, Carleson measures, sampling and interpolation sequences.

The aim of this note is to present a connection between these techniques, by introducing an analogous concept of associated weights for $1 \leq p < \infty$. The boundedness of composition operators is then studied by applying p-associated weights. Moreover, in the particular case of radial weights, we give conditions on the weights in order that bounded composition operators have to be generated by affine maps, thus extending the classical result that holds true for composition operators between Fock spaces.

1. Notation and preliminaries

Consider the space $\mathbb{C}^N$ endowed with the complex inner product and the norm defined by:

$$\langle z, w\rangle = \sum_{i=1}^{N} z_i \overline{w_i}, \qquad |z| = \sqrt{\langle z, z\rangle}, \qquad z = (z_1, \ldots, z_N), w = (w_1, \ldots, w_N).$$

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We denote by $\mathcal{H}(\mathbb{C}^N)$ the space of entire functions and we consider the usual Lebesgue measure on $\mathbb{C}^N = \mathbb{R}^{2N}$. We will use the notation $\mathcal{H}(\mathbb{C}^N;\mathbb{C}^N)$ for the space $\big(\mathcal{H}(\mathbb{C}^N)\big)^N$.

Let v be a *weight* on $\mathbb{C}^N$, namely a continuous function $v : \mathbb{C}^N \to]0,\infty[$. Define for $1 \le p < \infty$

$$\mathcal{F}^p_v = \{f \in \mathcal{H}(\mathbb{C}^N) \mid \int_{\mathbb{C}^N} |f(z)|^p v^p(z)dz < \infty\} \tag{1.1}$$

and

$$||f||_{p,v} = \left(\int_{\mathbb{C}^N} |f(z)|^p v^p(z)dz < \infty\right)^{\frac{1}{p}}, \qquad f \in \mathcal{F}^p_v.$$

The spaces $\mathcal{F}^p_v$ are subspaces of $L^p(\mathbb{C}^N, v^p dz)$, hence are normed spaces. If $p = 2$ the norm is induced by the inner product:

$$(f,g)_v = \int_{\mathbb{C}^N} f(z)\overline{g}(z)v^2(z)dz \qquad f,g \in \mathcal{F}^2_v.$$

If $p = \infty$ we consider

$$\mathcal{F}^\infty_v = \{f \in \mathcal{H}(\mathbb{C}^N) \mid \sup_{\mathbb{C}^N} |f(z)|v(z) < \infty\}, \tag{1.2}$$

endowed with the norm

$$||f||_{\infty,v} = \sup_{\mathbb{C}^N} |f(z)|v(z), \qquad f \in \mathcal{F}^\infty_v.$$

The space $\mathcal{F}^\infty_v$ is often also denoted by $H^\infty_v(\mathbb{C}^N)$. It is a Banach space and has been widely studied (see e.g. [3, 4, 16, 18] and the references therein).

If $v(z) = e^{-\frac{\alpha}{2}|z|^2}$, then $\mathcal{F}^p_v$ is the classical Fock space $\mathcal{F}^{p,\alpha}$ (see e.g. the monograph [30]).

Lemma 1.1. *Let $v : \mathbb{C}^N \to]0,\infty[$ be a continuous function and $1 \le p < \infty$. For every compact $K \subseteq \mathbb{C}^N$, the evaluation map*

$$\delta_z : \mathcal{F}^p_v \to \mathbb{C}, \qquad f \mapsto f(z)$$

is continuous uniformly with respect to $z \in K$. As a consequence, $\mathcal{F}^p_v$ is a Banach space and $\mathcal{F}^2_v$ is a Hilbert space.

Proof. Since every compact subset in $\mathbb{C}^N$ is covered by a finite number of polydisks, it is enough to prove that for every $z \in \mathbb{C}^N$ there exists $M > 0$ such that

$$\sup_{w \in P(z,1)} |f(w)| \le M||f||_{p,v},$$

where $P(z,1) = \{w = (w_1,\dots,w_N) \in \mathbb{C}^N \mid |z_i - w_i| \le 1, i = 1,\dots,N\}$.

Given $z \in \mathbb{C}^N$, let v_0 be the minimum of v on $P(z,2)$.

