OPERATORS ACTING IN SEQUENCE SPACES GENERATED BY DUAL BANACH SPACES OF DISCRETE CESÀRO SPACES

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Abstract. The dual spaces $d(p), 1 < p < \infty$, of the discrete Cesàro (Banach) spaces $\mathrm{ces}(q), 1 < q < \infty$, were studied by G. Bennett, A. Jagers and others. These (reflexive) dual Banach spaces induce the non-normable Fréchet spaces $d(p+) := \bigcap_{r>p} d(r)$, for $1 \le p < \infty$, and the (LB)-spaces $d(p-) := \bigcup_{1<r<p} d(r)$, for $1 < p \le \infty$, recently introduced and investigated in [11]. Here a detailed study is made of various aspects, such as the spectrum, continuity, compactness, mean ergodicity and supercyclicity of the Cesàro operator, multiplication operators and inclusion operators when they act on (and between) such spaces.

1. Introduction

Given an element $x = (x_n)_n$ of $\mathbb{C}^{\mathbb{N}}$ let $|x| := (|x_n|)_n$ and write $x \ge 0$ if $x = |x|$. By $x \le z$ we mean that $(z-x) \ge 0$. The sequence space $\mathbb{C}^{\mathbb{N}}$ is a Frèchet space for the locally convex topology of coordinatewise convergence. The *Cesàro operator* $C : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$, defined by

$$C(x) := (x_1, \frac{x_1 + x_2}{2}, \dots, \frac{x_1 + x_2 + \dots + x_n}{n}, \dots), \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}, \tag{1.1}$$

satisfies $0 \le |C(x)| \le C(|x|)$, for $x \in \mathbb{C}^{\mathbb{N}}$, and is a linear and topological isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself. Clearly $C(x) \ge 0$ whenever $x \ge 0$ in $\mathbb{C}^{\mathbb{N}}$. It is known that $C : \ell_p \longrightarrow \ell_p$ is continuous for every $1 < p < \infty$, [18, Theorem 326]. G. Bennett thoroughly investigated the spaces

$$\mathrm{ces}(p) := \{x \in \mathbb{C}^{\mathbb{N}} : C(|x|) \in \ell_p\}, \quad 1 < p < \infty, \tag{1.2}$$

which are reflexive Banach spaces relative to the norm

$$\|x\|_{\mathrm{ces}(p)} := \|C(|x|)\|_p, \quad x \in \mathrm{ces}(p), \tag{1.3}$$

where $\|\cdot\|_p$ denotes the usual norm in ℓ_p; see, for example, [9], as well as [8], [14], [16], [24] and the references therein. The dual Banach spaces $(\mathrm{ces}(p))'$, $1 < p < \infty$, are somewhat complicated, [19]. A more transparent *isomorphic* identification of $(\mathrm{ces}(p))'$ is presented in Corollary 12.17 of [9]. It is shown there that

$$d(p) := \{x \in \ell_\infty : \hat{x} := (\sup_{k \ge n} |x_k|)_n \in \ell_p\}, \quad 1 < p < \infty, \tag{1.4}$$

is a Banach space for the norm

$$\|x\|_{d(p)} := \|\hat{x}\|_p, \quad x \in d(p), \tag{1.5}$$

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which is isomorphic to $(\mathrm{ces}(p'))'$, where $\frac{1}{p} + \frac{1}{p'} = 1$. The duality is given by

$$\langle w, x\rangle := \sum_{n=1}^{\infty} w_n x_n, \quad w \in \mathrm{ces}(p'), \quad x \in d(p).$$

The family of Banach spaces $\mathrm{ces}(p)$, $1 < p < \infty$, induces the Fréchet spaces $\mathrm{ces}(p+) := \bigcap_{q>p} \mathrm{ces}(q)$ for $1 \leq p < \infty$ and induces the (LB)-spaces $\mathrm{ces}(p-) := \bigcup_{1<q<p} \mathrm{ces}(q)$ for $1 < p \leq \infty$. These non-normable sequence spaces in $\mathbb{C}^{\mathbb{N}}$ were introduced and studied in [3], [7], as well as the properties of various linear operators acting in them, [5], [7]. Similarly, the family of Banach spaces $d(p)$, $1 < p < \infty$, which were investigated in some detail in [10], generate the corresponding Fréchet spaces $d(p+) := \bigcap_{q>p} d(q)$ for $1 \leq p < \infty$ and the (LB)-spaces $d(p-) := \bigcup_{1<q<p} d(q)$ for $1 < p \leq \infty$. The spaces $d(p+)$ and $d(p-)$ were introduced and studied in [11], where it is shown that they are rather different to their counterparts $\mathrm{ces}(p+)$ and $\mathrm{ces}(p-)$, respectively.

