ORDER SPECTRUM OF THE CESÀRO OPERATOR IN BANACH LATTICE SEQUENCE SPACES

J. BONET AND W.J. RICKER

Abstract. The discrete Cesàro operator C acts continuously in various classical Banach sequence spaces within $\mathbb{C}^{\mathbb{N}}$. For the coordinatewise order, many such sequence spaces X are also complex Banach lattices (eg. c_0, ℓ^p for $1 < p \leq \infty$, and ces(p) for $p \in \{0\} \cup (1, \infty)$). In such Banach lattice sequence spaces, C is always a positive operator. Hence, its order spectrum is well defined within the Banach algebra of all regular operators on X. The purpose of this note is to show, for every X belonging to the above list of Banach lattice sequence spaces, that the order spectrum $\sigma_{\mathrm{o}}(C)$ of C *coincides* with its usual spectrum $\sigma(C)$ when C is considered as a continuous linear operator on the Banach space X.

1. Introduction

Let E be a complex Banach lattice and $\mathcal{L}(E)$ denote the unital Banach algebra of all continuous linear operators from E into itself, equipped with the operator norm $\|\cdot\|_{\mathrm{op}}$. The unit is the identity operator $I : E \longrightarrow E$. Associated with each $T \in \mathcal{L}(E)$ is its *spectrum*

$$\sigma(T) := \{\lambda \in \mathbb{C} : (\lambda I - T) \text{ is not invertible in } \mathcal{L}(E)\}$$

and its resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$. An operator $T \in \mathcal{L}(E)$ is called *regular* if it is a finite linear combination of *positive operators*. The complex vector space of all regular operators is denoted by $\mathcal{L}^r(E)$; it is also a unital Banach algebra for the norm

$$\|T\|_r := \inf\{\|S\|_{\mathrm{op}} : S \in \mathcal{L}(E), S \geq 0, |T(z)| \leq S(|z|) \ \forall\, z \in E\}, \quad T \in \mathcal{L}^r(E). \tag{1.1}$$

Again $I : E \longrightarrow E$ is the unit. Moreover, $\|T\|_{\mathrm{op}} \leq \|T\|_r$ for $T \in \mathcal{L}^r(E)$, with equality whenever $T \geq 0$ (i.e., if T is a positive operator). The spectrum of $T \in \mathcal{L}^r(E)$, considered as an element of the Banach algebra $\mathcal{L}^r(E)$, is denoted by $\sigma_{\mathrm{o}}(T)$ and is called its *order spectrum*. Then $\rho_{\mathrm{o}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{o}}(T)$ is the *order resolvent* of T. Clearly

$$\sigma(T) \subseteq \sigma_{\mathrm{o}}(T), \quad T \in \mathcal{L}^r(E). \tag{1.2}$$

From the usual formula for the spectral radius, [?, Ch.I, §2, Proposition 8], it follows that the spectral radii for $T \in \mathcal{L}^r(E)$ satisfy $r(T) = r_{\mathrm{o}}(T)$ whenever $T \geq 0$. Standard references for the above concepts and facts are [?], [?], [?], for example.

It is clear from (??) that $r(T) \leq r_{\mathrm{o}}(T)$ for $T \in \mathcal{L}^r(E)$. So, if $r(T) < r_{\mathrm{o}}(T)$, then (??) cannot be an equality. This is the strategy applied in [?, pp.79-80] to exhibit a regular operator for which $\sigma(T) \subsetneqq \sigma_{\mathrm{o}}(T)$. For an example of a *positive operator* T satisfying $\sigma(T) \subsetneqq \sigma_{\mathrm{o}}(T)$, see [?, pp.283-284]. In the contrary direction, a rich supply of classical operators T for which the equality

$$\sigma(T) = \sigma_{\mathrm{o}}(T) \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary 47A10, 47B37, 47B65, 47L10; Secondary 46A45, 46B45, 47C05.

Key words and phrases. Banach algebra, Banach sequence space, Cesàro operator, spectrum, order spectrum.

is satisfied arise in harmonic analysis, [?, Theorem 3.4].

The aim of this note is to contribute two further classes of operators T which satisfy (??). In Section ?? it is shown that in any *Banach function space* E, all multiplication operators T by L^∞-functions are regular operators and satisfy (??). This is a consequence of the fact that the algebra of such multiplication operators is maximal commutative. Let $\mathbb{N} := \{1, 2, \ldots\}$. The remaining three sections deal with the classical *Cesàro operator* $C : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ defined by

$$C(x) := \left(\tfrac{1}{n}\textstyle\sum_{k=1}^{n} x_k\right)_{n=1}^{\infty} \quad x = (x_n)_{n=1}^{\infty} \in \mathbb{C}^\mathbb{N}, \tag{1.4}$$

which is clearly a *positive operator* for the coordinatewise order in the positive cone of $\mathbb{C}^\mathbb{N} = \mathbb{R}^\mathbb{N} \oplus i\mathbb{R}^\mathbb{N}$. Section ?? establishes some general results for determining the regularity of linear operators in *Banach lattice sequence spaces.* These results are designed to apply to the particular operators $(C-\lambda I)^{-1}$, where C is given in (??). In Section ?? we will consider the restriction of C to the Banach lattice sequence spaces c_0 and $\ell^p, 1 < p \leq \infty$, and show that (??) is satisfied in all cases (with C in place of T). Section ?? is devoted to proving the same fact, but now when C acts in the discrete Cesàro spaces $\mathrm{ces}(p), 1 < p < \infty$, and in $\mathrm{ces}(0)$.