Let $f \in \mathcal{F}^p_v$ and $x \in P(z,1)$. Clearly $|f|$ is separately subharmonic in each variable, hence for every $r_1,\dots,r_N > 0$:

$$\begin{aligned}|f(x_1,\cdot,x_N)| &\le \frac{1}{2\pi}\int_0^{2\pi} |f(x_1 + r_1e^{i\theta_1}, x_2,\dots,x_N)|d\theta_1 \le \dots \le \\ &\le \frac{1}{(2\pi)^N}\int_{[0,2\pi]^N} f(x_1 + r_1e^{i\theta_1},\dots,x_1 + r_Ne^{i\theta_N})d\theta_1\dots d\theta_N.\end{aligned}$$

By multiplying both sides by $2^N r_1 \cdot \dots \cdot r_N$ and by integrating with respect to $r_1, \dots, r_N$ on $[0,1]^N$, we get that:

$$|f(x)| \le \frac{1}{\pi^n}\int_{P(x,1)}|f(w)|dw \le \frac{1}{\pi^n}\int_{P(z,2)}|f(w)|dw \le \frac{1}{v_0}\frac{1}{\pi^n}\int_{P(z,2)}|f(w)|v(w)dw$$

$$\le \frac{1}{v_0}\frac{1}{\pi^n}\left(\int_{P(z,2)}(|f(w)|v(w))^p dw\right)^{1/p}\left(\int_{P(z,2)}dw\right)^{1/p'} \le \frac{1}{v_0}\frac{1}{\pi^n}(4\pi)^{n/p'}||f||_{p,v}.$$

Finally we prove that $\mathcal{F}^p_v$ is a closed subspace of $L^p(\mathbb{C}^N, v^p dz)$. Let $(f_n)_n$ be a sequence in $\mathcal{F}^p_v$ converging to a function $f \in L^p(\mathbb{C}^N, v^p dz)$. Let K be a compact subset of $\mathbb{C}^N$. There exists $C_K > 0$ such that for every $n, m \in \mathbb{N}$:

$$\forall z \in K \qquad |f_n(z) - f_m(z)| \le C_K ||f_n - f_m||_{p,v}.$$

Hence $(f_n)_n$ is a Cauchy sequence with respect to the compact-open topology and therefore it converges to an entire function. □

Remark 1.2. By the Riesz Representation Theorem, for every $z \in \mathbb{C}^N$ there exists $K_z \in \mathcal{F}^2_v$ such that

$$\forall f \in \mathcal{F}^2_v \qquad \delta_z(f) = \langle K_z, f\rangle = \int_{\mathbb{C}^N} K_z(w)\overline{f(w)}v^2(w)dw. \tag{1.3}$$

Observe that, for every $z \in \mathbb{C}^N$,

$$||\delta_z||_{(\mathcal{F}^2_v)'} = \sup_{||f||_{2,v}\le 1}|f(z)| = \sup_{||f||_{2,v}\le 1}|\langle K_z, f\rangle| \le ||K_z||_{2,v}. \tag{1.4}$$

Considering in particular $f = K_z \cdot ||K_z||^{-1}_{2,v}$, we get that

$$||\delta_z||_{(\mathcal{F}^2_v)'} \ge |\langle K_z, K_z\rangle| \cdot ||K_z||^{-1}_{2,v} = ||K_z||_{2,v}. \tag{1.5}$$

Hence

$$||\delta_z||_{(\mathcal{F}^2_v)'} = ||K_z||_{2,v} = \sqrt{\langle K_z, K_z\rangle} = \sqrt{K_z(z)}. \tag{1.6}$$

Several examples of spaces $\mathcal{F}^p_v(\mathbb{C})$ are known in literature. In no way pretending to be exhaustive, we refer to [1, 11, 14, 19, 20, 22, 23]. In particular we recall in the next example the spaces $\mathcal{F}^p_v$ that have been first considered in [23].

Example 1.3. Let $v = e^{-\varphi}$ where $\varphi \in C^2(\mathbb{C}^N)$ is a real valued function on $\mathbb{C}^N$ such that

$$m\omega_0 \le dd^c\varphi \le M\omega_0$$

holds uniformly pointwise on $\mathbb{C}^N$ for some positive constants m and M, where d is the usual exterior derivative, $d^c = i(\partial - \overline{\partial})$ and $\omega_0 = dd^c|\cdot|^2$ is the standard Euclidean Kähler form on $\mathbb{C}^N$. The associated spaces $\mathcal{F}^p_v$ have been thoroughly studied in [23].

If $\varphi(z) = -\frac{\alpha}{2}|z|^2$ one gets the classical Fock spaces and for $\varphi(z) = |z|^2 - m\log|z|$, with $m \in \mathbb{N}$, one gets the Fock-Sobolev spaces $\mathcal{F}^{p,m}$ (see e.g. [12]).

In the case of example 1.3, the following characterization of composition operators holds. For sake of completeness we give a proof with the arguments of [1].

