NUCLEAR WEIGHTED COMPOSITION OPERATORS ON WEIGHTED BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract. We characterize nuclear weighted composition operators $W_{\psi,\varphi}f = \psi(f \circ \varphi)$ on weighted Banach spaces of analytic functions with sup-norms. Consequences about nuclear composition operators on Bloch type spaces are presented. They extend previous work by Fares and Lefèvre on nuclear composition operators on Bloch spaces. Examples of nuclear as well as compact and non-nuclear weighted composition operators are given.

1. Introduction and preliminaries

1.1. **Introduction.** There is a vast literature about composition operators defined on Banach spaces of analytic functions on the disc, where the function theoretic properties of the symbol are related with the functional analytic properties of the composition operator. We refer to the monographs [7, 16]. Properties as compactness, weak compactness and nuclearity have been widely studied in this context; see for instance [3, 4, 5, 10, 14, 16, 17, 20] and the references therein. Our inspiration and main motivation come from very recent results of Fares and Lefèvre [10], who characterize the nuclear composition operators on the classical and little Bloch spaces $\mathcal{B}$ and $\mathcal{B}_0$. Our aim is to characterize the nuclearity of weighted composition operators defined on weighted Banach spaces H_v^∞ and H_v^0 of analytic functions, when v is a normal weight in the sense of Shields and Williams [18]; normal weights include all stantard weights $v_p(z) = (1 - |z|)^p$, $p > 0$. Since a composition operator on a Bloch space can be represented as a weighted composition operator on a weighted Banach space of analytic functions defined by a standard weight, we apply our results to extend the characterization of Fares and Lefèvre to all Bloch spaces $\mathcal{B}_p$ and $\mathcal{B}_p^0$ for $p \geq 1$.

Fares and Lefèvre also provide in [10] a non-trivial example of an analytic self map on the unit disc (which has a contact point with the unit circle) such that the corresponding composition operator is nuclear [10, Example 1, Theorem 3.3]. We give an example of a non-trivial composition operator on $H_{v_p}^\infty$ which is nuclear, based in examples given by Bonet, Domański, Lindström and Taskinen [4]. Moreover, we also exhibit a composition operator which is compact and non-nuclear on $\mathcal{B}_p$, for every $p > 1$. Such an example is based in Taskinen [20], where many examples of compact and non-nuclear composition operators on $H_{v_p}^\infty$ are presented,

2010 *Mathematics Subject Classification.* Primary 47B10, 47B33.

The research of the authors was partially supported by the project MTM2016-76647-P. The research of the first author was partially supported by the project GV Prometeo 2017/102.

Accepted in Proceedings of the American Mathematical Society, DOI: 10.1090/proc/15223.

and are close to the examples of compact and non-nuclear composition operators of Shapiro and Taylor [17] for the Hardy space.

Necessary notation and preliminaries are given below. In Section 2 we present the characterization of nuclear weighted composition operators in our main results Theorem 2.2 and Corollary 2.3. Some examples are also given in this section. In Section 3 we apply our results to the Bloch type spaces.

1.2. **Notation and Preliminaries.** We denote by $H(\mathbb{D})$ the space of analytic functions from the unit disc $\mathbb{D}$ to $\mathbb{C}$ endowed with the topology of unform convergence on the compact subsets of $\mathbb{D}$. A *weight* is a function $v : \mathbb{D} \to \mathbb{R}_+$ which is radial ($v(z) = v(|z|)$ for each $z \in \mathbb{D}$), positive, non-increasing and continuous. A weighted composition operator is defined by $W_{\psi,\varphi}(f) = \psi(f \circ \varphi)$, where $\psi, \varphi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The functions φ and ψ are called *symbol* and *multiplier* of the operator $W_{\psi,\varphi}$.

The weighted Banach spaces of analytic functions with sup-norms are defined as follows:

$$H_v^\infty := H_v^\infty(\mathbb{D}) := \{f \in H(\mathbb{D}) \,:\, \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}, \tag{1.1}$$

$$H_v^0 := H_v^0(\mathbb{D}) := \{f \in H(\mathbb{D}) \,:\, \lim_{|z|\to 1^-} v(z)|f(z)| = 0\}, \tag{1.2}$$

endowed with the norm $\|\cdot\|_v$. If $\limsup_{|z|\to 1} v(z) > 0$, then H_v^∞ is isomorphic to H^∞ and $H_v^0 = \{0\}$. The weight v is called *typical* when it satisfies $\lim_{|z|\to 1} v(z) = 0$.