The purpose of this paper is to undertake an investigation of certain natural *linear operators* (e.g., the Cesàro operator, inclusion maps, multiplication operators) acting in the spaces $d(p+)$, $p \in [1, \infty)$, and $d(p-)$, $1 < p \leq \infty$, and to determine various properties of such operators, e.g. their spectrum, compactness, mean ergodicity, etc. We point out that a detailed investigation of the Cesàro operator C acting on the Banach spaces $\mathrm{ces}(p)$, resp. $d(p)$, for $1 < p < \infty$, was carried out in [6], [14], resp. [10], and on the Fréchet spaces $\mathrm{ces}(p+)$ for $1 \leq p < \infty$ in [5]. For the (LB)-spaces $\mathrm{ces}(p-)$, $1 < p \leq \infty$, see [7]. Here we treat C when it is acting in the spaces $d(p+)$, $p \in [1, \infty)$, and $d(p-)$, $1 < p \leq \infty$. Its spectrum is determined in Theorem 3.2 for $d(p+)$ and in Theorem 3.6 for $d(p-)$. For the mean ergodic properties of C, see Proposition 3.5 (for $d(p+)$) and Proposition 3.8 (for $d(p-)$). The properties of multiplication operators on $\mathrm{ces}(p+)$, resp. $\mathrm{ces}(p-)$, can be found in [5], resp. [7]. Here we are concerned with such operators when they act in the spaces $d(p+)$ and $d(p-)$, especially their spectrum, compactness and mean ergodicity; see Section 4. For a *characterization* of multiplication operators in $d(p+)$, resp. $d(p-)$, we refer to Theorem 4.8, resp. Theorem 4.7. Curiously, when acting in $d(p+)$, the space of multipliers is *independent* of p; see Remark 4.9. Theorem 4.13 identifies precisely which multiplication operators, both in $d(p+)$ and in $d(p-)$, are compact; in both cases the space of such operators is "*independent*" of p. The spectra of multiplication operators in $d(p+)$, resp. $d(p-)$, are identified in Theorem 4.16, resp. Theorem 4.17. The mean ergodic properties of multiplication operators are recorded in Theorem 4.20 and Theorem 4.21. An interesting feature (see Theorem 4.10) is that algebra of all multiplication operators, both in $d(p+)$ and in $d(p-)$, is *maximal abelian*, that is, it coincides with its commutant algebra. The final section 5 is devoted to analyzing various operators acting between *different* spaces. As a sample, we mention that C maps $d(p+)$ continuously into $d(q+)$ if and only if $1 \leq p \leq q < \infty$ (see Theorem 5.8), whereas it is a compact operator if and only if $p < q$ (cf. Theorem 5.11). Moreover, C maps $\mathrm{ces}(p-)$ continuously into $d(q-)$ if and only if $p \leq q$ (see Theorem 5.9), whereas it is compact if and only if $p < q$ (cf. Theorem 5.12). Such results rely on a detailed knowledge of the continuity/compactness properties of inclusion maps between members within the family of spaces $\{\ell_{p+}, \text{ces}(q+), d(r+) : 1 \leq p, q, r < \infty\}$ and also between members within the family of spaces $\{\ell_{p-}, \text{ces}(q-), d(r-) : 1 < p, q, r \leq \infty\}$; see, for example, Theorem 5.3 and Theorem 5.6.

2. Preliminaries

Let X, Y be locally convex Hausdorff spaces (briefly, lcHs'). The identity operator on X is denoted by I and $\mathcal{L}(X, Y)$ is the space of all continuous linear operators from X into Y; briefly $\mathcal{L}(X)$ if $X = Y$. The null space and the range of $T \in \mathcal{L}(X)$ are denoted by Ker(T) and Im(T), respectively. Let Γ_X be any system of continuous seminorms generating the topology of X. Then $\mathcal{L}_s(X)$ denotes $\mathcal{L}(X)$ equipped with the strong operator topology τ_s which is given by the family of seminorms $q_x : T \longmapsto q(Tx)$, for every $x \in X$ and $q \in \Gamma_X$. Furthermore, $\mathcal{L}_b(X)$ denotes $\mathcal{L}(X)$ endowed with the topology τ_b of uniform convergence on the bounded subsets of X, i.e., generated by the seminorms $q_B : T \longmapsto \sup_{x \in B} q(Tx)$, for every $q \in \Gamma_X$ and every bounded set $B \subseteq X$. If Γ_X can be taken to be countable and X is complete, then X is called a *Fréchet space*. The *dual operator* of $T \in \mathcal{L}(X, Y)$ is denoted by $T' : Y' \longrightarrow X'$, where $X' = \mathcal{L}(X, \mathbb{C})$ is the topological dual space of X. The *strong topology* in X (resp. in X') is denoted by $\beta(X, X')$ (resp. by $\beta(X', X)$) and we write X_β (resp. X'_β). If X is a barrelled space, then $X_\beta = X$, [25, Remark p. 271]. Given lcHs' X, Y and $T \in \mathcal{L}(X, Y)$, its dual operator $T' \in \mathcal{L}(Y'_\beta, X'_\beta)$, [25, Proposition 23.30(b)]. Our basic references for functional analysis and operators in lcHs' are [15], [20], [21], [22], [25].