2. Multiplication operators

Let (Ω, Σ, μ) be a *localizable measure space* (in the sense of [?, 64A]), that is, the associated measure algebra is a complete Boolean algebra and, for every measurable set $A \in \Sigma$ with $\mu(A) > 0$ there exists $B \in \Sigma$ such that $B \subseteq A$ and $0 < \mu(B) < \infty$ (i.e., μ has the finite subset property). All σ-finite measures are localizable, [?, 64H Proposition]. Every Banach function space E (of $\mathbb{C}$-valued functions) over (Ω, Σ, μ) is a complex Banach lattice for the pointwise μ-a.e. order. Given any $\varphi \in L^\infty(\mu)$, the multiplication operator $M_\varphi : E \longrightarrow E$ defined by $f \longmapsto \varphi f$, for $f \in E$, belongs to $\mathcal{L}(E)$ and satisfies $\|M_\varphi\|_{\mathrm{op}} = \|\varphi\|_\infty$. Define a unital, commutative subalgebra of $\mathcal{L}(E)$ by

$$\mathcal{M}_E(L^\infty(\mu)) := \{M_\varphi : \varphi \in L^\infty(\mu)\};$$

the unit is the identity operator $I = M_{\mathbf{1}}$ where $\mathbf{1}$ is the constant function 1 on Ω. Recall that the *commutant* of $\mathcal{M}_E(L^\infty(\mu))$ is defined by

$$\mathcal{M}_E(L^\infty(\mu))^c := \{A \in \mathcal{L}(E) : AM_\varphi = M_\varphi A \;\; \forall\, \varphi \in L^\infty(\mu)\} \subseteq \mathcal{L}(E).$$

It is known that $\mathcal{M}_E(L^\infty(\mu))$ is a *maximal commutatitive*, unital subalgebra of $\mathcal{L}(E)$, that is, $\mathcal{M}_E(L^\infty(\mu)) = \mathcal{M}_E(L^\infty(\mu))^c$, [?, Proposition 2.2]. Moreover, also the *bicommutant* $\mathcal{M}_E(L^\infty(\mu))^{cc} = \mathcal{M}_E(L^\infty(\mu))$.

Proposition 2.1. *Let (Ω, Σ, μ) be a localizable measure space and E be a Banach function space over (Ω, Σ, μ).*

(i) $\mathcal{M}_E(L^\infty(\mu)) \subseteq \mathcal{L}^r(E)$.
(ii) *$\mathcal{M}_E(L^\infty(\mu))$ is inverse closed in $\mathcal{L}(E)$. That is, if $T \in \mathcal{M}_E(L^\infty(\mu))$ is invertible in $\mathcal{L}(E)$ (i.e., there exists $S \in \mathcal{L}(E)$ satisfying $ST = I = TS$), then necessarily $S \in \mathcal{M}_E(L^\infty(\mu))$.*
(iii) *For every $T \in \mathcal{M}_E(L^\infty(\mu))$ we have $\sigma_{\mathrm{o}}(T) = \sigma(T)$.*

Proof. (i) Let $\varphi \in L^\infty(\mu)$. Then $\varphi = [(\mathrm{Re}\,\varphi)^+ - (\mathrm{Re}\,\varphi)^-] + i[(\mathrm{Im}\,\varphi)^+ - (\mathrm{Im}\,\varphi)^-]$ with all four functions $(\mathrm{Re}\,\varphi)^+, (\mathrm{Re}\,\varphi)^-, (\mathrm{Im}\,\varphi)^+, (\mathrm{Im}\,\varphi)^-$ belonging to the positive cone $L^\infty(\mu)^+$ of $L^\infty(\mu)$. Since $M_\varphi = [M_{(\mathrm{Re}\,\varphi)^+} - M_{(\mathrm{Re}\,\varphi)^-}] + i[M_{(\mathrm{Im}\,\varphi)^+} - M_{(\mathrm{Im}\,\varphi)^-}]$ is a linear combination of positive operators, it is clear that $M_\varphi \in \mathcal{L}^r(E)$.

(ii) Since $\mathcal{M}_E(L^\infty(\mu))$ is maximal commutative in $\mathcal{L}(E)$, it follows that $\mathcal{M}_E(L^\infty(\mu))$ is inverse closed in $\mathcal{L}(E)$, [**?**, Ch.II, §15, Theorem 4].

(iii) In view of (**??**) it suffices to show that $\rho(T) \subseteq \rho_{\rm o}(T)$. Suppose that $T = M_\varphi$ with $\varphi \in L^\infty(\mu)$. Fix $\lambda \in \rho(T)$. Then $\lambda I - T = M_{(\lambda\mathbf{1}-\varphi)}$ belongs to $\mathcal{M}_E(L^\infty(\mu))$ because $(\lambda\mathbf{1} - \varphi) \in L^\infty(\mu)$. Since $M_{(\lambda\mathbf{1}-\varphi)}$ is invertible in $\mathcal{L}(E)$, it follows from part (ii) that actually $(\lambda I - T)^{-1} \in \mathcal{M}_E(L^\infty(\mu))$ and hence, by part (i), that also $(\lambda I - T)^{-1} \in \mathcal{L}^r(E)$. □

Remark 2.2. We point out that $\|T\|_{\rm op} = \|T\|_r$ for each $T \in \mathcal{M}_E(L^\infty(\mu))$. Indeed, let $\varphi \in L^\infty(\mu)$ satisfy $T = M_\varphi$, in which case $\|M_\varphi\|_{\rm op} = \|\varphi\|_\infty$. Define $S := \|\varphi\|_\infty I$ and note that $S \geq 0$ with $\|S\|_{\rm op} = \|\varphi\|_\infty$. Moreover,

$$|M_\varphi(f)| = |\varphi f| \leq \|\varphi\|_\infty |f| = S(|f|), \quad f \in E,$$

and so $\|T\|_r \leq \|S\|_{\rm op} = \|\varphi\|_\infty = \|T\|_{\rm op}$; see (**??**). The reverse inequality $\|T\|_{\rm op} \leq \|T\|_r$ always holds.