Many results on weighted Banach spaces of analytic functions are formulated in terms of the so-called *associated weights* which are defined by

$$\widetilde{v}(z) := \big(\sup\big\{|f(z)| : f \in H_v^\infty,\ \|f\|_v \le 1\big\}\big)^{-1}.$$

The function $\widetilde{v}$ is also a weight and if we take $\widetilde{v}$ instead of v, both the spaces H_v^0 and H_v^∞ and the norm $\|\cdot\|_v$ do not change. If v is typical, then $\widetilde{v}$ is typical and

$$\widetilde{v}(z) := \big(\sup\big\{|f(z)| : f \in H_v^0,\ \|f\|_v \le 1\big\}\big)^{-1}.$$

A weight v is called *essential* if there is a constant $C > 0$ such that

$$v(z) \le \widetilde{v}(z) \le Cv(z), \text{ for each } z \in \mathbb{D}.$$

The weights $v_p(z) = (1 - |z|^2)^p,\ 0 < p < \infty$, are called *standard weights*. They clearly satisfy $\widetilde{v_p} = v_p$, hence they are essential weights. More details about associated weights and essential weights can be seen in [2].

The boundedness and compactness of composition operators were studied in [4]. Later, in [5], Contreras and Hernández-Díaz extended this study to weighted composition operators. They proved that if v, w are two typical weights, then the weighted composition operator $W_{\psi,\varphi} : H_v^\infty \to H_w^\infty$ is bounded if and only if

$$\|W_{\psi,\varphi}\| = \sup_{z\in\mathbb{D}} \frac{|\psi(z)|w(z)}{\widetilde{v}(\varphi(z))} < \infty. \tag{1.3}$$

Moreover, the operator $W_{\psi,\varphi} : H_v^0 \to H_w^0$ is bounded if and only if $\psi \in H_w^0$ and (1.3) holds. We also have that $(H_v^0)'' = H_v^\infty$ isometrically and the bi-transpose $(W_{\psi,\varphi})'' = W_{\psi,\varphi} : H_v^\infty \to H_w^\infty$. It was shown by Shields and Williams [19] that the topological isomorphisms $H_v^0 \simeq c_0$ and $H_v^\infty \simeq \ell_\infty$ hold when the weight v is normal. More general results were obtained by Lusky [11, 12, 13].

We use the duality of [18], but modifying the notation slightly. From now on, let $v, u : \mathbb{D} \to \mathbb{R}_+$ be two functions such that v is a typical weight as before and u is a radial, positive and continuous function which satisfies

$$\int_0^1 u(r)dr < \infty. \tag{1.4}$$

We also consider

$$A_u^1 := A_u^1(\mathbb{D}) := \{f \in H(\mathbb{D}) \,:\, |f|_u := \int_{\mathbb{D}} |f(z)|u(z)dA(z) < \infty\}, \tag{1.5}$$

where dA denotes the normalized Lebesgue measure on $\mathbb{D}$. The following definition can be found in [18, p. 291]:

Definition 1.1. We say that the weight v is *normal* if there are $k > \varepsilon > 0$ such that

$$\lim_{r\to 1} \frac{v(r)}{(1-r)^\varepsilon} = \lim_{r\to 1} \frac{(1-r)^k}{v(r)} = 0. \tag{1.6}$$

Every normal weight is essential, as follows for instance from Proposition 2 in [9].

The pair of functions $\{v, u\}$ is called *normal* if v is a normal weight, u is a function as in (1.4) and if, for some k satisfying (1.6), there is $\alpha > k - 1$ with

$$v(r)u(r) = (1 - r^2)^\alpha, \qquad 0 \le r < 1. \tag{1.7}$$

Now, we consider some k, ε, α satisfying (1.6) and (1.7). We use the following pairing between H_v^∞ and A_u^1: given $f \in H_v^\infty, g \in A_u^1$, we define

$$\begin{aligned}(f, g) &:= \int_{\mathbb{D}} f(z)g(\overline{z})v(z)u(z)dA(z) = \int_{\mathbb{D}} f(\overline{z})g(z)v(z)u(z)dA(z) \\ &= \int_{\mathbb{D}} f(z)g(\overline{z})(1-|z|^2)^\alpha dA(z).\end{aligned} \tag{1.8}$$

Following [18], we denote, for $\alpha > -1$, the kernel function or reproducing kernel for the point ξ by

$$K_\xi(z) := \frac{1+\alpha}{(1-\xi z)^{2+\alpha}}, \qquad z, \xi \in \mathbb{D}. \tag{1.9}$$

The following lemma is relevant in our proofs below.

Lemma 1.2 (Lemma 10 of [18]). *If K_ξ is the function given in* (1.9) *and $\{v, u\}$ is a normal pair, then we have*

(i) $K_\xi \in H_v^0 \cap A_u^1$;
(ii) $g(\xi) = (K_\xi, g)$, *for all* $g \in A_u^1$;
(iii) $f(\xi) = (f, K_\xi)$, *for all* $f \in H_v^\infty$.

We recall from [18], Theorem 2, p. 296 that the paring (1.8) defines a duality such that $(H_v^0)' = A_u^1$. More precisely, for all $g \in A_u^1$ the functional

$$\lambda_g(f) := (f, g), \quad f \in H_v^0, \tag{1.10}$$

satisfies $\lambda_g \in (H_v^0)'$ and $\|\lambda_g\| \le |g|_u$ (the norm in (1.5)). Moreover, for each $\lambda \in (H_v^0)'$ there exists a unique $g \in A_u^1$ with $\lambda = \lambda_g$ and $|g|_u \le C\|\lambda\|$ for some constant $C > 0$ independent of λ.