Proposition 1.4. *Let $v = e^{-\varphi_1}$ and $w = e^{-\varphi_2}$ with $\varphi_i \in C^2(\mathbb{C}^N), i = 1, 2$, real valued functions of the type considered in Example 1.3. Let $\psi \in \mathcal{H}(\mathbb{C}^N; \mathbb{C}^N)$. The composition operator $C_\psi : \mathcal{F}^p_v \to \mathcal{F}^q_w$, $f \mapsto f \circ \psi$ with $1 \le p \le q < \infty$, is continuous if and only if there exists $r > 0$ (or equivalently for all $r > 0$) such that*

$$\sup_{z\in\mathbb{C}^N} \int_{\psi^{-1}(B(z,r))} \frac{w^q(w)}{v^q(\psi(w))} dw < \infty.$$

In particular, if $\frac{w}{v\circ\psi}$ is q - integrable on $\mathbb{C}^N$, then C_ψ is continuous.

Proof. Let λ be the pull-back measure defined by

$$\lambda(E) = \int_{\psi^{-1}(E)} e^{-q\varphi_2}(z)dz$$

for any Borel subset of $\mathbb{C}^N$. By [1, Section 2, Corollary 3.1], C_ψ is a bounded composition operator if and only if the measure $\mu = e^{q\varphi_1} \cdot \lambda$ is a (p, q)-Carleson measure, or, equivalently by [1, Lemma 2.1], if

$$\sup_{z\in\mathbb{C}^N} \mu(B(z, r)) < \infty \qquad \text{for some/all} \quad r > 0.$$

The assertion follows by observing that

$$\mu(B(z, r)) = \int_{B(z,r)} e^{q\varphi_1(w)} d\lambda(w) = \int_{\psi^{-1}(B(z,r))} e^{q\varphi_1(\psi(z))} e^{-q\varphi_2(z)} dz < \infty.$$

2. p-associated weights and composition operators

Associated weights were introduced in [4], in order to study properties of the space $H^\infty_v(\mathbb{C})$. Given a continuous weight v, the associated weight is defined as

$$\widetilde{v}(z) = \sup\{|f(z)| \mid f \in H^\infty_v(\mathbb{C}^N), (||f||_{v,\infty} \le 1\})^{-1}.$$

It is known that $\widetilde{v}$ is continuous and $0 < v \le \widetilde{v}$ (see [4, Properties 1.2]). The weight v is said to be essential if $v \simeq \widetilde{v}$, namely there exist $c, C > 0$ such that

$$cv \le \widetilde{v} \le Cv.$$

We will consider an analogous definition for every $1 \le p < \infty$.

Definition 2.1. Let $v : \mathbb{C}^N \to]0, \infty[$ be a continuous weight and $1 \le p < \infty$. The p-associated weight $\widetilde{v}_p$ is defined by

$$\widetilde{v}_p(z) = \frac{1}{||\delta_z||_{(\mathcal{F}^p_v)'}}, \qquad z \in \mathbb{C}^N. \tag{2.1}$$

The weight v is said to be p-essential if $v \simeq \widetilde{v}_p$.

Remark 2.2. Observe that if v is a 2-essential weight, then, by (1.6),

$$v(z) \simeq \frac{1}{\sqrt{K_z(z)}}. \tag{2.2}$$

Example 2.3. Let $v = e^{-\varphi}$ be the weight considered in Example 1.3.

By [23, Corollary 3.7],

$$f(z) = \delta_z(f) = \int_{\mathbb{C}^N} K(z, w)\overline{f(w)} e^{-2\varphi(w)} dw \qquad \forall f \in \mathcal{F}^p_v, z \in \mathbb{C}^N,$$

where K_z is the reproducing kernel. Then $K_z \in \mathcal{F}_v^q$ for every $1 \leq q \leq \infty$ (see [1, Equ. (2)]) and $||K_z||_{q,v} \simeq v(z)^{-1}$. Finally [23, Proposition 3.8] yields that $||\delta_z||_{(\mathcal{F}_v^p)'} = ||K_z||_{p',v} \simeq v(z)^{-1}$, where p' is the conjugate exponent of p, and therefore v is p-essential.

Example 2.4. Let $v(z) = e^{-|z|^\alpha}$, with $\alpha > 2$, $x \in \mathbb{C}$. It has been proved in [14, Corollary 8] that, for large values of $|z|$,

$$||\delta_z||_{(\mathcal{F}_v^2)'} = ||K_z||_{2,v} \simeq e^{-|z|^\alpha}|z|^{\alpha-2}.$$

Hence v is not a 2-essential weight.

Remark 2.5. If v is a continuous weight such that $\mathcal{F}_v^p \hookrightarrow H_v^\infty$ continuously, then clearly there exists $C > 0$ such that $\widetilde{v} \leq C\widetilde{v}_p$. Hence, if v is p-essential, it is essential. The converse does not hold. For example, the weight v defined as $v(z) = e^{-|z|^\alpha}$, with $\alpha > 2$, is essential (see e.g. [9, Lemma 1] or [8, Remark 3.11]), but it is not 2-essential.