3. The Cesàro operator in Banach sequence spaces

We begin with some preliminaries. Equipped with the topology of pointwise convergence $\mathbb{C}^\mathbb{N}$ is a locally convex Fréchet space. Let $A = (a_{nm})_{n,m=1}^\infty$ be any lower triangular (infinite) matrix, i.e., $a_{nm} = 0$ whenever $m > n$. Then A induces the continuous linear operator $T_A : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ defined by

$$T_A(x) := \left(\textstyle\sum_{m=1}^\infty a_{nm} x_m\right)_{n=1}^\infty, \quad x \in \mathbb{C}^\mathbb{N}. \tag{3.1}$$

For $x \in \mathbb{C}^\mathbb{N}$ define $|x| := (|x_n|)_{n=1}^\infty$. Then also $|x| \in \mathbb{C}^\mathbb{N}$. A vector subspace $X \subseteq \mathbb{C}^\mathbb{N}$ is called *solid* (or an *ideal*) if $y \in X$ whenever $x \in X$ and $y \in \mathbb{C}^\mathbb{N}$ satisfy $|y| \leq |x|$. It is always assumed that X contains the vector space consisting of all elements of $\mathbb{C}^\mathbb{N}$ which have only finitely many non-zero coordinates. In addition, it is assumed that X has a norm $\|\cdot\|_X$ with respect to which it is a complex *Banach lattice* for the *coordinatewise order* and such that the natural inclusion $X \subseteq \mathbb{C}^\mathbb{N}$ is continuous. Under the previous requirements X is called a *Banach lattice sequence space.*

Lemma 3.1. *Let $A = (a_{nm})_{n,m=1}^\infty$ be a lower triangular matrix with all entries non-negative real numbers and $X \subseteq \mathbb{C}^\mathbb{N}$ be a Banach lattice sequence space such that $T_A(X) \subseteq X$. Let $B = (b_{nm})_{n,m=1}^\infty$ be any matrix such that*

$$|b_{nm}| \leq a_{nm}, \quad n, m \in \mathbb{N}. \tag{3.2}$$

Then the restricted operator $T_A : X \longrightarrow X$ belongs to $\mathcal{L}(X)$. Moreover, $T_B : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ satisfies $T_B(X) \subseteq X$ and the restricted operator $T_B : X \longrightarrow X$ also belongs to $\mathcal{L}(X)$. In addition, $\|T_B\|_{\rm op} \leq \|T_A\|_{\rm op}$.

Proof. Condition (**??**) implies that B is also a lower triangular matrix. Moreover, the continuity of both $T_A : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ and of the inclusion map $X \subseteq \mathbb{C}^\mathbb{N}$ imply, via the Closed Graph Theorem in the Banach space X, that the restricted operator $T_A \in \mathcal{L}(X)$.

Given $x \in X$ we have for each $n \in \mathbb{N}$, via (**??**), that

$$(T_B(x))_n = \left|\textstyle\sum_{m=1}^\infty b_{nm} x_m\right| \leq \textstyle\sum_{m=1}^\infty |b_{nm}| \cdot |x_m| \leq \textstyle\sum_{m=1}^\infty a_{nm}|x_m| = (T_A(|x|))_n.$$

Since X is solid and $T_A(|x|) \in X$, these inequalities and (**??**) imply that $T_B(x) \in X$. Moreover, as $\|\cdot\|_X$ is a lattice norm it follows that

$$\begin{aligned} \|T_B(x)\|_X &= \|(\textstyle\sum_{m=1}^\infty b_{nm} x_m)_{n=1}^\infty\|_X \leq \|(\textstyle\sum_{m=1}^\infty a_{nm}|x_m|)_{n=1}^\infty\|_X \\ &= \|T_A(|x|)\|_X \leq \|T_A\|_{\rm op}\|x\|_X, \end{aligned}$$

for each $x \in X$, where the stated series are actually finite sums. Hence, $\|T_B\|_{\rm op} \le \|T_A\|_{\rm op}$ and the proof is complete. □

Since the operator T_A as given in Lemma **??** satisfies $T_A \ge 0$, it is clearly regular.