Recall that an operator $T \in \mathcal{L}(X, Y)$, with X, Y lcHs', is called *bounded* (resp. *compact*) if there exists a neighbourhood U of $0 \in X$ such that $T(U)$ is a bounded (resp. a relatively compact) subset of Y. If Y is Montel (i.e., each bounded set is relatively compact), then $T \in \mathcal{L}(X, Y)$ is compact if and only if it is bounded. All of the Fréchet spaces considered in this paper are of the type described in the following result, [5, Lemma 25].

Lemma 2.1. *Let $X := \text{proj}_k X_k$ and $Y := \text{proj}_m Y_m$ be Fréchet spaces such that $X = \bigcap_{k \in \mathbb{N}} X_k$ with each $(X_k, \|\cdot\|_k)$ a Banach space (resp. $Y = \bigcap_{m \in \mathbb{N}} Y_m$ with each $(Y_m, \|\cdot\|_m)$ a Banach space). Moreover, it is assumed that X is dense in X_k and that $X_{k+1} \subseteq X_k$ with a continuous inclusion for each $k \in \mathbb{N}$ (resp. $Y_{m+1} \subseteq Y_m$ with a continuous inclusion for each $m \in \mathbb{N}$). Let $T : X \longrightarrow Y$ be a linear operator.*

(i) *T is continuous if and only if for each $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that T has a unique continuous linear extension $T_{k,m} : X_k \longrightarrow Y_m$.*

(ii) *Assume that $T \in \mathcal{L}(X, Y)$. Then T is bounded if and only if there exists $k_0 \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$, the operator T has a unique continuous linear extension $T_{k_0,m} : X_{k_0} \longrightarrow Y_m$.*

Let $Y = \text{ind}_m Y_m$ be a *regular* (LB)-space, [20, p. 83]. Then a set $B \subseteq Y$ is bounded if and only if there exists $m \in \mathbb{N}$ such that $B \subseteq Y_m$ and B is bounded in the Banach space Y_m. All of the (LB)-spaces considered in this paper are of the type described in the following result, [7, Lemma 17].

Lemma 2.2. *Let $X = \text{ind}_k X_k$ and $Y = ind_m Y_m$ be two (LB)-spaces with increasing unions of Banach spaces $X = \bigcup_{k \in \mathbb{N}} X_k$ and $Y = \bigcup_{m \in \mathbb{N}} Y_m$. Let $T : X \longrightarrow Y$ be a linear operator.*

(i) *T is continuous if and only if for each $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T(X_k) \subseteq Y_m$ and the restriction $T : X_k \longrightarrow Y_m$ is continuous.*

(ii) *Assume that Y is regular. Then T is bounded if and only if there exists $m \in \mathbb{N}$ such that $T(X_k) \subseteq Y_m$ and $T : X_k \longrightarrow Y_m$ is continuous for all $k \geq m$.*

The following result, [22, §42.1(6), p. 202], is a version of Schauder's theorem in the Banach space setting.

Lemma 2.3. *Let X, Y be Montel spaces. Then $T \in \mathcal{L}(X, Y)$ is compact if and only if $T' \in \mathcal{L}(Y'_\beta, X'_\beta)$ is compact.*

For $T \in \mathcal{L}(X)$, with X a lcHs, the *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{pt}(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If the space X needs to be stressed, then we also write $\sigma(T; X), \sigma_{pt}(T; X)$ and $\rho(T; X)$. Unlike for Banach spaces, it may happen that $\rho(T) = \emptyset$ or that $\rho(T)$ is not open in $\mathbb{C}$. This is why some authors (eg. [28]) prefer the subset $\rho^*(T)$ of $\rho(T)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that the open disc $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\} \subseteq \rho(T)$ and $\{R(\mu, T) : \mu \in B(\lambda, \delta)\}$ is an equicontinuous subset of $\mathcal{L}(X)$. If X is a Fréchet space or an (LB)-space, then it suffices that such sets are bounded in $\mathcal{L}_s(X)$. Define $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$, which is a closed set containing $\sigma(T)$. If X happens to be a Banach space, then $\sigma^*(T) = \sigma(T)$. An advantage of $\rho^*(T)$, if it is non-empty, is that it is open and the resolvent map $\lambda \longmapsto R(\lambda, T)$ is analytic from $\rho^*(T)$ into $\mathcal{L}_b(X)$, [1, Proposition 3.4]. In [1, Remark 3.5(vi), p. 265] an example of an operator $T \in \mathcal{L}(X)$, with X a Fréchet space, is presented which satisfies $\overline{\sigma(T)} \subseteq \sigma^*(T)$ properly.