Definition 1.3. Let X, Y be two Banach spaces. A linear operator $T : X \to Y$ is said to be nuclear if there are two sequences $(x'_n) \subset X'$ and $(y_n) \subset Y$ such that $\sum_n \|x'_n\|\|y_n\| < \infty$ and

$$T = \sum_{n=1}^{\infty} x'_n \otimes y_n,$$

where $x'_n \otimes y_n : X \to Y$ is defined to be the mapping $x \mapsto x'_n(x)y_n$.

Hence, a nuclear operator is the (absolutely convergent) sum of rank one operators. We also need the notion of absolutely summing operator.

Definition 1.4. Let X, Y be two Banach spaces. A linear operator $T : X \to Y$ is said to be absolutely summing if there is $C > 0$ such that for all finite sequences $(x_j)_{j=1}^n \subset X$ we have

$$\sum_{j=1}^{n} \|Tx_j\| \leq C \sup_{\|x'\|\leq 1} \sum_{j=1}^{n} |x'(x_j)| = C \sup_{|\varepsilon_j|=1} \Big\| \sum_{j=1}^{n} \varepsilon_j x_j \Big\|.$$

It is easy to see that any nuclear operator is compact and absolutely summing. By Lemma 2.1 in [10], a linear operator $T : X \to Y$, with $X = c_0$ or $X = \ell_\infty$, is absolutely summing if and only if it is nuclear. We refer the reader to [8] and [15] for more information about nuclear and absolutely summing operators.

2. Nuclear weighted composition operators

In this section we characterize when the weighted composition operator $W_{\psi,\varphi}$ is nuclear on the spaces H_v^∞ and H_v^0 when v is a normal weight.

Before proving our characterization we need the following result, which is of special interest in this setting. Only here we specify with superindices where the operator is acting for the convenience of the reader.

Proposition 2.1. *Let $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If v, w are typical weights and the weighted composition operator $W_{\psi,\varphi}^{0,\infty} : H_v^0 \to H_w^\infty$ is continuous, we have*

(1) *The operator $W_{\psi,\varphi}^{\infty,\infty} : H_v^\infty \to H_w^\infty$ is continuous. Moreover, in this case, $W_{\psi,\varphi}^{0,\infty}$ is compact if and only if its bi-transpose satisfies $(W_{\psi,\varphi}^{0,\infty})'' = W_{\psi,\varphi}^{\infty,\infty}$.*
(2) *If $\psi \in H_w^0$, then the operator $W_{\psi,\varphi}^{0,0} : H_v^0 \to H_w^0$ is continuous and its bi-transpose satisfies $(W_{\psi,\varphi}^{0,0})'' = W_{\psi,\varphi}^{\infty,\infty}$. Moreover, the operator $W_{\psi,\varphi}^{0,0}$ is compact if and only if $W_{\psi,\varphi}^{\infty,\infty}(H_v^\infty) \subseteq H_w^0$.*

Proof. (1) Let $f \in H_v^\infty$. Then there is a sequence (f_j) of polynomials such that $f_j \to f$ in the compact-open topology and $\|f_j\|_v \leq \|f\|_v$ (see for instance [1]). Since $W_{\psi,\varphi} : H_v^0 \to H_w^\infty$ is continuous, $(W_{\psi,\varphi}(f_j))$ is bounded in H_w^∞ and, hence it is also bounded in $H(\mathbb{D})$. So, there is a subsequence of $(W_{\psi,\varphi}(f_j))$, which we denote the same, that converges to $h \in H(\mathbb{D})$ in the compact-open topology (that we denote *co*). Moreover, $h \in H_w^\infty$ since $w(z)|f_j(\varphi(z))|\psi(z) \leq M$ for some constant $M > 0$ and every $z \in \mathbb{D}$ and $j \in \mathbb{N}$. But, as $W_{\psi,\varphi} : (H(\mathbb{D}), co) \to (H(\mathbb{D}), co)$ is also continuous, $W_{\psi,\varphi}(f_j) \to W_{\psi,\varphi}(f)$ in $(H(\mathbb{D}), co)$ and hence, also pointwise. This implies that $W_{\psi,\varphi}(f) = h \in H_w^\infty$. We have shown that $W_{\psi,\varphi}(H_v^\infty) \subseteq H_w^\infty$. An application of the closed graph theorem gives the continuity of $W_{\psi,\varphi}^{\infty,\infty}$.