Corollary 3.2. *Let $A = (a_{nm})_{n,m=1}^\infty$ be a lower triangular matrix with non-negative real entries and $X \subseteq \mathbb{C}^\mathbb{N}$ be a Banach lattice sequence space such that $T_A(X) \subseteq X$. Let $B = (b_{nm})_{n,m=1}^\infty$ be any matrix satisfying (**??**). Then the operator $T_B \in \mathcal{L}(X)$ is necessarily regular, that is, $T_B \in \mathcal{L}^r(X)$.*

Proof. Define the non-negative real numbers $s_{nm} := (\mathrm{Re}\, b_{nm})^+, u_{nm} := (\mathrm{Re}\, b_{nm})^-, v_{nm} := (\mathrm{Im}\, b_{nm})^+$ and $w_{nm} := (\mathrm{Im}\, b_{nm})^-$ for each $n, m \in \mathbb{N}$. Then $b_{nm} = (s_{nm} - u_{nm}) + i(v_{nm} - w_{nm})$ and $\{s_{nm}, u_{nm}, v_{nm}, w_{nm}\} \subseteq [0, a_{nm}]$ for $n, m \in \mathbb{N}$. Setting $S := (s_{nm})_{n,m=1}^\infty, U := (u_{nm})_{n,m=1}^\infty, V := (v_{nm})_{n,m=1}^\infty$ and $W := (w_{nm})_{n,m=1}^\infty$ it is clear from the definition (**??**) that each operator $T_S \ge 0, T_U \ge 0, T_V \ge 0$ and $T_W \ge 0$ (in X) belongs to $\mathcal{L}(X)$; see Lemma **??**. Since $T_B = (T_S - T_U) + i(T_V - T_W)$, it follows that $T_B \in \mathcal{L}^r(X)$. □

Together with appropriate estimates, Corollary **??** will be the main ingredient required to establish (**??**) for C (in place of T) when it acts in various classical Banach lattice sequence spaces X.

Let $\Sigma_0 := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. We recall the formula for the inverses $(C - \lambda I)^{-1} : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ whenever $\lambda \in \mathbb{C} \setminus \Sigma_0$, [**?**, p.266]. Namely, for $n \in \mathbb{N}$ the n-th row of the lower triangular matrix determining $(C - \lambda I)^{-1}$ has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^n (1 - \frac{1}{k\lambda})}, \quad 1 \le m < n, \quad \text{and} \quad \frac{n}{1 - n\lambda} = \frac{1}{(\frac{1}{n} - \lambda)}, \quad m = n, \tag{3.3}$$

with all other entries in row n being 0. We write

$$(C - \lambda I)^{-1} = T_{D_\lambda} - \frac{1}{\lambda^2} T_{E_\lambda}, \tag{3.4}$$

where the diagonal matrix $D_\lambda = (d_{nm}(\lambda))_{n,m=1}^\infty$ is given by

$$d_{nn}(\lambda) := \frac{1}{(\frac{1}{n} - \lambda)} \quad \text{and} \quad d_{nm}(\lambda) := 0 \quad \text{if } n \ne m. \tag{3.5}$$

Setting $\gamma[\lambda] := \mathrm{dist}(\lambda, \Sigma_0) > 0$ it is routine to check that

$$|d_{nn}(\lambda)| \le \frac{1}{\gamma[\lambda]}, \quad n \in \mathbb{N}, \quad \lambda \in \mathbb{C} \setminus \Sigma_0. \tag{3.6}$$

Moreover, $E_\lambda = (e_{nm}(\lambda))_{n,m=1}^\infty$ is the lower triangular matrix given by $e_{1m}(\lambda) = 0$, for $m \in \mathbb{N}$, and for all $n \ge 2$ by

$$e_{nm}(\lambda) := \begin{cases} \frac{1}{n \prod_{k=m}^n (1 - \frac{1}{k\lambda})} & \text{if} \quad 1 \le m < n \\ 0 & \text{if} \quad m \ge n. \end{cases} \tag{3.7}$$

Lemma 3.3. *Let $X \subseteq \mathbb{C}^\mathbb{N}$ be any Banach lattice sequence space. For each $\lambda \in \mathbb{C} \setminus \Sigma_0$ the diagonal operator T_{D_λ}, with $D_\lambda = (d_{nm}(\lambda))_{n,m=1}^\infty$ given by (**??**), is regular in X, that is, $T_{D_\lambda} \in \mathcal{L}^r(X)$.*

Proof. Fix $\lambda \notin \Sigma_0$ and let $A := \frac{1}{\gamma[\lambda]} I$, where I is the identity matrix in $\mathbb{C}^\mathbb{N}$, in which case $T_A(X) \subseteq X$ is clear. It follows from (**??**) that the matrix $B := D_\lambda$ satisfies (**??**). Hence, the regularity of T_{D_λ} in X follows from Corollary **??**. □

Remark 3.4. (i) Since any Banach lattice sequence space $X \subseteq \mathbb{C}^\mathbb{N}$ is a Banach function space over the σ-finite measure space $(\mathbb{N}, 2^\mathbb{N}, \mu)$, relative to counting measure μ, and the function $n \longmapsto d_{nn}(\lambda)$ on $\mathbb{N}$ belongs to $L^\infty(\mu)$ by (**??**), the regularity of $T_{D_\lambda} \in \mathcal{L}(X)$ also follows from Proposition **??**(i).

(ii) For appropriate X and $\lambda \notin \Sigma_0$, it is clear from (**??**) and Lemma **??** that the regularity of $(C-\lambda I)^{-1} \in \mathcal{L}(X)$ is completely determined by the matrix E_λ.

The following inequalities will be needed in the sequel. For $\alpha < 1$ we refer to [**?**, Lemma 7] and for general $\alpha \in \mathbb{R}$ to [**?**, Lemma 3.2(i)].