The following two facts are important for determining the spectra of operators in certain Fréchet spaces and (LB)-spaces. The first result occurs in [2, Lemma 2.1] .

Lemma 2.4. *Let $X = \bigcap_{n \in \mathbb{N}} X_n$ be a Fréchet space which is the intersection of a sequence of Banach spaces $(X_n, \|\cdot\|_n)$, for $n \in \mathbb{N}$, satisfying $X_{n+1} \subseteq X_n$ with $\|x\|_n \leq \|x\|_{n+1}$ for each $n \in \mathbb{N}$ and $x \in X_{n+1}$. Let $T \in \mathcal{L}(X)$ satisfy the following condition:*

(A) *For each $n \in \mathbb{N}$ there exists $T_n \in \mathcal{L}(X_n)$ such that the restriction of T_n to X (resp. of T_n to X_{n+1}) coincides with T (resp. with T_{n+1}).*

Then $\sigma(T; X) \subseteq \bigcup_{n \in \mathbb{N}} \sigma(T_n; X_n)$ and $R(\lambda, T)$ coincides with the restriction of $R(\lambda, T_n)$ to X for each $n \in \mathbb{N}$ and $\lambda \in \bigcap_{n \in \mathbb{N}} \rho(T_n; X_n)$.

Moreover, if $\bigcup_{n \in \mathbb{N}} \sigma(T_n; X_n) \subseteq \overline{\sigma(T; X)}$, then

$$\sigma^*(T; X) = \overline{\sigma(T; X)}.$$

Concerning (LB)-spaces, the following result is Lemma 5.2 of [4].

Lemma 2.5. *Let $X = \mathrm{ind}_n X_n$ be a Hausdorff inductive limit of a sequence of Banach spaces $(X_n, \|\cdot\|_n)$, for $n \in \mathbb{N}$. Let $T \in \mathcal{L}(X)$ satisfy the following condition:*

(A) *For each $n \in \mathbb{N}$ the restriction T_n of T to X_n maps X_n into itself and $T_n \in \mathcal{L}(X_n)$. Then the following properties are satisfied.*

(i) $\sigma_{pt}(T; X) = \bigcup_{n \in \mathbb{N}} \sigma_{pt}(T_n; X_n)$.

(ii) $\sigma(T; X) \subseteq \bigcap_{m \in \mathbb{N}} (\bigcup_{n=m}^{\infty} \sigma(T_n; X_n))$. *Moreover, if $\lambda \in \bigcap_{n=m}^{\infty} \rho(T_n; X_n)$ for some $m \in \mathbb{N}$, then $R(\lambda, T_n)$ coincides with the restriction of $R(\lambda, T)$ to X_n for each $n \geq m$.*

(iii) *If* $\bigcup_{n=m}^{\infty} \sigma(T_n; X_n) \subseteq \overline{\sigma(T, X)}$ *for some* $m \in \mathbb{N}$, *then*

$$\sigma^*(T; X) = \overline{\sigma(T; X)}.$$

We conclude this section with the following useful result.

Proposition 2.6. *Let* X *be a reflexive lcHs and* $T \in \mathcal{L}(X)$.

(i) $\sigma(T; X) = \sigma(T'; X'_\beta)$.
(ii) $\sigma^*(T; X) = \sigma^*(T'; X'_\beta)$.

Proof. (i) We will require the following
Fact. *If* $S \in \mathcal{L}(X)$ *is an isomorphism, then* $S' \in \mathcal{L}(X'_\beta)$ *is also an isomorphism.*

By assumption there exists $R \in \mathcal{L}(X)$ such that $SR = RS = I_X$ (the identity operator on X). Noting that both $R', S' \in \mathcal{L}(X'_\beta)$ it follows that $S'R' = R'S' = I_{X'_\beta}$, that is, $S' \in \mathcal{L}(X'_\beta)$ is an isomorphism. This proves the Fact.

Since X is reflexive, also X'_β is reflexive, [21, §23.5(5), p. 303]. Moreover $T'' = T$. Accordingly, it suffices to show that $\sigma(T'; X'_\beta) \subset \sigma(T; X)$ as the reverse containment will follow by "the same" argument via duality. But, if $\lambda \notin \sigma(T; X)$, then $(\lambda I_X - T) \in \mathcal{L}(X)$ is an isomorphism. By the above Fact, $(\lambda I_X - T)' = (\lambda I_{X'_\beta} - T')$ is also an isomorphism, that is, $\lambda \notin \sigma(T'; X'_\beta)$.

