Cesàro and Volterra integral operators on Korenblum-type spaces of analytic functions

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Joint work with A.A. Albanese and W.J. Ricker

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Aim of the lecture

AIM

Investigate the behaviour of the Cesàro operator C, as well as Volterra integral operator V_g, acting on certain Banach, Fréchet and (LB) spaces of analytic functions.

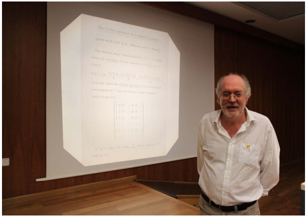
In the first part we report on joint work in progress with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany) about the Cesàro operator.

Ernesto Cesàro (1859-1906)

Vito Volterra (1860-1940)

Albanese and Ricker

Angela Albanese



Werner Ricker

The discrete Cesàro operator

The *Cesàro operator* C is defined for a sequence $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ of complex numbers by

$$\mathrm{C}(x) = \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

Proposition.

The operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself with

$$C^{-1}(y) = (ny_n - (n-1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^{\mathbb{N}}, \tag{1}$$

where we set $y_{-1} := 0$.

Recall that $\mathbb{C}^{\mathbb{N}}$ is a Fréchet space for the topology of coordinatewise convergence.

The Cesàro operator is defined for analytic functions on the disc $\mathbb{D}$ by

$$Cf = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_n \right) z^n, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}).$$

The Cesàro operator acts continuously and has the integral representation

$$Cf(z) = \frac{1}{z} \int_0^z \frac{f(\rho)}{1-\rho} \, d\rho, \quad f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

Indeed, for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, we have

$$
\begin{aligned}
Cf(z) &= \frac{1}{z}\int_0^z \frac{f(\rho)}{1-\rho}\, d\rho = \frac{1}{z}\int_0^z \left(\sum_{n=0}^{\infty} a_n \rho^n\right)\left(\sum_{m=0}^{\infty} \rho^m\right) \\
&= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{1}{z}\int_0^z \rho^{n+m}\, d\rho = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{z^{n+m}}{n+m+1} \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=n} \frac{z^k}{k+1} = \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\sum_{n=0}^{k} a_n\right) z^k.
\end{aligned}
$$

Theorem. Hardy. 1920.

Let $1 < p < \infty$. The Cesàro operator maps the Banach space ℓ^p continuously into itself, which we denote by $\mathrm{C}^{(p)} : \ell^p \to \ell^p$, and $\|(\mathrm{C}^{(p)})\| = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$, for all $n \in \mathbb{N}$.

In particular, **Hardy's inequality** holds:

$$\|(\mathrm{C}^{(p)})\|_p \leq p' \|x\|_p, \quad x \in \ell^p.$$

Clearly C is not continuous on ℓ_1, since $\mathrm{C}(e_1) = (1, 1/2, 1/3, ...)$.

Proposition.

The Cesàro operators $\mathsf{C}^{(\infty)} \colon \ell^\infty \to \ell^\infty$, $\mathsf{C}^{(c)} \colon c \to c$ and $\mathsf{C}^{(0)} \colon c_0 \to c_0$ are continuous, and $\|\mathsf{C}^{(\infty)}\| = \|\mathsf{C}^{(c)}\| = \|\mathsf{C}^{(0)}\| = 1$.

Moreover, $\lim \mathsf{C}x = \lim x$ for each $x \in c$.

Spectrum and point spectrum

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ (resp. $\mathcal{K}(X)$) is the space of all continuous (resp. compact) linear operators on X.

The **resolvent set** $\rho(T, X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The elements of $\sigma_{pt}(T, X)$ are called eigenvalues of T.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

(i) $\sigma(\mathsf{C};\ell^\infty) = \sigma(\mathsf{C};c_0) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C};\ell^\infty) = \{(1,1,1,...)\}$.

(iii) $\sigma_{pt}(\mathsf{C};c_0) = \emptyset$.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

Let $1 < p < \infty$ and $1/p + 1/p' = 1$.

(i) $\sigma(\mathsf{C};\ell^p) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C};\ell^p) = \emptyset$.

In particular, C is not compact in the spaces $\ell^p, 1 < p \leq \infty$, or in the space c_0.