Proposition 2.6. *Let v and w be continuous weights on $\mathbb{C}^N$, $1 \leq p, q < \infty$, and $\psi \in \mathcal{H}(\mathbb{C}^N;\mathbb{C}^N)$. If the composition operator $C_\psi : \mathcal{F}_v^p \to \mathcal{F}_w^q$ is continuous, then*

$$\sup_{z\in\mathbb{C}^N} \frac{\widetilde{w}_q(z)}{\widetilde{v}_p(\psi(z))} < \infty.$$

Proof. If C_ψ is continuous, then

$$(C_\psi)' : (\mathcal{F}_w^q)' \to (\mathcal{F}_v^p)'$$

is continuous. Hence there exists $C > 0$ such that

$$\forall \xi \in (\mathcal{F}_w^p)' \qquad ||(C_\psi)'\xi||_{(\mathcal{F}_v^p)'} \leq C||\xi||_{(\mathcal{F}_w^q)'}. \tag{2.3}$$

In particular,

$$\forall z \in \mathbb{C}^N \qquad ||(C_\psi)'\delta_z||_{(\mathcal{F}_v^p)'} \leq C||\delta_z||_{(\mathcal{F}_w^q)'}. \tag{2.4}$$

By observing that

$$(C_\psi)'(\delta_z)(f) = \delta_z(C_\psi f) = \delta_{\psi(z)}(f), \tag{2.5}$$

we get

$$\forall z \in \mathbb{C}^N \qquad ||\delta_{\psi(z)}||_{(\mathcal{F}_v^p)'} \leq C||\delta_z||_{(\mathcal{F}_w^q)'}, \tag{2.6}$$

therefore there exists a costant $\widetilde{C} > 0$ such that

$$\forall z \in \mathbb{C}^N \qquad \frac{1}{\widetilde{v}_p(\psi(z))} \leq \widetilde{C}\frac{1}{\widetilde{w}_q(z)} \qquad \square. \tag{2.7}$$

As a consequence of the definition of p-essential weights and of Remark 2.5, we get.

Corollary 2.7. *Let v and w be continuous weights on $\mathbb{C}^N$, $1 \leq p, q < \infty$, and $\psi \in \mathcal{H}(\mathbb{C}^N;\mathbb{C}^N)$ such that the composition operator $C_\psi : \mathcal{F}_v^p \to \mathcal{F}_w^q$ is continuous.*

(1) If v is p-essential and w is q-essential, then $\sup_{z\in\mathbb{C}^N} \frac{w(z)}{v(\psi(z))} < \infty$.
(2) If $\mathcal{F}_w^p \hookrightarrow H_w^\infty$ continuously, then $\sup_{z\in\mathbb{C}^N} \frac{w(z)}{\widetilde{v}_p(\psi(z))} < \infty$.

Remark 2.8. Proposition 2.6 and Corollary 2.7 should be compared with the characterization of bounded operators between spaces of type H_v^∞ in terms of associated weights (see [7, Proposition 5]).

3. Radial weights

Motivated by the results in the previous section, we investigate the boundedness of the quotient $\frac{w(z)}{v(\psi(z))}$, where w and v are continuous radial weights on $\mathbb{C}^N$ and $\psi \in \mathcal{H}(\mathbb{C}^N;\mathbb{C}^N)$. In the following, if $v : \mathbb{C}^N \to]0,\infty[$ is a radial weight, with an abuse of notation, for any $r > 0$ we will write $v(r)$ meaning $v(z)$ for any $z \in \mathbb{C}^N$ with $|z| = r$, and we say that v is (strictly) decreasing meaning that $v(r)$ is (strictly) decreasing on $[0,\infty[$.

Proposition 3.1. *Let $v, w : \mathbb{C}^N \to]0,\infty[$ be radial continuous decreasing weights and let $\psi \in \mathcal{H}(\mathbb{C}^N;\mathbb{C}^N)$. If for some $\alpha > 1$*

$$\lim_{r\to\infty} \frac{w(r)}{v(\alpha r)} = +\infty \tag{3.1}$$

and

$$\sup_{z\in\mathbb{C}^N} \frac{w(z)}{v(\psi(z))} < +\infty, \tag{3.2}$$

then ψ is affine.