(ii) By the reflexivity of X it suffices to show that $\sigma^*(T'; X'_\beta) \subseteq \sigma^*(T; X)$. If $\lambda \notin \sigma^*(T; X)$, then there exists $\delta > 0$ such that $B(\lambda, \delta) \subseteq \rho(T; X)$ and $\{(\mu I_X - T)^{-1} : \mu \in B(\lambda, \delta)\} \subseteq \mathcal{L}(X)$ is equicontinuous. Since reflexive spaces are barrelled, [25, Proposition 23.22], it follows from [22, §39.3(6), p. 138] that $\{((\mu I_X - T)^{-1})' : \mu \in B(\lambda, \delta)\} \subseteq \mathcal{L}(X'_\beta)$ is equicontinuous. But, it was shown in the proof of part (i) that

$$((\mu I_X - T)^{-1})' = (\mu I_{X'_\beta} - T')^{-1} \tag{2.1}$$

and so $\{(\mu I_{X'_\beta} - T')^{-1} : \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X'_\beta)$. That is, $\lambda \notin \sigma^*(T'; X'_\beta)$. □

3. The Cesàro operator in $d(p+)$ and $d(p-)$

The aim of this section is to investigate the spectrum and certain operator theoretic aspects of the Cesàro operator when it acts in the Fréchet spaces $d(p+)$, $1 \le p < \infty$, and in the (LB)-spaces $d(p-)$, $1 < p \le \infty$. We begin with the Fréchet space setting.

The reflexive Banach spaces $d(p)$ for $1 < p < \infty$ satisfy $d(r) \subseteq d(s)$ with $\|\cdot\|_{d(s)} \le \|\cdot\|_{d(r)}$ on $d(r)$ whenever $1 < r \le s < \infty$; see (1.4), (1.5) and [10, Proposition 2.7(i)]. Given $1 \le p < \infty$, consider any strictly decreasing sequence $\{p_k\}_{k\in\mathbb{N}} \subseteq (p, \infty)$ satisfying $p_k \downarrow p$, in which case $d(p_{k+1}) \subseteq d(p_k)$ for all $k \in \mathbb{N}$. Then $d(p+) := \bigcap_{q>p} d(q) = \bigcap_{k\in\mathbb{N}} d(p_k)$ is a Fréchet space relative to the increasing sequence of *norms* on $d(p+)$ defined by $\|\cdot\|_k : x \longmapsto \|x\|_{d(p_k)}$ for $k \in \mathbb{N}$; see (1.5). Since each Banach space $d(q)$, $1 < q < \infty$, is solid in $\mathbb{C}^{\mathbb{N}}$ (i.e., if $x \in d(q)$ and $y \in \mathbb{C}^{\mathbb{N}}$ satisfy $|y| \le |x|$, then $y \in d(q)$), it is clear that $d(p+)$ is also *solid* in $\mathbb{C}^{\mathbb{N}}$. The space $d(p+)$ is actually a *Fréchet-Schwartz* space (but, it is *not* nuclear) in which the canonical vectors $e_n := (\delta_{nk})_k$, for $n \in \mathbb{N}$, form an unconditional basis. In particular, $d(p+)$ is dense in $d(p_k)$ for all $k \in \mathbb{N}$. Moreover, with continuous inclusions, it is the case that $d(p) \subseteq d(p+) \subseteq \mathbb{C}^{\mathbb{N}}$ for $1 < p < \infty$ (also $d(1+) \subseteq \mathbb{C}^{\mathbb{N}}$) and that $d(p+) \subseteq d(q+)$ whenever $1 \le p \le q < \infty$. As a consequence of Lemma 2.1, for each

$1 \le p \le q < \infty$ the Cesàro operator $C \in \mathcal{L}(d(p+), d(q+))$. All of the above facts, and more, can be found in Sections 3, 4 of [11].

In order to determine the spectra of C acting in $d(p+)$ we first recall the Banach space case. The following facts are recorded in [10]; see Proposition 3.2 (and its proof), Remark 3.3 and Corollary 3.5.

Proposition 3.1. *Let $1 < p < \infty$. Then $C \in \mathcal{L}(d(p))$ satisfies $\|C\|_{\mathrm{op}} = p'$ and its spectra are given by*

$$\sigma(C; d(p)) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| \le \tfrac{p'}{2}\} \text{ and } \sigma_{pt}(C; d(p)) = \emptyset. \tag{3.1}$$

Moreover,

$$\{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| < \tfrac{p'}{2}\} \subseteq \sigma_{pt}(C'; (d(p))') \tag{3.2}$$

and $\overline{\mathrm{Im}(C - \lambda I)} \neq d(p)$ whenever $|\lambda - \frac{p'}{2}| < \frac{p'}{2}$.