More about the spectrum and point spectrum

Theorem

The Cesàro operator satisfies

$$(a) \quad \sigma(\mathsf{C}, H(\mathbb{D})) = \sigma_{pt}(\mathsf{C}, H(\mathbb{D})) = \{\frac{1}{m} \colon m \in \mathbb{N}\}.$$

Persson showed in 2008 the following facts:

For every $m \in \mathbb{N}$ the operator $(\mathsf{C} - \frac{1}{m} I) \colon H(\mathbb{D}) \to H(\mathbb{D})$ is not injective because $\mathrm{Ker}(\mathsf{C} - \frac{1}{m} I) = \mathrm{span}\{e_m\}$, where $e_m(z) = z^{m-1}(1-z)^{-m}$, $z \in \mathbb{D}$, and it is not surjective because the function $f_m(z) := z^{m-1}$, $z \in \mathbb{D}$, does not belong to the range of $(\mathsf{C} - \frac{1}{m} I)$.

The growth spaces

For $\gamma > 0$ the growth classes $A^{-\gamma}$ and $A_0^{-\gamma}$ are the Banach spaces defined by

$$A^{-\gamma} = \{f \in H(\mathbb{D}) \colon \|f\|_{-\gamma} := \sup_{z \in \mathbb{D}} (1-|z|)^{\gamma} |f(z)| < \infty\}.$$

$$A_0^{-\gamma} = \{f \in H(\mathbb{D}) \colon \lim_{|z| \to 1} (1-|z|)^{\gamma} |f(z)| = 0\}.$$

$A_0^{-\gamma}$ is the closure of the polynomials on $A^{-\gamma}$.

The Cesàro operator acts continuously on $A^{-\gamma}$. Its spectrum on these (and many other spaces of analytic functions on the disc) has been studied by Aleman and Persson 2008-2010.

The result of Aleman and Persson

Theorem. Aleman, Persson.

Let $\gamma > 0$. The Cesàro operator $\mathsf{C}_{\gamma,0} \colon A_0^{-\gamma} \to A_0^{-\gamma}$ has the following properties.

(i) $\sigma_{pt}(\mathsf{C}_{\gamma,0}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\}$.

(ii) $\sigma(\mathsf{C}_{\gamma,0}) = \sigma_{pt}(\mathsf{C}_{\gamma,0}) \cup \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2\gamma}\right| \leq \frac{1}{2\gamma}\right\}$.

(iii) If $\left|\lambda - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma}$ (equivalently $\mathrm{Re}\left(\frac{1}{\lambda}\right) > \gamma$), then $\mathrm{Im}(\lambda I - \mathsf{C}_{\gamma,0})$ is a closed subspace of $A_0^{-\gamma}$ and has codimension 1.

Moreover, the Cesàro operator $\mathsf{C}_\gamma : A^{-\gamma} \to A^{-\gamma}$ satisfies

(iv) $\sigma_{pt}(\mathsf{C}_\gamma) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m \leq \gamma\}$, and

(v) $\sigma(\mathsf{C}_\gamma) = \sigma(\mathsf{C}_{\gamma,0})$.

The norm

The continuity of C_γ and $C_{\gamma,0}$ as established by Aleman and Persson gives no quantitative estimate for their operator norm.

Theorem.

(i) Let $\gamma \geq 1$. Then $\|C_\gamma^n\| = \|C_{\gamma,0}^n\| = 1$ for all $n \in \mathbb{N}$.

(ii) Let $0 < \gamma < 1$. Then $\|C_\gamma^n\| = \|C_{\gamma,0}^n\| = 1/\gamma^n$ for all $n \in \mathbb{N}$.

Definition of the *-spectrum.

X **is a Hausdorff locally convex space (lcs).**

- $\rho^*(T)$ consists of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that each $\mu \in B(\lambda, \delta) := \{z \in \mathbb{C} \colon |z - \lambda| < \delta\}$ belongs to $\rho(T)$ and the set $\{R(\mu, T) \colon \ \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X)$.
- $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$.
- $\sigma^*(T)$ is a closed set containing $\sigma(T)$. If $T \in \mathcal{L}(X)$ with X a Banach space, then $\sigma(T) = \sigma^*(T)$. There exist continuous linear operators T on a Fréchet space X such that $\overline{\sigma(T)} \subset \sigma^*(T)$ properly.

More about the spectrum

Notation:

$\Sigma := \{\frac{1}{m} : m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$.

Proposition.

(i) $\sigma(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \sigma_{pt}(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \Sigma$.