Proof. We show that $\sup_{z\in\mathbb{C}^N,|z|\geq 1} \frac{|\psi(z)|}{|z|}$ is bounded. Otherwise, there would exist $(z_k)_{k\in\mathbb{N}}$ such that $|z_k| \to \infty$ and

$$\forall k \in \mathbb{N} \quad |\psi(z_k)| \geq k|z_k|.$$

Then, for large k,

$$\sup_{k\in\mathbb{N}} \frac{w(z_k)}{v(\alpha z_k)} \leq \sup_{k\in\mathbb{N}} \frac{w(|z_k|)}{v(kz_k)} \leq \sup_{k\in\mathbb{N}} \frac{\widetilde{w}(z_k)}{v(\psi(z_k))} < +\infty,$$

against the hypothesis. Now the assertion follows with the argument of [10, Theorem 1]. For sake of completeness we give some hints of the proof. Assume that $\psi = (\psi_1, \ldots, \psi_n)$, with $\psi_i : \mathbb{C}^N \to \mathbb{C}$. For any $\zeta \in \mathbb{C}^N$, $||\zeta|| = 1$, consider the entire function on $\mathbb{C}$

$$\psi_i^\zeta(\lambda) := \psi_i(\lambda\zeta), \qquad \lambda \in \mathbb{C}.$$

If $\psi_i(z) = \sum_{s=0}^\infty \sum_{|\alpha|=s} c_\alpha z^\alpha$, since

$$\sup_{|\lambda|\geq 1} \frac{|\psi_i^\zeta(\lambda)|}{|\lambda|} < \infty, \tag{3.3}$$

for every $\zeta \in \mathbb{C}^N$, $||\zeta|| = 1$, we get

$$\sum_{|\alpha|=s} c_\alpha z^\alpha = 0 \qquad s \geq 2,$$

and therefore ψ_i is affine. □

By combining Proposition 3.1 and Proposition 2.6 we immediately get:

Corollary 3.2. *Let $v, w : \mathbb{C}^N \to]0,\infty[$ be radial continuous weights, $1 \leq p, q < \infty$. Assume that v is p-essential, w is q-essential and that for some $\alpha > 1$*

$$\lim_{r\to\infty} \frac{w(r)}{v(\alpha r)} = +\infty \tag{3.4}$$

If $\psi \in \mathcal{H}(\mathbb{C}^N;\mathbb{C}^N)$ satisfies that the composition operator $C_\psi : \mathcal{F}_v^p \to \mathcal{F}_w^q$ is continuous, then ψ is affine.

Example 3.3. The weights $v(z) = e^{-\gamma|z|^2 - m\log|z|}$ and $w(z) = e^{-\beta|z|^2 - k\log|z|}$ are p-essential for any $1 \leq p < \infty$ (see Example 1.3). Moreover condition (3.1) is satisfied, simply by choosing $\alpha > 1$ such that $\alpha^2\gamma > \beta^2$. Hence we get that if a composition operator between Fock-Sobolev spaces is continuous, then ψ is affine. Thus Corollary 3.2 covers some of the results in [13].

Corollary 3.4. *Let $v, w : \mathbb{C} \to]0,\infty[$ be radial continuous decreasing p-essential weights and assume that for some $\alpha > 1$*

$$\lim_{r\to\infty} \frac{w(r)}{v(\alpha r)} = \infty. \tag{3.5}$$

Let $\psi \in \mathcal{H}(\mathbb{C})$ non-constant. The composition operator $C_\psi : \mathcal{F}^p_v \to \mathcal{F}^p_w$ is continuous if and only if

$$\sup_{z\in\mathbb{C}} \frac{w(z)}{v(\psi(z))} < \infty.$$

In this case, ψ is affine.

Proof. The necessity part follows from Proposition 2.6. Conversely, if $\sup_{z\in\mathbb{C}} \frac{w(z)}{v(\psi(z))} < \infty$, then by Proposition 2.6, we get that $\psi(z) = az + b$, with $a \neq 0$. Therefore, for every $f \in \mathcal{F}^p_v$:

$$\int_{\mathbb{C}} |f(\psi(z)|^p w^p(z)dz = \frac{1}{a}\int_{\mathbb{C}} |f(z|^p w^p(\frac{z-b}{a})dz \leq C^p \int_{\mathbb{C}} |f(z)|^p v^p(z)dz. \qquad \square$$

Proposition 3.5. *Let $\psi(z) = Az + b$, with A a $N \times N$ complex matrix and $b \in \mathbb{C}^N$ and let $v : \mathbb{C}^N \to]0,\infty[$ be a radial continuous strictly decreasing weight.*

(1) If $||A|| = \sup\{|Az| : |z| = 1\} < 1$, then $\sup_{z\in\mathbb{C}^N} \frac{v(z)}{v(Az+b)} < +\infty$.