Let us assume now that $W_{\psi,\varphi}^{0,\infty}$ is compact. An application of Gantmacher-Nakamura's theorem, see for instance [6, Theorem 5.5 (p. 185)], gives that the bi-transpose of $W^{0,\infty}_{\psi,\varphi}:\ H^0_v \to H^\infty_w$ acts $(W^{0,\infty}_{\psi,\varphi})'':\ (H^\infty_v, w^*) \to (H^\infty_w, w)$ continuously. On the other hand, the operator $W^{\infty,\infty}_{\psi,\varphi} : (H^\infty_v, \tau_p) \to (H^\infty_w, \tau_p)$ is continuous if we endow the corresponding spaces with the pointwise topology τ_p, which is weaker than the w^* topology in H^∞_v and H^∞_w. Altogether gives that $(W^{0,\infty}_{\psi,\varphi})''$ and $W^{\infty,\infty}_{\psi,\varphi}$ are both $w^*-\tau_p$ continuous and both of them agree in H^0_v, which is w^*-dense in H^∞_v. Hence we have $W^{\infty,\infty}_{\psi,\varphi} = (W^{0,\infty}_{\psi,\varphi})''$. Conversely, if $W^{\infty,\infty}_{\psi,\varphi} = (W^{0,\infty}_{\psi,\varphi})''$, we have $(W^{0,\infty}_{\psi,\varphi})''(H^\infty_v) \subseteq H^\infty_w$. Since weakly compact weighted composition operators on H^∞_v are always compact by [3, Theorem 1], [5, Theorem 5.2], by Gantmacher-Nakamura's theorem again we get that $W^{0,\infty}_{\psi,\varphi}$ is compact.

The first part in (2) is completely analogous to Propositions 3.1 and 3.2 of [5] (see also [3]). The equivalence of the compactness of $W^{0,0}_{\psi,\varphi}$ to the fact that $W^{\infty,\infty}_{\psi,\varphi}(H^\infty_v) \subseteq H^0_w$ follows also from Gantmacher-Nakamura's theorem. □

Theorem 2.2. *Let $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If w is a typical weight and v is a normal weight such that $\{v, u\}$ is a normal pair associated to $\alpha > -1$ as in* (1.6)*, then the following conditions are equivalent:*

(a) *The weighted composition operator $W_{\psi,\varphi} : H^0_v \to H^\infty_w$ is nuclear;*
(b) *The weighted composition operator $W_{\psi,\varphi} : H^\infty_v \to H^\infty_w$ is nuclear;*
(c) $Q_\alpha := \displaystyle\int_{\mathbb{D}} \Big(\sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)|}{|1-\overline{\xi}\varphi(z)|^{\alpha+2}} \Big) \frac{(1-|\xi|^2)^\alpha}{v(\xi)} dA(\xi) < \infty.$

If in addition $\psi \in H^0_w$, then the above statements are equivalent to each one of the following:

(d) *The weighted composition operator $W_{\psi,\varphi} : H^0_v \to H^0_w$ is nuclear;*
(e) *The weighted composition operator $W_{\psi,\varphi} : H^\infty_v \to H^0_w$ is nuclear.*

Proof. First, we observe that, since $H^0_v \simeq c_0$ by [19], we can apply [10, Lemma 2.1] to conclude that (a) is equivalent to the weighted composition operator $W_{\psi,\varphi}$: $H^0_v \to H^\infty_w$ being absolutely summing.

First we prove that (c) holds if and only if $W_{\psi,\varphi} : H^0_v \to H^\infty_w$ is absolutely summing. Let us assume that (c) holds. Since the polynomials are dense in H^0_v, it is enough to consider polynomials $f_1, \dots, f_N$ in the definition of absolutely summing, as it is done in [10], p. 5. For each $\eta > 1$ we can select $z_1, \dots, z_N \in \mathbb{D}$ such that, by Lemma 1.2 and (1.8), we have

$$\begin{aligned}
\sum_{j=1}^N \big\|W_{\psi,\varphi}(f_j)\big\|_w &= \sum_{j=1}^N \sup_{z\in\mathbb{D}} w(z)\big|\psi(z) f_j(\varphi(z))\big| \le \eta \sum_{j=1}^N |\psi(z_j) f_j(\varphi(z_j))| w(z_j) \\
&= \eta \sum_{j=1}^N |\psi(z_j)| \big|\big(f_j, K_{\varphi(z_j)}\big)\big| w(z_j) \\
&= \eta \sum_{j=1}^N |\psi(z_j)| \Big| \int_{\mathbb{D}} f_j(\xi) \frac{(1+\alpha)(1-|\xi|^2)^\alpha}{|1-\overline{\xi}\varphi(z_j)|^{\alpha+2}} dA(\xi) \Big| w(z_j) \\
&\le \eta(1+\alpha) \Big(\sup_{\zeta\in\mathbb{D}} \sum_{j=1}^N |f_j(\zeta)| v(\zeta) \Big) \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{|\psi(z)| w(z)}{|1-\overline{\xi}\varphi(z)|^{\alpha+2}} \frac{(1-|\xi|^2)^\alpha}{v(\xi)} dA(\xi).
\end{aligned}$$

Since the last integral is Q_α and

$$\sum_{j=1}^{N} |f_j(\zeta)|v(\zeta) = \sup_{|\varepsilon_j|=1} \Big| \sum_{j=1}^{N} \varepsilon_j f_j(\zeta) \Big| v(\zeta),$$

we get

$$\begin{aligned} \sum_{j=1}^{N} \big\| W_{\psi,\varphi}(f_j) \big\|_w &\leq \eta(1+\alpha) Q_\alpha \sup_{|\varepsilon_j|=1} \Big(\sup_{\zeta \in \mathbb{D}} \Big| \sum_{j=1}^{N} \varepsilon_j f_j(\zeta) \Big| v(\zeta) \Big) \\ &= \eta(1+\alpha) Q_\alpha \sup_{|\varepsilon_j|=1} \Big\| \sum_{j=1}^{N} \varepsilon_j f_j \Big\|_v . \end{aligned}$$

But $\eta > 1$ is arbitrary, and hence we obtain that $W_{\psi,\varphi}$ is absolutely summing.