(ii) Fix $m \in \mathbb{N}$. Let $x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^{\mathbb{N}}$ where $x_n^{(m)} := 0$ for $n \in \{1, \dots, m-1\}$, $x_m^{(m)} := 1$ and $x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!}$ for $n > m$. Then the eigenspace

$$\operatorname{Ker}\left(\frac{1}{m} I - \mathsf{C}\right) = \operatorname{span}\{x^{(m)}\} \subseteq \mathbb{C}^{\mathbb{N}}$$

is 1-dimensional.

(iii) $\sigma^*(\mathsf{C}, H(\mathbb{D})) = \Sigma \cup \{0\} = \sigma^*(\mathsf{C}; \mathbb{C}^{\mathbb{N}})$.

The Fréchet growth spaces

Let $\gamma \geq 0$.

$$A_+^{-\gamma} := \cap_{\mu>\gamma} A^{-\mu} = \cap_{\mu>\gamma} A_0^{-\mu}.$$

The space $A_+^{-\gamma}$ is Fréchet when it is endowed with the lc-topology generated by the fundamental sequence of seminorms

$$\|f\|_k := \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma+\frac{1}{k}} |f(z)|.$$

It is a Fréchet-Schwartz space because the inclusion $A^{-\mu_1} \hookrightarrow A^{-\mu_2}$ is compact for all $0 < \mu_1 < \mu_2$. In particular, every bounded subset of $A_+^{-\gamma}$ is relatively compact, i.e. the space is Montel

Let $0 < \gamma \le \infty$

$$A_{-}^{-\gamma} := \cup_{\mu<\gamma} A^{-\mu} = \cup_{\mu<\gamma} A_0^{-\mu},$$

and it is endowed with the finest locally convex topology such that all the inclusions $A^{-\mu} \hookrightarrow A^{-\gamma},\ \mu < \gamma$, are continuous.
In particular, $A_{-}^{-\gamma}$ is the (DFS)-space

$$A_{-}^{-\gamma} := \operatorname*{ind}_k A^{-(\gamma-\frac{1}{k})} = \operatorname*{ind}_k A_0^{-(\gamma-\frac{1}{k})}.$$

As a consequence $A_{-}^{-\gamma}$ is a Montel space, too.

The (LB) growth spaces. The Korenblum space

The **Korenblum space** $A_-^{-\infty}$ was introduced by Korenblum in 1975 is usually denoted by

$$A^{-\infty} = \cup_{0<\gamma<\infty} A^{-\gamma} = \cup_{n\in\mathbb{N}} A^{-n}.$$

All these spaces play an important role in the study of interpolation and sampling of holomorphic functions on the disc.

The Cesàro operators $\mathrm{C}\colon A_-^{-\gamma} \to A_-^{-\gamma}$ and $\mathrm{C}\colon A_+^{-\gamma} \to A_+^{-\gamma}$ are continuous because C acts continuously in every step.

Theorem

(1) Let $\gamma \in]0,\infty[$.

(a) $\sigma_{pt}(\mathsf{C}, A_+^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m \leq \gamma\}$.

(b) $\sigma(\mathsf{C}, A_+^{-\gamma}) = \{0\} \cup \{\frac{1}{m} \colon m,\ m \leq \gamma\} \cup \{\lambda \in \mathbb{C} \colon |\lambda - \frac{1}{2\gamma}| < \frac{1}{2\gamma}\}$.

(c) $\sigma^*(\mathsf{C}, A_+^{-\gamma}) = \overline{\sigma(\mathsf{C}, A_+^{-\gamma})}$.

(2) Let $\gamma = 0$.

(a) $\sigma_{pt}(\mathsf{C}, A_+^{-0}) = \emptyset$.

(b) $\sigma(\mathsf{C}, A_+^{-0}) = \{0\} \cup \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > 0\}$.

(c) $\sigma^*(\mathsf{C}, A_+^{-0}) = \overline{\sigma(\mathsf{C}, A_+^{-0})}$.

Theorem

(1) Let $\gamma \in]0, \infty[$.

(a) $\sigma_{pt}(C, A_-^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\}$.

(b) $\sigma(C, A_-^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\} \cup \{\lambda \in \mathbb{C} \colon |\lambda - \frac{1}{2\gamma}| \leq \frac{1}{2\gamma}\}$.

(c) $\sigma^*(C, A_-^{-\gamma}) = \sigma(C, A_-^{-\gamma})$.

(2) For the Korenblum space $A^{-\infty}$ (i.e. $\gamma = \infty$) we have:

(a) $\sigma(C, A^{-\infty}) = \sigma_{pt}(C, A^{-\infty}) = \{\frac{1}{m} \colon m \in \mathbb{N}\}$.