The spectra of C acting in $d(p+), p \ge 1$, can now be determined. They should be compared with the case of C when it acts in the spaces ces$(p+)$, $p \ge 1$, [5, Theorem 3].

Theorem 3.2. (i) *Let $1 < p < \infty$. The following statements are valid.*

(a1) $\sigma_{pt}(C; d(p+)) = \emptyset$.

(a2) $\sigma(C; d(p+)) = \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| < \frac{p'}{2}\} \cup \{0\}$.

(a3) $\sigma^*(C; d(p+)) = \overline{\sigma(C; d(p+))} = \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| \le \frac{p'}{2}\}$.

(ii) *For $p = 1$ the following statements are valid.*

(b1) $\sigma_{pt}(C; d(1+)) = \emptyset$.

(b2) $\sigma(C; d(1+)) = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\} \cup \{0\}$.

(b3) $\sigma^*(C; d(1+)) = \overline{\sigma(C; d(1+))} = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\}$.

Proof. By using Lemma 2.4 above, together with Proposition 3.1 above in place of [5, Theorem 1], a careful examination of the proof of Theorem 3 given in [5] for C acting in the spaces ces$(p+)$, shows that it can be adapted to also apply to C acting in the spaces $d(p+)$.

However, there is one point which needs to be clarified. For the proof of the fact that $0 \in \sigma(C; \mathrm{ces}(p+))$, as given on p. 1537 of [5] it is important that the vector $y := \sum_{i=1}^{\infty} \frac{1}{2i-1} e_{2i-1}$ defined there belongs to ces$(p+)$. Fortunately, the same vector y also belongs to the smaller space $d(p+) \subseteq \mathrm{ces}(p+)$. Indeed, since $\hat{y} = (1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}, \frac{1}{7}, \ldots)$ belongs to ℓ_q for all $1 < q < \infty$, it follows that $y \in d(r)$ for all $1 < r < \infty$ (cf. (1.4)) and hence, that $y \in d(p+), p \ge 1$. Moreover, for each $q > 1$, the vector $x := C^{-1}(y) = ((-1)^{n+1})_n$ satisfies $|x| = (1)_n \notin d(q)$ as $d(q) \subseteq \ell_q$. Accordingly, $x \notin d(p+)$ for $p \ge 1$. With these observations the proof that $0 \in \sigma(C; d(p+))$ follows the lines of that given in [5] for proving that $0 \in \sigma(C; \mathrm{ces}(p+))$. □

Since $d(p+)$ is isomorphic to the power series Fréchet space $\Lambda_0^\infty(\alpha)$ for each $1 \le p < \infty$, with $\alpha := (\log(n))_n$, [11, Corollary 4.5(ii)], its strong dual $(d(p+))'_\beta$ (which is the (DFS)-space ces$(p'-)$, [11, Remark 4.4]) is a sequence space [25, p. 357]. It is then routine to verify that the dual operator $C' \in \mathcal{L}((d(p+))'_\beta)$ is given by the formula

$$C'(y) := (\textstyle\sum_{k=n}^{\infty} \frac{y_k}{k})_n, \quad y = (y_n)_n \in (d(p+))'_\beta. \tag{3.3}$$

Using this observation and Proposition 3.1 above, the proof of Proposition 4 in [5] can be modified to yield the following result.

Proposition 3.3. (i) *For each* $1 < p < \infty$ *it is the case that*

$$\{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| < \tfrac{p'}{2}\} \subseteq \sigma_{pt}(C'; (d(p+))'_\beta).$$

(ii) *If* $p = 1$, *then*

$$\{\lambda \in \mathbb{C} : Re(\lambda) > 0\} \subseteq \sigma_{pt}(C'; (d(1+))'_\beta).$$

Remark 3.4. Since the spectrum of a compact operator acting in a Fréchet space is necessarily a compact subset of $\mathbb{C}$, [15, Theorem 9.10.2], it follows from parts (a2) and (b2) of Theorem 3.2 that the operator $C \in \mathcal{L}(d(p+))$ *fails* to be compact for every $p \geq 1$. Since $d(p+)$ is a Fréchet-Schwartz space, it is also Montel, [25, Remark 24.24(b)], and hence, there is no distinction between compact and bounded operators in $d(p+)$; see Section 2. □