Lemma 3.5. *Let $\lambda \in \mathbb{C}\setminus\Sigma_0$ and set $\alpha := Re(\frac{1}{\lambda})$. Then there exist positive constants $P(\alpha)$ and $Q(\alpha)$ such that*

$$\frac{P(\alpha)}{n^\alpha} \leq \prod_{k=1}^{n} |1 - \frac{1}{k\lambda}| \leq \frac{Q(\alpha)}{n^\alpha}, \quad n \in \mathbb{N}. \tag{3.8}$$

4. The classical spaces $\ell^p, 1 < p \leq \infty$, and c_0

For each $1 < p \leq \infty$ let $C_p \in \mathcal{L}(\ell^p)$ denote the Cesàro operator as given by (**??**) when it is restricted to ℓ^p. As a consequence of Hardy's inequality, [**?**, Theorem 326], it is known that $\|C_p\|_{\text{op}} = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (with $p' := 1$ when $p = \infty$). Concerning the spectrum of C_p we have

$$\sigma(C_p) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| \leq \tfrac{p'}{2}\}, \quad 1 < p \leq \infty. \tag{4.1}$$

Various proofs of (**??**) are known for $1 < p < \infty$, [**?**], [**?**], [**?**], [**?**], [**?**]; see the discussion on p.268 of [**?**]. For the case $p = \infty$ we refer to [**?**, Theorem 4], for example.

Remark 4.1. For each $\lambda \neq 0$ set $\alpha := \text{Re}(\frac{1}{\lambda})$. Then, for any $b > 0$ we have

$$\alpha < \tfrac{1}{b} \text{ and only if } |\lambda - \tfrac{b}{2}| > \tfrac{b}{2}.$$

The corresponding results for $\alpha > \frac{1}{b}$ and $\alpha = \frac{1}{b}$ also hold.

Proposition 4.2. *For each $1 < p < \infty$ the order spectrum of the positive operator $C_p \in \mathcal{L}(\ell^p)$ satisfies*

$$\sigma_o(C_p) = \sigma(C_p). \tag{4.2}$$

Proof. Via (**??**) it suffices to verify that $\rho(C_p) \subseteq \rho_o(C_p)$.

With the notation of (**??**) and (**??**) it is shown on p.269 of [**?**], as a consequence of (**??**) in Lemma **??** above, that for every $\lambda \neq 0$ satisfying $\alpha := \text{Re}(\frac{1}{\lambda}) < 1$ there exists a constant $\beta(\lambda) > 0$ such that

$$|e_{nm}(\lambda)| \leq \frac{\beta(\lambda)}{n^{1-\alpha}m^\alpha}, \quad 1 \leq m \leq n, \quad n \in \mathbb{N}. \tag{4.3}$$

Set $B := E_\lambda$ and let A be the lower triangular matrix whose entries $a_{nm}(\lambda) \geq 0$ are given by the right-side of (**??**) for each $n \in \mathbb{N}$ and $1 \leq m \leq n$ (and 0 otherwise). According to (**??**) the matrices A and B satisfy (**??**). Let $X := \ell^p$ for $p \in (1, \infty)$ fixed. Then Corollary **??** implies that E_λ will be regular (i.e., $T_{E_\lambda} \in \mathcal{L}^r(\ell^p)$) whenever $T_A(\ell^p) \subseteq \ell^p$. Note that $T_A \in \mathcal{L}(\mathbb{C}^\mathbb{N})$ is given by

$$x \longmapsto \beta(\lambda) \left(\frac{1}{n^{1-\alpha}} \sum_{m=1}^{n} \frac{x_m}{m^\alpha}\right)_{n=1}^{\infty} := \beta(\lambda) G_\lambda(x), \quad x \in \mathbb{C}^\mathbb{N}. \tag{4.4}$$

So, if $\text{Re}(\frac{1}{\lambda}) < 1$, then (**??**) implies that $T_A \in \mathcal{L}(\ell^p)$ whenever $G_\lambda : \ell^p \longrightarrow \ell^p$ is continuous.

Let now $\lambda \in \rho(C_p)$, that is, $|\lambda - \frac{p'}{2}| > \frac{p'}{2}$. Then $\alpha := \text{Re}(\frac{1}{\lambda}) < \frac{1}{p'}$, because of Remark **??**, and hence, $(1-\alpha)p > 1$. Then the Proposition on p.269 of [**?**] yields that indeed $G_\lambda \in \mathcal{L}(\ell^p)$. As noted above, this implies that $T_{E_\lambda} \in \mathcal{L}^r(\ell^p)$. Combined with (**??**) and Lemma **??** it follows that $(C_p - \lambda I)^{-1} \in \mathcal{L}^r(\ell^p)$, that is, $\lambda \in \rho_o(C_p)$. This completes the proof of (**??**). □

Recall that $\|C_\infty\|_{\text{op}} = 1$ and, from (**??**) for $p = \infty$, that

$$\sigma(C_\infty) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \le \tfrac{1}{2}\} \tag{4.5}$$

Proposition 4.3. *The order spectrum of the positive operator $C_\infty \in \mathcal{L}(\ell^\infty)$ satisfies*

$$\sigma_{\text{o}}(C_\infty) = \sigma(C_\infty).$$

Proof. Again by (**??**) it suffices to prove that $\rho(C_\infty) \subseteq \rho_{\text{o}}(C_\infty)$.