(b) $\sigma^*(C, A^{-\infty}) = \{\frac{1}{m} \colon m \in \mathbb{N}\} \cup \{0\}$.

Mean ergodic properties. Definitions

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Banach space, an operator T is power bounded if and only if $\sup_n ||T^n|| < \infty$.

If X is a barrelled space, an operator T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded. This is a consequence of the uniform boundedness principle.

Mean ergodic properties. Definitions

For $T \in \mathcal{L}(X)$**, we set** $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$.

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} T^m x, \quad x \in X, \tag{2}$$

exist in X.

If T is mean ergodic, then one then has the direct decomposition

$$X = \mathrm{Ker}(I - T) \oplus \overline{(I - T)(X)}.$$

Mean ergodic properties. Definitions

Uniformly mean ergodic operators

If $\{T_{[n]}\}_{n=1}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$ to $P \in \mathcal{L}(X)$, then T is called *uniformly mean ergodic*.

Theorem. Lin. 1974.

Let T a (continuous) operator on a Banach space X which satisfies $\lim_{n\to\infty} ||T^n/n|| = 0$. The following conditions are equivalent:

(1) T is uniformly mean ergodic.

(4) $(I - T)(X)$ is closed.

Proposition.

- The Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded and uniformly mean ergodic.
- The Cesàro operator $\mathsf{C}^{(p)}\colon \ell^p \to \ell^p$, $1 < p < \infty$, is not power bounded and not mean ergodic.
- The Cesàro operator $\mathsf{C}^{(0)}\colon c_0 \to c_0$ is power bounded, not mean ergodic.

Hypercyclicity. Definitions

Hypercyclic operator

$T \in \mathcal{L}(X)$, with X separable, is called **hypercyclic** if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X.

Supercyclic operator

If, for some $z \in X$, the projective orbit $\{\lambda T^n z \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*.

Clearly, hypercyclicity always implies supercyclicity.

Proposition.

- The Cesàro operator $C \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded, uniformly mean ergodic and not supercyclic.
- The Cesàro operator $\mathrm{C}^{(p)} \colon \ell^p \to \ell^p$, $1 < p < \infty$, is not power bounded, not mean ergodic and not supercyclic.
- The Cesàro operator $\mathrm{C}^{(0)} \colon c_0 \to c_0$ is power bounded, not mean ergodic and not supercyclic.

Theorem. Albanese, Bonet, Ricker.

(i) Let $0 < \gamma < 1$. Both of the operators C_γ and $\mathsf{C}_{\gamma,0}$ fail to be power bounded and are not mean ergodic. Moreover,

$$\mathrm{Ker}(I - \mathsf{C}_\gamma) = \mathrm{Ker}(I - \mathsf{C}_{\gamma,0}) = \{0\},$$

and $\mathrm{Im}(I - \mathsf{C}_\gamma)$ (resp. $\mathrm{Im}(I - \mathsf{C}_{\gamma,0})$) is a proper closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$).

(ii) Both of the operators C_1 and $\mathsf{C}_{1,0}$ are power bounded but not mean ergodic. Moreover, $\mathrm{Im}(I - \mathsf{C}_1)$ (resp. $\mathrm{Im}(I - \mathsf{C}_{1,0})$) is not a closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$).

Theorem continued

(iii) Let $\gamma > 1$. Both of the operators C_γ and $\mathsf{C}_{\gamma,0}$ are power bounded and uniformly mean ergodic. Moreover, $\mathrm{Im}(I - \mathsf{C}_\gamma)$ (resp. $\mathrm{Im}(I - \mathsf{C}_{\gamma,0})$) is a proper closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$). In addition,

$$\mathrm{Im}(I - \mathsf{C}_\gamma) = \{h \in A^{-\gamma} : h(0) = 0\}. \tag{3}$$

Moreover, with $\varphi(z) := 1/(1-z)$, for $z \in \mathbb{D}$, the linear projection operator $P_\gamma : A^{-\gamma} \to A^{-\gamma}$ given by

$$P_\gamma(f) := f(0)\varphi, \qquad f \in A^{-\gamma},$$

is continuous and satisfies $\lim_{n\to\infty}(\mathsf{C}_\gamma)_{[n]} = P_\gamma$ in the operator norm.