Let X be a lcHs. An operator $T \in \mathcal{L}(X)$ is called *power bounded* if $\{T^n : n \in \mathbb{N}\}$ is an equicontinuous subset of $\mathcal{L}(X)$. Given $T \in \mathcal{L}(X)$, the averages $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$, for $n \in \mathbb{N}$, are called the Cesàro means of T. It is straight-forward to check that $\frac{T^n}{n} = T_{[n]} - \frac{(n-1)}{n} T_{[n-1]}$ for $n \geq 2$. The operator T is called *mean ergodic* (resp. *uniformly mean ergodic*) if $\{T_{[n]}\}_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$), [23]. If X is separable, then $T \in \mathcal{L}(X)$ is called *supercyclic* if there exists $x \in X$ such that the projective orbit $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}_0\}$ is dense in X, where $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

By replacing the use of Proposition 4 in [5] with Proposition 3.3 above, an analogous proof to that of Proposition 5 in [5] for the spaces ces($p+$) yields a proof of the following result for the spaces $d(p+)$.

Proposition 3.5. *The Cesàro operator* $C \in \mathcal{L}(d(p+))$, *for* $p \geq 1$, *is not power bounded, not mean ergodic and not supercyclic.*

We now turn our attention to C acting in the (LB)-spaces $d(p-)$. Given $1 < p \leq \infty$, for the remainder of this section let $\{p_k\}_{k\in\mathbb{N}} \subseteq (1, p)$ be any strictly increasing sequence satisfying $p_k \uparrow p$, in which case $d(p_k) \subseteq d(p_{k+1})$ for all $k \in \mathbb{N}$. Then $d(p-) = \bigcup_{1<q<p} d(q) = \mathrm{ind}_k d(p_k)$ is an (LB)-space, that is, a countable inductive limit of Banach spaces, [25, pp. 290-291]. Clearly $d(p-)$ is solid in $\mathbb{C}^{\mathbb{N}}$. Since the (inclusion) linking maps $d(r) \subseteq d(s)$ are *compact* whenever $1 < r < s < \infty$, [10, Proposition 5.2(iii)], it follows that $d(p-), p \in (1, \infty]$, is actually a (DFS)-space, [25, p. 304], (but, it is *not* nuclear) which is isomorphic to the strong dual $(\mathrm{ces}(p'+))'_\beta$ of the Fréchet-Schwartz space ces($p'+$), [11, Proposition 4.3(i)]. We note that each space ces($q+$), $1 \leq q < \infty$, is isomorphic to the fixed power series Fréchet space $\Lambda_0^1(\alpha)$, with $\alpha = (\log(n))_n$, of type 0 and order 1, [3, Corollary 3.2]. The canonical vectors $\{e_n : n \in \mathbb{N}\}$ form an unconditional basis for each space $d(p-), p \in (1, \infty]$, [11, Theorem 4.6]. As a consequence of Lemma 2.2, for each $1 < p \leq q \leq \infty$ the Cesàro operator $C \in \mathcal{L}(d(p-), d(q-))$, [11, Proposition 4.8(ii)(a)]. For further properties of the spaces $d(p-), 1 < p \leq \infty$, see Section 4 of [11]. Since $d(p-) = \mathrm{ind}_k\, d(p_k)$ with $1 < p_k \uparrow p$, we point out that $C \in \mathcal{L}(d(p-))$ satisfies all of the assumptions of Lemma 2.5 with $T_n := C|_{d(p_n)} \in \mathcal{L}(d(p_n))$ for $n \in \mathbb{N}$.

The spectra of C acting in $d(p-), p \in (1, \infty]$, can now be identified. They should be compared with C when it acts in the spaces ces($p-$), $p \in (1, \infty]$, [7, Section 3].

Theorem 3.6. *Let* $p \in (1, \infty]$.

(i) $\sigma_{pt}(C; d(p-)) = \emptyset$.

(ii) $\{0\} \cup \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| < \frac{p'}{2}\} \subseteq \sigma(C; d(p-)) \subseteq \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\}$.

(iii) $\sigma^*(C; d(p-)) = \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\}$.

(iv) $\{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| < \frac{p'}{2}\} \subseteq \sigma_{pt}(C'; (d(p-))'_\beta), \quad 1 < p < \infty$.

Proof. (i) This follows from Lemma 2.5(i) and Proposition 3.1.

(ii) An analogous proof as that given for Proposition 8 in [7] applies here. The use of Lemma 5 and Theorem 6 in [7] needs to be replaced, respectively, with Lemma 2.5 and Proposition 3.1 above.