Now, we assume that $W_{\psi,\varphi} : H_v^0 \to H_w^\infty$ is absolutely summing. By Pietsch's theorem [8, 2.12], there is a Borel probability measure ν on the $\sigma(A_u^1, H_v^0)$-compact unit ball U of A_u^1 such that

$$\big\| W_{\psi,\varphi}(f) \big\|_w \leq M \int_U |\zeta(f)| d\nu(\zeta),$$

for each $f \in H_v^0$ and some constant $M > 0$ independent of f. For $\xi \in \mathbb{D}$ arbitrary, we get for $K_\xi \in H_v^0$,

$$\big\| W_{\psi,\varphi}(K_\xi) \big\|_w = \big\| \psi(K_\xi \circ \varphi) \big\|_w \leq M \int_U |\zeta(K_\xi)| d\nu(\zeta).$$

Integrating over $\mathbb{D}$ and applying Fubini's theorem, since ν is a probability measure, we have (using (1.7))

$$\begin{aligned} \int_{\mathbb{D}} \sup_{z \in \mathbb{D}} \frac{(1+\alpha) w(z) |\psi(z)|}{|1 - \xi\varphi(z)|^{\alpha+2}} \frac{(1-|\xi|^2)^\alpha}{v(\xi)} dA(\xi) &= \int_{\mathbb{D}} \big\| \psi(K_\xi \circ \varphi) \big\|_w u(\xi) dA(\xi) \\ \leq M \int_U \int_{\mathbb{D}} |\zeta(K_\xi)| u(\xi) dA(\xi) d\nu(\zeta) &\leq M \sup_{\zeta \in U} \int_{\mathbb{D}} |\zeta(K_\xi)| u(\xi) dA(\xi). \end{aligned}$$

Now, since $K_\xi \in H_v^0$ and $\zeta \in A_u^1$ we have, by Lemma 1.2, $\zeta(K_\xi) = \big(K_\xi, \zeta\big) = \zeta(\xi)$. Thus,

$$\begin{aligned} &\int_{\mathbb{D}} \sup_{z \in \mathbb{D}} \frac{w(z)|\psi(z)|}{|1 - \overline{\xi}\varphi(z)|^{\alpha+2}} \frac{(1-|\xi|^2)^\alpha}{v(\xi)} dA(\xi) \\ &\quad \leq \frac{M}{\alpha+1} \sup_{\zeta \in U} \int_{\mathbb{D}} |\zeta(\xi)| u(\xi) dA(\xi) = \frac{M}{\alpha+1} \sup_{\zeta \in U} |\zeta|_u = \frac{M}{\alpha+1} < \infty. \end{aligned}$$

This shows that (a) is equivalent to (c).

The implication (a)⇒(b) follows from Proposition 2.1 (1) and the fact that the transpose of a nuclear operator is nuclear [15, 3.1.8]. The implication (b)⇒(a) is obvious because H_v^0 is a topological subspace of H_v^∞ and nuclear operators are an ideal.

Assume now that $\psi \in H_w^0$. The implication (a)⇒(d) follows because $W_{\psi,\varphi}(H_v^0) \subseteq H_w^0$ [5] and in this case $W_{\psi,\varphi} : H_v^0 \to H_w^\infty$ is absolutely summing if and only if $W_{\psi,\varphi} : H_v^0 \to H_w^0$ is absolutely summing. The implication (d)⇒(e) is a consequence of Proposition 2.3 (2), using again that the transpose of a nuclear mapping is nuclear, and (e)⇒(b) follows by an analogous argument to the one of the implication (b)⇒(a), which finishes the proof. □

The case of standard weights follows as a consequence.