(2) If $||A|| > 1$ and

$$\forall\beta > 1 \qquad \lim_{r\to\infty} \frac{w(r)}{v(\beta r)} = +\infty.$$

then

$$\sup_{z\in\mathbb{C}^N} \frac{v(z)}{v(Az+b)} = +\infty.$$

Proof Assume $v(z) = \exp(-\varphi(|z|))$, with $\varphi : [0,\infty[\to [0,\infty[$ continuous and strictly increasing.

(1) Set $\varepsilon = 1 - ||A||$. For $|z| > \frac{|b|}{\varepsilon}$ we get

$$\varphi(|Az+b|) \leq \varphi(||A||\cdot|z| + |b|) \leq \varphi(||A||\cdot|z| + \varepsilon|z|) \leq \varphi(|z|).$$

It is enough to take $C := \max\{\varphi(|Az+b|) : |z| \leq \frac{|b|}{\varepsilon}\}$ to get

$$\varphi(|Az+b| \leq \varphi(|z|) + C, \qquad z \in \mathbb{C}^N.$$

(2) Take $\varepsilon > 0$ such that $1 + 2\varepsilon < ||A||$. There exists $\overline{z} \in \mathbb{C}^N$ such that $|\overline{z}| = 1$ and $|A\overline{z}| \geq 1 + 2\varepsilon$. For every $n \in \mathbb{N}$ such that $n > \frac{|b|}{\varepsilon}$

$$|A(n\overline{z}) + b| \geq |nA\overline{z}| - |b| \geq n(1+2\varepsilon) - |b| \geq n(1+\varepsilon) + (n\varepsilon - |b|) \geq n(1+\varepsilon).$$

Assume that there exists $C > 0$ such that

$$\varphi(|Az+b|) \leq C + \varphi(|z|) \qquad \forall\, z \in \mathbb{C}^N.$$

Then, for every $n > \frac{|b|}{\varepsilon}$:

$$\varphi(n(1+\varepsilon)) \leq \varphi(|A(n\overline{z})+b|) \leq C + \varphi(n),$$

hence

$$\varphi(n(1+\varepsilon)) - \varphi(n) \leq C$$

This contradicts

$$\lim_{r\to\infty} \varphi((1+\varepsilon)r) - \varphi(r) = +\infty. \qquad \square$$

We discuss now the case $N = 1$ and $\psi(z) = az + b$ with $|a| = 1$. If $b = 0$, then clearly $\frac{v(az)}{v(z)} = 1$ for every $z \in \mathbb{C}$. If $b \neq 0$, we consider some cases of particular interest.

Proposition 3.6. *Let $\psi(z) = az + b$ with $a \in \mathbb{C}$, $|a| = 1$ and $b \in \mathbb{C}$, $b \neq 0$. Let $v(z) = \exp(-\varphi(|z|))$, with $\varphi : [0,\infty[\to [0,\infty[$ continuous and strictly increasing. If*

$$\lim_{r\to\infty} \varphi(r+|b|) - \varphi(r) = +\infty, \tag{3.6}$$

then

$$\sup_{z\in\mathbb{C}} \frac{v(z)}{v(az+b)} = +\infty.$$

In particular $\sup_{z\in\mathbb{C}} \frac{v(z)}{v(az+b)} = +\infty$ if $v(z) = e^{-\alpha|z|^p}$, $p \geq 1$, if $v(z) = e^{-e^{|z|}}$ or if $v(z) = e^{-|z|^2 - m\log|z|}$, $m \in \mathbb{N}$.

Proof. Assume $a = e^{it}$ and $b = |b|e^{i\theta}$. If $z = re^{i(\theta-t)}$ with $r > 0$, then

$$|az+b| = |(r+|b|)e^{i\theta}| = r + |b|,$$

hence

$$\lim_{r\to\infty} \varphi(|z+b|) - \varphi(|z|) = +\infty,$$

and therefore $\sup_{z\in\mathbb{C}} \frac{v(z)}{v(az+b)} = +\infty$.

If $\varphi(r) = r^p$, $p \geq 1$, we have

$$\alpha(r+|b|)^p = \alpha r^p \left(1 + \frac{|b|}{r}\right)^p \geq \alpha r^p + \alpha p|b|r^{p-1} \geq \alpha r^p + \alpha p|b|r^{p-1} = \varphi(r) + \alpha p|b|r^{p-1}.$$

Therefore

$$\lim_{r\to\infty} \varphi(r+|b|) - \varphi(r) = +\infty.$$

Consider $\varphi(r) = e^r$. Then $e^{r+|b|} = e^{|b|}e^r = e^{|b|}e^r$ and therefore $\lim_{r\to\infty} e^{r+|b|} - e^r = \lim_{r\to\infty} e^r(e^{|b|} - 1) = +\infty$.