Fix $\lambda \in \rho(C_\infty)$. According to (**??**), for $b = 1$ the condition in Remark **??** is satisfied with $\alpha := \text{Re}(\frac{1}{\lambda})$. Hence, the inequalities (**??**) are valid and so $A := (a_{nm}(\lambda))_{n,m=1}^\infty \ge 0$ and $B := E_\lambda$ can again be defined exactly as in the proof of Proposition **??**. Then (**??**) is satisfied with $X := \ell^\infty$. Arguing as in the proof of Proposition **??** (via Corollary **??**) it remains to verify that $T_A : \ell^\infty \longrightarrow \ell^\infty$ is continuous, where T_A is given by (**??**). To this effect, since $(1 - \alpha) > 0$ by Remark **??**, it follows that

$$\sup_{n\in\mathbb{N}} \textstyle\sum_{m=1}^\infty |a_{nm}(\lambda)| = \beta(\lambda) \sup_{n\in\mathbb{N}} \frac{1}{n^{1-\alpha}} \sum_{m=1}^\infty \frac{1}{m^\alpha} < \infty; \tag{4.6}$$

this has been verified on p.778 of [**?**] (put $w(n) = 1$ there for all $n \in \mathbb{N}$) by considering each of the cases $\alpha < 0$, $\alpha = 0$ and $0 < \alpha < 1$ separately. But, condition (**??**) is known to imply that $T_A \in \mathcal{L}(\ell^\infty)$, [**?**, Ex.2, p.220]. The proof that $\lambda \in \rho_{\text{o}}(C_\infty)$ is thereby complete. □

To conclude this section we consider the Cesàro operator C, as given by (**??**), when it is restricted to c_0; denote this operator by C_0. It is shown in [**?**, Theorem 3], [**?**], that $\|C_0\|_{\text{op}} = 1$ and

$$\sigma(C_0) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \le \tfrac{1}{2}\}. \tag{4.7}$$

Proposition 4.4. *The order spectrum of the positive operator $C_0 \in \mathcal{L}(c_0)$ satisfies*

$$\sigma_{\text{o}}(C_0) = \sigma(C_0).$$

Proof. Since (**??**) shows that $\sigma(C_0) = \sigma(C_\infty)$, the entire proof of Proposition **??** can be easily adapted (now for $X := c_0$ and fixed $\lambda \in \rho(C_0)$), using the same notation, *up to the stage* where (**??**) is shown to be valid. In *addition* to the validity of (**??**) it is also true that

$$\lim_{n\to\infty} a_{nm}(\lambda) = \tfrac{\beta(\lambda)}{m^\alpha} \lim_{n\to\infty} \tfrac{1}{n^{1-\alpha}} = 0, \quad m \in \mathbb{N}, \tag{4.8}$$

because $\alpha := \text{Re}(\frac{1}{\lambda})$ satisfies $(1 - \alpha) > 0$. The two conditions (**??**) and (**??**) together are known to imply that $T_A \in \mathcal{L}(c_0)$, [**?**, Theorem 4.51-C]. Again via Corollary **??** and Lemma **??** we can conclude that $T_{E_\lambda} \in \mathcal{L}^r(c_0)$ and hence, also $(C_0 - \lambda I)^{-1}$ is regular on c_0. □

5. The discrete Cesàro spaces $\text{ces}(p), 1 < p < \infty$, and $\text{ces}(0)$

For $1 < p < \infty$ the discrete Cesàro spaces are defined by

$$\text{ces}(p) := \{x \in \mathbb{C}^\mathbb{N} : \|x\|_{\text{ces}(p)} := \big(\textstyle\sum_{n=1}^\infty (\frac{1}{n} \sum_{k=1}^n |x_k|)^p\big)^{1/p} < \infty\}.$$

In view of (**??**) we see that $\|x\|_{\text{ces}(p)} = \|C(|x|)\|_{\ell^p}$ for $x \in \text{ces}(p)$. It is known that each space $\text{ces}(p), 1 < p < \infty$, is a reflexive Banach lattice sequence space for the norm $\|\cdot\|_{\text{ces}(p)}$ and the coordinatewise order. The spaces $\text{ces}(p)$ have been thoroughly treated in [**?**]. According to Theorem **??** of [**?**] the restriction of C (see (**??**)) to $\text{ces}(p)$, denoted here by $C_{(p)}$, is continuous with $\|C_{(p)}\|_{\text{op}} = p'$ and

$$\sigma(C_{(p)}) = \Big\{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| \le \tfrac{p'}{2}\Big\}, \quad 1 < p < \infty. \tag{5.1}$$

Proposition 5.1. *For each $1 < p < \infty$ the order spectrum of the positive operator $C_{(p)} \in \mathcal{L}(\text{ces(p)})$ satisfies*

$$\sigma_{\text{o}}(C_{(p)}) = \sigma(C_{(p)}). \tag{5.2}$$

Proof. In view of (**??**) it suffices to verify that $\rho(C_{(p)}) \subseteq \rho_{\text{o}}(C_{(p)})$.

We decompose the set $\rho(C_{(p)})$ into two disjoint parts, namely the set

$$\rho_1 := \{\lambda \in \mathbb{C} \setminus \{0\} : \ \text{Re}\left(\tfrac{1}{\lambda}\right) \leq 0\} = \{u \in \mathbb{C} \setminus \{0\} : \ \text{Re}(u) \leq 0\} \tag{5.3}$$

and its complement $\rho_2 := \rho(C_{(p)}) \setminus \rho_1$.