Corollary 2.3. *Let $\varphi, \psi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If $v_p(z) = (1-|z|^2)^p$, $p > 0$, is a standard weight, then the following conditions are equivalent:*

(a) *The weighted composition operator $W_{\psi,\varphi} : H^0_{v_p} \to H^\infty_{v_p}$ is nuclear;*
(b) *The weighted composition operator $W_{\psi,\varphi} : H^\infty_{v_p} \to H^\infty_{v_p}$ is nuclear;*
(c) $Q_p := \displaystyle\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^p|\psi(z)|}{|1-\bar{\xi}\varphi(z)|^{p+2}} dA(\xi) < \infty.$

If in addition $\psi \in H^0_{v_p}$, then the above statements are equivalent to each one of the following:

(d) *The weighted composition operator $W_{\psi,\varphi} : H^0_{v_p} \to H^0_{v_p}$ is nuclear;*
(e) *The weighted composition operator $W_{\psi,\varphi} : H^\infty_{v_p} \to H^0_{v_p}$ is nuclear.*

Proof. It is obvious that $\{v_p, 1\}$ is a normal pair for each $p > 0$, and one can take $\alpha = p$ in (1.7). Hence, the result follows from Theorem 2.2. □

If the multiplier ψ is constant and equal to 1 and the symbol satisfies that $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$, then the composition operator $C_\varphi = W_{1,\varphi}$ is trivially nuclear on the space $\mathcal{H}^0_{v_p}$. We give a non-trivial example for composition operators and standard weights.

Example 2.4. We consider, for $0 < \gamma < 1$, the self-map $\varphi \in H(\mathbb{D})$ of [4, Example 4.2]. We observe that $\overline{\varphi(\mathbb{D})} \cap \partial\mathbb{D} = \{1\}$.

We show that the composition operator $C_\varphi = W_{1,\varphi}$ is nuclear on the space $\mathcal{H}^0_{v_p}$ for $0 < \gamma \le \frac{p}{p+2}$, where $v_p(z) = (1-|z|^2)^p$ is the standard weight. It is enough to show that the integral Q_p in (c) of Corollary 2.3 is finite. Indeed, from [4, Example 4.2] we know that there is some $c > 0$ such that

$$1 - |\varphi(z)| \ge c|1-z|^\gamma, \quad z \in \mathbb{D}.$$

Therefore, if $\gamma(p+2) \le p$ we have

$$\begin{aligned} Q_p &= \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^p}{|1-\bar{\xi}\varphi(z)|^{p+2}} dA(\xi) \le \sup_{z\in\mathbb{D}} \frac{(1-|z|)^p}{(1-|\varphi(z)|)^{p+2}} \\ &\le \frac{1}{c} \sup_{z\in\mathbb{D}} \frac{(1-|z|)^p}{(1-|z|)^{\gamma(p+2)}} \le \frac{1}{c} \sup_{z\in\mathbb{D}} (1-|z|)^{p-\gamma(p+2)} < \infty. \end{aligned}$$

Taskinen proved in [20, Proposition 4.1] that whenever $0 \le p \le q$ there exists an analytic self map $\varphi : \mathbb{D} \to \mathbb{D}$ such that the composition operator $C_\varphi : H^\infty_{v_p} \to H^\infty_{v_q}$ is compact but not nuclear. For weighted composition operators, we provide below a condition on the multiplier ψ under which C_φ is nuclear whenever the composition operator $W_{\psi,\varphi}$ is nuclear.

Proposition 2.5. *Let $v_p(z) = (1-|z|^2)^p$ be the standard weight for $p > 0$. Assume that $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic self map which extends continuously to some $z_0 \in \partial\mathbb{D}$ with $\varphi(z_0) = z_0$ and that $\psi \in H^\infty_{v_p}$. If there is $r > 0$ such that*

(1) *ψ is bounded away from 0 in $\mathbb{D} \cap B(z_0, r)$ and*
(2) *$\overline{\varphi(\mathbb{D} \setminus B(z_0, r))} \subset \mathbb{D}$,*

then the composition operator $C_\varphi : H^\infty_{v_p} \to H^\infty_{v_p}$ is nuclear whenever $W_{\psi,\varphi} : H^\infty_{v_p} \to H^\infty_{v_p}$ is nuclear.

Proof. We can assume that $z_0 = 1$ without loss of generality. Now, let

$$G_\omega(z) = \frac{(1-|z|^2)^p}{|1-\overline{\omega}\varphi(z)|^{p+2}},$$

and put $F(\omega) := \sup_{z\in\mathbb{D}} G_\omega(z)|\psi(z)|$, which belongs to $L^1(\mathbb{D})$. We want to show that also $G(\omega) := \sup_{z\in\mathbb{D}} G_\omega(z)$ belongs to $L^1(\mathbb{D})$. We observe that

$$0 \le G(\omega) = \sup_{z\in\mathbb{D}} G_\omega(z) \le \sup_{z\in\mathbb{D}\cap B(1,r)} G_\omega(z) + \sup_{z\in\mathbb{D}\setminus B(1,r)} G_\omega(z).$$

By hypothesis (1), there is $c_1 > 0$ such that $|\psi(z)| > c_1$ for each $z \in B(1,r)\cap\mathbb{D}$. Hence

$$\sup_{z\in\mathbb{D}\cap B(1,r)} G_\omega(z) \le \sup_{z\in\mathbb{D}\cap B(1,r)} \frac{(1-|z|^2)^p}{|1-\overline{\omega}\varphi(z)|^{p+2}}\frac{|\psi(z)|}{c_1}$$

$$\le \frac{1}{c_1}\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^p}{|1-\overline{\omega}\varphi(z)|^{p+2}}|\psi(z)| = \frac{1}{c_1}F(\omega). \tag{2.1}$$