Theorem

The Cesàro operator C acting on $H(\mathbb{D})$ is power bounded, uniformly mean ergodic and not supercyclic, hence not hypercyclic.

As a consequence, C is not supercyclic on the spaces $A^{-\gamma}$, $\gamma \geq 0$, and $A_0^{-\gamma}, 0 < \gamma \leq \infty$.

Proposition

Let $\gamma \in [0, \infty[$.

The following conditions are equivalent:

(a) C is power bounded on $A_+^{-\gamma}$.

(b) C is (uniformly) mean ergodic on $A_+^{-\gamma}$.

(c) $1 \leq \gamma < \infty$.

Ergodicity of C on the (LB) growth spaces

Proposition

Let $\gamma \in]0, \infty]$.

The following conditions are equivalent:

(a) C is power bounded on $A_{-}^{-\gamma}$.

(b) C is (uniformly) mean ergodic on $A_{-}^{-\gamma}$.

(c) $1 < \gamma \leq \infty$.

To Volterra operators

The Cesàro operator has the integral representation for $f \in H(\mathbb{D})$

$$Cf(z) = \frac{1}{z}\int_0^z \frac{f(\rho)}{1-\rho}\,d\rho = \frac{1}{z}\int_0^z f(\rho)g'(\rho)d\rho.$$

with $g(z) = -\log(1-z), z \in \mathbb{D}$. The **Volterra operators** are defined by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta.$$

This operator on spaces analytic functions on the disc was considered by **Pommerenke, Aleman, Siskakis, Constantin, Pau, Peláez and Rättyä**, among others. The work by **Bassallote, Contreras, Hernández-Mancera, Martín and Paul** in 2012 is important for us.

The **Volterra operator** V_g with symbol $g \in H(\mathbb{C})$ is defined on $H(\mathbb{C})$ by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (z \in \mathbb{C}).$$

Question

How it acts on $A_+^{-\gamma}$ and $A_-^{-\gamma}$?

What is the spectrum in case it is continuous?

The case of V_g on $A^{-\gamma}$ was investigated by **Malman** in 2018.

A result

Proposition.

Let $g \in H(\mathbb{D})$ be an analytic function.

(1) Let $0 \leq \gamma < \infty$. The operator $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is continuous if and only if $g \in A_+^{-0}$.

(2) Let $0 < \gamma < \infty$. The operator $V_g : A_-^{-\gamma} \to A_-^{-\gamma}$ is continuous if and only if $g \in A_+^{-0}$.

(3) The operator $V_g : A^{-\infty} \to A^{-\infty}$ is continuous if and only if $g \in A^{-\infty}$.

A few comments

The Volterra operator $V_g, g \in H(\mathbb{D})$, is continuous (respectively compact) on $A^{-\alpha}$ and, equivalently on $A_0^{-\alpha}$, if and only if g belongs to the Bloch space $\mathcal{B}$ (respectively to the little Bloch space $\mathcal{B}_0$). This was proved by Hu and Stević.

Recall that a function $g \in H(\mathbb{D})$ belongs to $\mathcal{B}$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|)|g'(z)| < \infty$, and g belongs to $\mathcal{B}_0$ if and only if $\lim_{|z|\to 1}(1-|z|)|g'(z)| = 0$.

Define the weight $v_{log}(z) := (\log(e/(1-|z|))^{-1}, \ z \in \mathbb{D}$. We say that $g \in H(\mathbb{D})$ has **logarithmic mean growth** if $\sup_{z\in\mathbb{D}} v_{log}(z)|g(z)| < \infty$. It is well known that every analytic function g in the Bloch space $\mathcal{B}$ has logarithmic mean growth.

A few comments

There are analytic functions $g \notin \mathcal{B}$ which have logarithmic mean growth; examples are due to Girela, González and Peláez. It is easy to see that every analytic function with logarithmic mean growth belongs to A_+^{-0}. Therefore the Volterra operator V_g for such symbols g is continuous on $A_+^{-\gamma}$ and $A_-^{-\gamma}$ for each γ, but it is not continuous on each $A^{-\alpha}$ for each $\alpha > 0$.

The space $H_{v_{log}}$ of all analytic functions with logarithmic mean growth is a Banach space for the norm $||g||_{\log} := \sup_{z \in \mathbb{D}} v_{log}(z)|g(z)|$, and it is contained but different from A_+^{-0}. Indeed, if they were equal, the closed graph theorem would imply that the space is simultaneously a Banach space and a Fréchet Schwartz space, hence finite dimensional. A contradiction.