Finally, if $\varphi(r) = r^2 + m\log r$,

$$\begin{aligned} &\lim_{r\to\infty} \left((r+|b|)^2 + m\log(r+|b|) - r^2 - m\log r\right) = \\ = &\lim_{r\to\infty} 2|b|r + |b|^2 + m\log\left(1 + \frac{|b|}{r}\right) = +\infty. \qquad \square \end{aligned}$$

Example 3.7. Let $\psi(z) = az + b$ with $a \in \mathbb{C}$, $|a| = 1$ and $b \in \mathbb{C}$, $b \neq 0$. Let $v(z) = \exp(-\varphi(|z|))$, with $\varphi(r) = r^p$, $0 < p \leq 1$. If $|z| \geq 1$:

$$\begin{aligned} \varphi(|az+b|) &= \alpha|az+b|^p \leq \alpha(|z|+|b|)^p = \alpha|z|^p\left(1+\frac{|b|}{|z|}\right)^p \leq \\ &\leq \alpha|z|^p\left(1+p\frac{|b|}{|z|}\right) = \alpha|z|^p + \alpha p|b||z|^{p-1} \leq \varphi(|z|) + \alpha p|b|. \end{aligned}$$

Then

$$\varphi(|az+b|) \le \varphi(|z|) + \alpha p|b| + \max_{|z|\le 1}\varphi(|az+b|).$$

Therefore $\sup_{z\in\mathbb{C}} \frac{v(z)}{v(az+b)} < +\infty$. This should be compared with the behaviour of the weight when $p > 1$.

Proposition 3.8. *Let $\psi(z) = Az + b$ where A is a $N\times N$ complex matrix, $b \in \mathbb{C}^N$, $1 \le p \le \infty$, $v(z) = \exp(-\varphi(|z|))$, with $\varphi : [0,\infty[\to [0,\infty[$ continuous and strictly increasing, such that v is p-essential.*

(1) If $||A|| < 1$, then $C_\psi : \mathcal{F}_v^\infty \to \mathcal{F}_v^\infty$ is continuous.
(2) If $||A|| < 1$ and for some $\alpha > 1$

$$\lim_{r\to\infty} \frac{v(r)}{v(\alpha r)} = \infty,$$

then $C_\psi : \mathcal{F}_v^p \to \mathcal{F}_v^p$ is continuous for any $1 \le p < \infty$.
(3) If $||A|| > 1$ and

$$\forall \beta > 1 \qquad \lim_{r\to\infty} \frac{v(r)}{v(\beta r)} = +\infty,$$

then $C_\psi : \mathcal{F}_v^p \to \mathcal{F}_v^p$ is not continuous for any $1 \le p \le \infty$.
(4) If $N = 1$, $\psi(z) = az + b$, $|a| = 1$, and

$$\lim_{r\to\infty} \varphi(r+|b|) - \varphi(r) = +\infty, \tag{3.7}$$

then $C_\psi : \mathcal{F}_v^\infty \to \mathcal{F}_v^\infty$ is continuous if and only if $b = 0$.
(5) If $N = 1$, $\psi(z) = az + b$, $|a| = 1$, and (3.7) holds, then $C_\psi : \mathcal{F}_v^p \to \mathcal{F}_v^p$ is continuous if and only if $b = 0$, for any $1 \le p < \infty$.

Proof. It is enough to apply Proposition 3.6, recalling that, for $p = \infty$ C_ψ is continuous if and only if $\sup_{z\in\mathbb{C}^N} \frac{v(z)}{v(Az+b)} < \infty$ (see [7]) and for $1 \le p < \infty$, if C_ψ is continuous then $\sup_{z\in\mathbb{C}^N} \frac{v(z)}{v(Az+b)} < \infty$ and the converse holds if for some $\alpha > 1$ $\lim_{r\to\infty} \frac{v(r)}{v(\alpha r)} = +\infty$. If (3.7) holds, this condition is satisfied for any $\alpha > 1$, since $\varphi(\alpha r) - \varphi(r) \ge \varphi(r+|b|) - \varphi(r)$ for $r > \frac{|b|}{\alpha-1}$. □.

Example 3.9. If $p = \infty$, Proposition 3.8 can be applied when $\varphi(r) = r^q$, $q > 0$, $\varphi(r) = r^2 + m\log r$, $m \in \mathbb{N}$ or $\varphi(r) = e^r$, since $e^{-\varphi}$ is essential by [8, Remark 3.11] and by [4, Corollary 16].

If $1 \le p < \infty$, Proposition 3.8 can be applied when $\varphi(r) = \alpha r^2$ or $\varphi(r) = r^2 + m\log r$ by example 2.3.