First fix $\lambda \in \rho_1$. Then $\lambda \notin \Sigma_0$ and so we may consider $E_\lambda = (e_{nm}(\lambda))_{n,m=1}^\infty$ and $D_\lambda = (d_{nm}(\lambda))_{n,m=1}^\infty$ as specified by (**??**) and (**??**), respectively. It is shown on p.72 of [**?**] that

$$|e_{nm}(\lambda)| \leq \tfrac{1}{n}, \quad 1 \leq m < n, \quad n \in \mathbb{N}. \tag{5.4}$$

Warning: In [**?**] the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ is used rather than $\mathbb{N} = \{1, 2, 3, \ldots\}$ which is used here and so the inequalities from [**?**] are slightly different when they are stated here. Back to our proof, it is clear from (**??**) that the matrix $A = (c_{nm})_{n,m=1}^\infty$ for the Cesàro operator C is lower triangular with its n-th row, for each $n \in \mathbb{N}$, given by $c_{nm} := \frac{1}{n}$ for $1 \leq m \leq n$ and $c_{nm} := 0$ for $m > n$. Setting $B := E_\lambda$ it is clear from (**??**) that (**??**) is satisfied for the pair A, B in the space $X := \text{ces}(p)$. Since $C_{(p)} = T_A : \text{ces}(p) \longrightarrow \text{ces}(p)$ is continuous, it follows from Corollary **??** that $T_{E_\lambda} \in \mathcal{L}^r(\text{ces}(p))$ and hence, via Lemma **??** and (**??**), that also $(C_{(p)} - \lambda I)^{-1} \in \mathcal{L}^r(\text{ces}(p))$.

Consider now the set ρ_2. From (**??**) it is routine to establish that a non-zero point $z \in \mathbb{C}$ belongs to $\sigma(C_{(p)})$ if and only if $\text{Re}(\frac{1}{z}) \geq \frac{1}{p'}$. From the case of equality in Remark **??**, it follows that $\rho_2 = \bigcup_{0<\alpha<1/p'} \Gamma_\alpha$, where

$$\Gamma_\alpha := \left\{z \in \mathbb{C} \setminus \{0\} : \text{Re}\left(\tfrac{1}{z}\right) = \alpha\right\} = \left\{z \in \mathbb{C} \setminus \{0\} : \left|z - \tfrac{1}{2\alpha}\right| = \tfrac{1}{2\alpha}\right\}. \tag{5.5}$$

Fix a point $\lambda \in \rho_2$. Then there exists a unique number $\alpha \in (0, \frac{1}{p'})$ such that $\lambda \in \Gamma_\alpha$, namely $\alpha := \text{Re}(\frac{1}{\lambda})$. In the notation of (**??**) it is shown on p.72 of [**?**] that

$$|e_{nm}(\lambda)| \leq e_{nm}\left(\tfrac{1}{\alpha}\right), \quad n, m \in \mathbb{N}. \tag{5.6}$$

Note that $e_{nm}(\frac{1}{\alpha}) \geq 0$ for all $n, m \in \mathbb{N}$ follows from (**??**) as $0 < \alpha < \frac{1}{p'}$ implies that $1 - \frac{1}{k(1/\alpha)} = (1 - \frac{\alpha}{k}) > 0$ for $m \leq k \leq n$. Setting $\widetilde{A} := E_{1/\alpha}$ and $\widetilde{B} := E_\lambda$ it is clear from (**??**) that (**??**) is satisfied for the pair $\widetilde{A}, \widetilde{B}$ in place of A, B. Moreover, $\frac{1}{\alpha} > p'$ implies that $\frac{1}{\alpha} \in \rho(C_{(p)})$, that is, $(C_{(p)} - \frac{1}{\alpha} I)^{-1} \in \mathcal{L}(\text{ces}(p))$. Since $T_{D_{1/\alpha}} \in \mathcal{L}(\text{ces}(p))$ by Lemma **??** (with $\frac{1}{\alpha}$ in place of λ), the identity $T_{E_{1/\alpha}} = \alpha^2(T_{D_{1/\alpha}} - (C_{(p)} - \frac{1}{\alpha}I)^{-1})$ shows that $T_{\widetilde{A}} \in \mathcal{L}(\text{ces}(p))$. Hence, Corollary **??** can be applied to conclude that $T_{\widetilde{B}} = T_{E_\lambda} \in \mathcal{L}^r(\text{ces}(p))$. It then follows from (**??**) and Lemma 3.3 that $(C_{(p)} - \lambda I)^{-1} \in \mathcal{L}^r(\text{ces}(p))$. □

The remaining space to consider is $\text{ces}(0) := \{x \in \mathbb{C}^{\mathbb{N}} : C(|x|) \in c_0\}$ equipped with the norm

$$\|x\|_{\text{ces}(0)} := \|C(|x|)\|_{c_0} = \sup_{n\in\mathbb{N}} \tfrac{1}{n} \textstyle\sum_{k=1}^n |x_k|, \quad x \in \text{ces}(0).$$

It is a Banach lattice sequence space for the norm $\|\cdot\|_{\mathrm{ces}(0)}$ and the coordinatewise order. According to [?, Theorem 6.4], the restriction of C (see ((??)) to ces(0), denoted here by $C_{(0)}$, is continuous with $\|C_{(0)}\|_{\mathrm{op}} = 1$ and

$$\sigma(C_{(0)}) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \leq \tfrac{1}{2}\}. \tag{5.7}$$

Proposition 5.2. *The order spectrum of the positive operator $C_{(0)} \in \mathcal{L}(\mathrm{ces}(0))$ satisfies*

$$\sigma_{\mathrm{o}}(C_{(0)}) = \sigma(C_{(0)}).$$

Proof. As usual it suffices to show that $\rho(C_{(0)}) \subseteq \rho_{\mathrm{o}}(C_{(0)})$.