By hypothesis (2), there is $0 < c_2 < 1$ such that $|\varphi(z)| < c_2$ when $z \in \mathbb{D}\setminus B(1,r)$. Then

$$\sup_{z\in\mathbb{D}\setminus B(1,r)} G_\omega(z) \le \sup_{z\in\mathbb{D}\setminus B(1,r)} \frac{(1-|z|^2)^p}{|1-\overline{\omega}\varphi(z)|^{p+2}} \le \frac{1}{(1-c_2)^{p+2}}. \tag{2.2}$$

From (2.1) and (2.2) we conclude that $G \in L^1(\mathbb{D})$. □

Let v_p be the standard weight, as above, for $p > 0$. Taskinen [20, Proposition 4.1] constructs an analytic self map $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\overline{\varphi(\mathbb{D})} \cap \partial\mathbb{D} = \{1\}$ and the corresponding composition operator $C_\varphi : H^\infty_{v_p} \to H^\infty_{v_q}$ is compact and not nuclear for every $p \ge q > 0$. With the same construction we give below an example of a weighted composition operator $W_{\psi,\varphi} : H^\infty_{v_p} \to H^\infty_{v_p}$ with $\psi = \varphi'$ which is compact and not nuclear. We observe that the symbol φ depends on $p > 0$.

Example 2.6. Let $\Delta := \mathbb{D} \cap \{z : \ Re(z) > 0\}$. The Riemann mapping

$$\tau(z) = \frac{2z^2+3z-2}{2z^2-3z-2}$$

maps Δ onto $\mathbb{D}$. We define the analytic function $g_p(z) = az(-\log(bz)+1)^{\frac{1}{3p}}$ on Δ, where $a, b > 0$ are taken in such a way that $g_p(\Delta) \subseteq \frac{1}{2}\Delta$. Let $\varphi_p := \tau \circ g_p \circ \tau^{-1}$. We observe:

(i) The mapping φ_p admits a continuous extension to 1 with $\varphi_p(1) = 1$ and $|\varphi_p(z)| \to 1$ if and only if $z \to 1$ [20, (39), p. 214]. As a consequence, $\overline{\varphi_p(\mathbb{D}\setminus B(1,r))} \subset \mathbb{D}$ for every $0 < r < 1$.

(ii) The function τ^{-1} is a conformal mapping from $\mathbb{D}$ to Δ. Since $\tau(0) = 1$ and $\tau'(0) \neq 0$, we get that τ extends conformally to a neighbourhood V of 0 and τ^{-1} extends conformally to a neighbourhood U of 1, and both τ' and $(\tau^{-1})'$ are bounded away from 0 in these neighbourhoods.

(iii) The mapping g_p extends continuously to 0 with $g_p(0) = 0$. The neighbourhood U of 1 in (ii) can be taken in such a way that $\tau^{-1}(U\cap\mathbb{D})$ is included in $B(0,t)\cap\Delta$ for t small enough. It is easy to check that $g_p'(z)$ is bounded away from 0 in $B(0,t)\cap\Delta$ for t small enough. In fact, $\lim_{z\to 0}|g_p'(z)| = \infty$. We can choose t even smaller to have also that $g_p(B(0,t)\cap\Delta)$ is included in $V\cap\mathbb{D}$, for V the neighbourhood of 0 taken in (ii).

(iv) The chain rule yields

$$\varphi_p'(z) = \tau'(g_p(\tau^{-1}(z)))g_p'(\tau^{-1}(z))(\tau^{-1})'(z)$$

is bounded away from 0 on $B(1,r)\cap\mathbb{D}$, for any $r>0$ such that $B(1,r)\subseteq U$.

For $p>1$, we show that $W_{\varphi_p',\varphi_p}: H_{v_p}^\infty \to H_{v_p}^\infty$ is compact using that the composition operator $C_{\varphi_p}: H_{v_p}^\infty \to H_{v_p}^\infty$ is compact. Indeed, by [4, Corollary 3.4], the compactness of $C_{\varphi_p}: H_{v_p}^\infty \to H_{v_p}^\infty$ is equivalent to

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^p}{(1-|\varphi_p(z)|^2)^p} = 0 \Longleftrightarrow \lim_{|z|\to 1}\frac{(1-|z|^2)^q}{(1-|\varphi_p(z)|^2)^q} = 0, \tag{2.3}$$

for any $q>0$. Hence, if C_{φ_p} is compact, by Schwarz-Pick lemma and (2.3) we have

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^p}{(1-|\varphi_p(z)|^2)^p}|\varphi_p'(z)| = \lim_{|z|\to 1}\frac{(1-|z|^2)|\varphi_p'(z)|}{(1-|\varphi_p(z)|^2)}\frac{(1-|z|^2)^{p-1}}{(1-|\varphi(z)|^2)^{p-1}} = 0.$$

Again Schwarz-Pick lemma implies $\varphi_p' \in H_{v_1}^\infty \hookrightarrow H_{v_p}^0$. Hence, we deduce that both of the operators $W_{\varphi_p',\varphi_p}: H_{v_p}^\infty \to H_{v_p}^\infty$ and $W_{\varphi_p',\varphi_p}: H_{v_p}^0 \to H_{v_p}^0$ are compact by [5, Corollary 4.5]. However, by [20, Proposition 4.1] and Proposition 2.5, neither $W_{\varphi_p',\varphi_p}: H_{v_p}^\infty \to H_{v_p}^\infty$ nor $W_{\varphi_p',\varphi_p}: H_{v_p}^0 \to H_{v_p}^0$ are nuclear operators.