We conclude with a result about composition operators acting between $\mathcal{F}_v^2$ spaces with different radial weights. The proof is based on the techniques of [25, Lemma 6.1].

Assume that v is a continuous weight on $\mathbb{C}$ such that $\int_0^\infty r^n v(r)dr < \infty$ for every $n \in \mathbb{N}$. If we set $a_n = \int_0^\infty r^{2n+1}v^2(r)dr$, then the reproducing kernel in $\mathcal{F}_v^2(\mathbb{C})$ is given by

$$K_z(w) = \sum_{n=0}^{\infty} a_n \overline{z}^n w^n, \qquad w \in \mathbb{C}, \tag{3.8}$$

and

$$||K_z||_{2,v} = \left(\sum_{n=0}^{\infty} a_n |z|^{2n} \right)^{\frac{1}{2}}. \tag{3.9}$$

(see e.g. [5, 6, 25]).

Theorem 3.10. *Let v and w be decreasing continuous radial weights on $\mathbb{C}$ such that $a_n = \int_0^\infty r^n v(r)dr < +\infty$, $b_n = \int_0^\infty r^n w(r)dr < +\infty$ for every $n \in \mathbb{N}$. Assume that $M := \sup_n \sqrt[2n]{b_n/a_n} < +\infty$. Let $\psi \in \mathcal{H}(\mathbb{C})$. If the composition operator $C_\psi : \mathcal{F}^2_v(\mathbb{C}) \to \mathcal{F}^2_w(\mathbb{C})$ is continuous, then ψ is affine.*

Proof. As in the proof of Theorem 2.6, we get that, if C_ψ is continuous, then there exists $C \geq 1$ such that

$$\forall z \in \mathbb{C}^N \qquad ||\delta_{\psi(z)}||_{(\mathcal{F}^2_v)'} \leq C ||\delta_z||_{\mathcal{F}^2_w)'}, \tag{3.10}$$

or equivalently

$$\forall z \in \mathbb{C}^N \qquad ||K_{\psi(z)}||_{2,v} \leq C ||K_z||_{2,w}. \tag{3.11}$$

Therefore we can apply (3.9) to get, for each $z \in \mathbb{C}$,

$$\left(\sum_{n=0}^{\infty} a_n |z|^{2n} \left(\frac{|\psi(z)|^{2n}}{|z|^{2n}} - C^2 \frac{b_n}{a_n} \right) \right) = \left(\sum_{n=0}^{\infty} a_n |\psi(z)|^{2n} - C^2 \sum_{n=0}^{\infty} b_n |z|^{2n} \right) \leq 0.$$

It is enough to prove that $\sup_{|z|\geq 1} \frac{|\psi(z)|}{|z|} < \infty$. Otherwise, there would exist a sequence $(z_k)_k$ of complex numbers such that $\lim_{k\to\infty} |z_k| = \infty$ and $\lim_{k\to\infty} \frac{|\psi(z_k)|}{|z_k|} = \infty$. Then there would exist $\nu \in \mathbb{N}$ such that for every $k \geq \nu$:

$$\frac{|\psi(z_k)|}{|z_k|} \geq C^2 \cdot (M+1),$$

with $M > 0$ as in the statement. Now, for every $k \in \mathbb{N}$ and $k \geq \nu$ we have

$$\frac{|\psi(z_k)|^{2n}}{|z_k|^{2n}} \geq C^{2n} \cdot (M+1)^{2n} \geq C^2 (M^{2n} + 1) \geq C^2 \left(\frac{b_n}{a_n} + 1 \right),$$

hence

$$\begin{aligned} 0 < C^2 a_1 |z_k| &\leq \sum_{n=0}^{\infty} a_n |z_k|^n \left(C^2 \left(\frac{b_n}{a_n} + 1 \right) - C^2 \frac{b_n}{a_n} \right) \\ &\leq \sum_{n=0}^{\infty} a_n |z_k|^n \left(\frac{|\psi(z_k)|^{2n}}{|z_k|^{2n}} - C^2 \frac{b_n}{a_n} \right) \leq 0. \end{aligned}$$

So we reach a contradiction. □.

Corollary 3.11. *Let v be a decreasing continuous weight on $\mathbb{C}$ such that $\int_0^\infty r^n v(r)dr < \infty$ and let $\psi \in \mathcal{H}(\mathbb{C})$. If the composition operator $C_\psi : \mathcal{F}^2_v(\mathbb{C}) \to \mathcal{F}^2_v(\mathbb{C})$ is continuous, then ψ is affine.*

Remark 3.12. Observe that Corollary 3.11 applies also to the weights considered in Example 2.4 and, more in general, those in [14], although they are not 2-essential.

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