Let the set ρ_1 be as in (??). For each $\alpha \in (0,1)$ let Γ_α be given by (??). Then (??) ensures that we have the disjoint partition $\rho(C_{(0)}) = \rho_1 \cup \rho_2$ with $\rho_2 := \bigcup_{0<\alpha<1} \Gamma_\alpha$.

For any given point $\lambda \in \rho_1$ the estimates (??) are again valid (see [?, p.72]) and so the argument in the proof of Proposition ?? can be easily adapted (now for $X := \mathrm{ces}(0)$) to again show that $(C_{(0)} - \lambda I)^{-1} \in \mathcal{L}^r(\mathrm{ces}(0))$.

Fix now $\lambda \in \rho_2$. Then there exists a unique $\alpha \in (0,1)$ such that $\lambda \in \Gamma_\alpha$, namely $\alpha := \mathrm{Re}(\frac{1}{\lambda})$. Then $\mathrm{Re}(1 - \frac{1}{k\lambda}) = (1 - \frac{\alpha}{k}) \geq 0$ for $k \in \mathbb{N}$. Arguing as at the bottom of p.396 in [?], now with $x \in \mathrm{ces}(0)$ in place of $a \in \mathrm{ces}(2)$ there, it follows that the 1-st coordinate of $E_\lambda(x)$ is 0 and, for $n \geq 2$, that the n-th coordinate of $E_\lambda(x)$ satisfies

$$|(E_\lambda(x))_n| \leq (E_{1/\alpha}(|x|))_n, \quad x \in \mathrm{ces}(0).$$

Substituting $x := (\delta_{rj})_{j=1}^\infty$ into the previous estimates, for each $r \in \mathbb{N}$, yields (??). Since $0 < \alpha < 1$ implies that $\frac{1}{\alpha} \in \rho(C_{(0)})$, the argument can be completed along the lines given in the proof of Proposition ?? to conclude that $(C_{(0)} - \lambda I)^{-1} \in \mathcal{L}^r(\mathrm{ces}(0))$. We again warn the reader that $\mathbb{N} = \{0, 1, 2, \ldots\}$ is used in [?]. □

Acknowledgement. The research of the first author (J. Bonet) was partially supported by the projects MTM2016-76647-P and GV Prometeo 2017/102 (Spain).

References

1. A.A. Albanese, J. Bonet, W.J. Ricker, Spectrum and compactness of the Cesàro operator on weighted ℓ_p spaces, *J. Aust. Math. Soc.*, **99** (2015), 287–314.
2. A.A. Albanese, J. Bonet, W.J. Ricker, Mean ergodicity and spectrum of the Cesàro operator on weighted c_0 spaces, *Positivity*, **20** (2016), 761–803.
3. W. Arendt, On the o-spectrum of regular operators and the spectrum of measures, *Math. Z.*, **178** (1981), 271–287.
4. G. Bennett, Factorizing the classical inequalities, *Mem. Amer. Math. Soc.* **120** (Nr. 576) (1996), 1–130.
5. F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer, Heidelberg-New York, 1973.
6. G.P. Curbera, W.J. Ricker, Spectrum of the Cesàro operator in ℓ^p, *Arch. Math. (Basel)*, **100** (2013), 267–271.
7. G.P. Curbera, W.J. Ricker, Solid extensions of the Cesàro operator on the Hardy space $H^2(\mathbb{D})$, *J. Math. Anal. Appl.*, **407** (2013), 387–397.
8. G.P. Curbera, W.J. Ricker, Solid extensions of the Cesàro operator on ℓ^p and c_0, *Integral Equ. Oper. Theory*, **80** (2014), 61–77.
9. B. de Pagter, W.J. Ricker, Algebras of multiplication operators in Banach function spaces, *J. Oper. Theory*, **42** (1999), 245–267.
10. D.H. Fremlin, *Topological Riesz Spaces and Measure Theory*, Cambridge University Press, Cambridge, 1974.

11. G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1964.
12. G. Leibowitz, Spectra of discrete Cesàro operators, *Tamkang J. Math.*, **3** (1972), 123–132.
13. G. Leibowitz, Discrete Hausdorff transformations, *Proc. Amer. Math. Soc.*, **38** (1973), 541–544.
14. J.B. Reade, On the spectrum of the Cesàro operator, *Bull. London Math. Soc.*, **17** (1985), 263–267.
15. B.E. Rhoades, Spectra of some Hausdorff matrices, *Acta Sci. Math. (Szeged)*, **32** (1971), 91–100.
16. B.E. Rhoades, Generalized Hausdorff matrices bounded on ℓ^p and c, *Acta Sci. Math. (Szeged)*, **43** (1981), 333–345.
17. H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer, Berlin-Heidelberg-New York, 1974.
18. H.H. Schaefer, On the o-spectrum of order bounded operators, *Math. Z.*, **154** (1977), 79–84.
19. A.E. Taylor, *Introduction to Functional Analysis*, Wiley, New York, 1958.

J. Bonet, Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, 46071 Valencia, Spain
Email: jbonet@mat.upv.es

W.J. Ricker: Math.-Geogr. Fakultät, Kath. Universität Eichstätt-Ingolstadt, 85072 Eichstätt, Germany
Email: werner.ricker@ku.de