3. Applications to composition operators on Bloch type spaces

Now we apply our results above to composition operators on Bloch type spaces. We begin with some definitions. For each $0<p<\infty$, the *Bloch space of order* p is

$$\mathcal{B}_p = \{f\in H(\mathbb{D}):\ \sup_{z\in\mathbb{D}}(1-|z|^2)^p|f'(z)|<\infty\},$$

and the *little Bloch space of order* p is

$$\mathcal{B}_p^0 = \{f\in H(\mathbb{D}):\lim_{|z|\to 1}(1-|z|^2)^p|f'(z)| = 0\}.$$

It is a well-known fact that $\mathcal{B}_p$ is a Banach space when it is endowed with the norm

$$\|f\|_p = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^p|f'(z)|,$$

and that $\mathcal{B}_p^0$ is a closed subspace of $\mathcal{B}_p$ for each $p>0$. For $p=1$, the spaces $\mathcal{B}_1$ and $\mathcal{B}_1^0$ are called the *classical Bloch space* and the *little Bloch space*, and they are denoted by $\mathcal{B}$ and $\mathcal{B}_0$.

All these spaces are continuously embedded in $H(\mathbb{D})$. It is a consequence of Schwarz-Pick lemma that an analytic self map $\varphi\in H(\mathbb{D})$ induces a bounded composition operator $C_\varphi:\mathcal{B}_p\to\mathcal{B}_p$ for each $p\geq 1$. Moreover, $C_\varphi:\mathcal{B}_p^0\to\mathcal{B}_p^0$ is bounded if and only if $\varphi\in\mathcal{B}_p^0$ for each $p\geq 1$, see [5] and [14].

The Bloch spaces are closely related to weighted Banach spaces of analytic functions. In fact, $f\in\mathcal{B}_p$ if and only if $f'\in H_{v_p}^\infty(\mathbb{D})$ for $v_p(z) = (1-|z|^2)^p$ and $p>0$. We have the isometric identification of $\mathcal{B}_p$ with the ℓ_1 sum $H_{v_p}^\infty\oplus_{\ell_1}\mathbb{C}$, via $g\to(g',g(0))$. With this identification, the composition operator C_φ on $\mathcal{B}_p$ satisfies $C_\varphi = W_{\varphi',\varphi}\oplus Id$, where Id is the identity, and hence it is nuclear if and only if $W_{\varphi',\varphi}$ is nuclear on $H_{v_p}^\infty$. An application of Theorem 2.2 with multiplier $\psi(z)=\varphi'(z)$ gives a characterization for the nuclear composition operators on Bloch type spaces. For the case $p=1$, the next result extends Theorem 2.2 and 2.3 of Fares and Lefèvre [10].

Theorem 3.1. *Let $\varphi \in H(\mathbb{D})$ be an analytic self map and consider the following conditions:*

(a) *The composition operator $C_\varphi : \mathcal{B}_p^0 \to \mathcal{B}_p$ is nuclear (or absolutely summing);*
(b) *The composition operator $C_\varphi : \mathcal{B}_p \to \mathcal{B}_p$ is nuclear;*
(c) $Q_p = \displaystyle\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^p|\varphi'(z)|}{|1-\overline{\omega}\varphi(z)|^{p+2}} dA(\omega) < \infty$;
(d) *The composition operator $C_\varphi : \mathcal{B}_p \to \mathcal{B}_p^0$ is nuclear;*
(e) *The composition operator $C_\varphi : \mathcal{B}_p^0 \to \mathcal{B}_p^0$ is nuclear.*

All conditions are equivalent for any $p > 1$. For $p = 1$, (a),(b) and (c) are equivalent and, if in addition $\varphi \in \mathcal{B}_0$, then all conditions are equivalent.

Example 2.6 and the representation $C_\varphi = W_{\varphi',\varphi} \oplus Id$ of any composition operator defined on $\mathcal{B}_p = H^\infty_{v_p} \oplus_{\ell_1} \mathbb{C}$ gives the following concluding example.

Example 3.2. Let $p > 1$ and let φ_p be the symbol considered in Example 2.6. The composition operator $C_{\varphi_p} : \mathcal{B}_p \to \mathcal{B}_p$ is compact, not nuclear, and it satisfies $C_\varphi(\mathcal{B}_p) \subseteq \mathcal{B}_p^0$ by Proposition 2.1.

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