Spectrum of Volterra operators

Now we investigate the **spectrum of the Volterra operator** when it acts continuously on Korenblum type spaces.

Aleman and Constantin in 2009 and **Aleman and Peláez** in 2012 investigated the spectra of Volterra operators on several spaces of of holomorphic functions on the disc.

We assume that $g \in H(\mathbb{C})$ be a non-constant entire function such that $g(0) = 0$ and V_g is the Volterra operator.

Proposition

Let $g \in H^\infty$ be a bounded analytic function with $g(0) = 0$.

1) If $\gamma \geq 0$, then $\sigma(V_g, A_+^{-\gamma}) = \sigma^*(V_g, A_+^{-\gamma}) = \{0\}$.

(2) If $0 < \gamma \leq \infty$, then $\sigma(V_g, A_-^{-\gamma}) = \sigma^*(V_g, A_-^{-\gamma}) = \{0\}$.

The same statements hold if $g \in \mathcal{B}_0$.

Results about the spectrum

We understand $0 \in \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \alpha\}$. Therefore, this set coincides with the disc $\{\lambda \in \mathbb{C} \mid |\lambda - \frac{c}{2\alpha}| \leq \frac{|c|}{2\alpha}\}$.

Proposition

Let $g(z) = c\log(1/(1-\overline{w}z)), z \in \mathbb{D}$ with $c, w \in \mathbb{C}, c \neq 0, |w| = 1$, For each $\gamma \geq 0$ the operator $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is continuous. Moreover $\sigma(V_g, A_+^{-\gamma}) = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) > \gamma\}$ and $\sigma^*(V_g, A_+^{-\gamma}) = \overline{\sigma(V_g, A_+^{-\gamma})}$.

If $\gamma = 0$, then $\sigma(V_g, A_+^{-0}) = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) > 0\}$ is the union of $\{0\}$ and an open half plane of $\mathbb{C}$ with 0 at the boundary, which depends on c. Therefore it is unbounded. If $\gamma > 0$, then $\sigma(V_g, A_+^{-0})$ is bounded but not closed.

Results about the spectrum

Proposition

Let $g(z) = c\log(1/(1-\overline{w}z)), z \in \mathbb{D}$ with $c, w \in \mathbb{C}, c \neq 0, |w| = 1$, For each $\gamma \in]0, \infty[$ the operator $V_g : A_-^{-\gamma} \to A_-^{-\gamma}$ is continuous, and $\sigma(V_g, A_-^{-\gamma}) = \sigma^*(V_g, A_-^{-\gamma}) = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \gamma\}$.

For $A^{-\infty}$ we have the following complement for symbols with logarithmic mean growth. Observe that in this case the Volterra operator need not act from each step into itself.

Proposition

If $g \in H(\mathbb{D})$ is an analytic function with $g(0) = 0$ such that

$$\sup_{z \in \mathbb{D}} v_{log}(z)|g(z)| < \infty, \ \text{with } v_{log}(z) := (\log(e/(1-|z|))^{-1}, \ z \in \mathbb{D},$$

then $V_g : A^{-\infty} \to A^{-\infty}$ is continuous and $\sigma(V_g, A^{-\infty}) = \sigma^*(V_g, A^{-\infty}) = \{0\}$.

Results about the spectrum

Now we investigate the spectrum of V_g on the Korenblum space $A^{-\infty}$ for symbols that are not of logarithmic mean growth.

Theorem

Let $g \in A^{-\infty}$ satisfy $g(0) = 0$. The following conditions are equivalent.

(1) The function g is of logarithmic mean growth.

(2) $\sigma^*(V_g, A^{-\infty}) = \{0\}$.

(3) $\sigma^*(V_g, A^{-\infty})$ is bounded.

Results about the spectrum

Proposition

If $g \in A^{-\infty}$ with $g(0) = 0$ satisfies $e^g \notin A^{-\infty}$, then $[0, +\infty[\subset \sigma(V_g, A^{-\infty})$.

This holds in particular for the functions $g_s(z) := -1 + 1/(1-z)^s, s > 0$.

Proposition

The function $g(z) = z/(1-z)$, which belongs to $A^{-\infty}$ satisfies

(1) $\{\lambda \in \mathbb{C} \mid \lambda \text{ not a negative real number}\} \subset \sigma(V_g, A^{-\infty})$, and

(2) $\sigma^*(V_g, A^{-\infty}) = \mathbb{C}$.

References

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