Again one point needs to be clarified. For the proof that $0 \in \sigma(C; \mathrm{ces}(p-))$, given on p. 10 in Proposition 8 of [7], it was required that the vector $y := (\frac{1-(-1)^n}{2n})_n$ should belong to $\mathrm{ces}(p-)$ for each $p \in (1, \infty]$ and satisfy $C^{-1}(y) = ((-1)^{n+1})_n \notin \mathrm{ces}(p-)$. The analoguous argument can be applied here after noting that y also belongs to the smaller space $d(p-), 1 < p \leq \infty$. Indeed, this is the *same* y as that in the proof of Proposition 3.2 above, where it was shown that $y \in d(r)$ for all $1 < r < \infty$ and hence, in particular, $y \in d(p-) = \bigcup_{1<r<p} d(r)$. Moreover, since $d(p-) \subseteq \mathrm{ces}(p-)$ and $C^{-1}(y) \notin \mathrm{ces}(p-)$, also $C^{-1}(y) \notin d(p-)$.

(iii) The proof of Proposition 10 given in [7] can be adapted to apply here. The use of Theorem 6 and Proposition 8 in [7] need to be replaced, respectively, with Proposition 3.1 above and part (ii) of this theorem.

(iv) Fix $1 < p < \infty$ and note that the natural inclusion $d(p-) \subseteq d(p)$ is continuous, as a consequence of Lemma 2.2(i). It follows that $(d(p))' \subseteq (d(p-))'_\beta \subseteq \mathbb{C}^{\mathbb{N}}$. Moreover, C' as given by (3.3) is the "same" operator in each of these three spaces. Hence,

$$\sigma_{pt}(C'; (d(p))') \subseteq \sigma_{pt}(C'; (d(p-))'_\beta). \tag{3.4}$$

But, it is shown in the proof of [10, Proposition 3.2(iii)] (see (3.7) there) that $\{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| < \frac{p'}{2}\} \subseteq \sigma_{pt}(C'; (d(p))')$. Combining this containment with (3.4) yields the desired result. □

Remark 3.7. Theorem 3.6(ii) shows that each spectrum $\sigma(C; d(p-)), 1 < p \leq \infty$, is an uncountable set. Accordingly, $C \in \mathcal{L}(d(p-))$ is *not* a compact operator, [15, Theorem 9.10.2], [17, p. 204]. As noted above, $d(p-)$ is the strong dual of a Fréchet-Schwartz space and hence, $d(p-)$ is a Montel space, [25, Proposition 24.25]. So, $C : d(p-) \longrightarrow d(p-)$ also *fails* to be a bounded operator.

Proposition 3.8. *Let* $p \in (1, \infty]$. *The Cesàro operator* $C \in \mathcal{L}(d(p-))$ *is not power bounded, not mean ergodic and not supercyclic.*

Proof. That C is not mean ergodic and not power bounded in $d(p-)$ can be established along the lines of the proof of Proposition 11 in [7] by simply replacing the space $\mathrm{ces}(p-)$ there with the space $d(p-)$ and noting that any vector of the form $x := \gamma(1, 1, 1, \ldots)$, with $\gamma \in \mathbb{C}$, belongs to $d(p-) = \bigcup_{1<q<p} d(q) \subseteq \bigcup_{1<q<p} \ell_q$, [10, Proposition 2.7(iii)], if and only if $\gamma = 0$.

Again by replacing $\mathrm{ces}(p-)$ with $d(p-)$, the proof of Proposition 14 (or Remark 15) in [7] can be routinely modified to yield that $C \in \mathcal{L}(d(p-))$ is not supercyclic. □

4. Multiplication operators in $d(p+)$ and $d(p-)$.

Given an element $a = (a_n)_n \in \mathbb{C}^{\mathbb{N}}$, the multiplication (or diagonal) operator $M^a : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ is defined coordinatewise by

$$M^a(x) := (a_n x_n)_n, \quad x \in \mathbb{C}^{\mathbb{N}}. \tag{4.1}$$

Clearly $M^a \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ and $M^a M^b = M^b M^a$ for all $a, b \in \mathbb{C}^{\mathbb{N}}$. Such operators acting on certain Banach spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$, interpreted to mean that $M^a(X) \subseteq X$, are investigated in [9], for example; see also [6], [10]. The classical Banach spaces X such as $\ell_p, 1 \leq p \leq \infty, c_0$ and c are treated in [27]. For X being one of the Fréchet spaces ℓ_{p+} or $\mathrm{ces}(p+)$ see [5] (also [13], [26] are relevant) and for the (LB)-spaces ℓ_{p-} or $\mathrm{ces}(p-)$ we refer to [7]. Our aim in this section is to study the case when X is one of the spaces $d(p+), p \in [1, \infty)$, or $d(p-), 1 < p \leq \infty$.

Given any pair $1 < p, q < \infty$, an element $a \in \mathbb{C}^{\mathbb{N}}$ is called a $(d(p), d(q))$-*multiplier* if $M^a(d(p)) \subseteq d(q)$, where M^a is given by (4.1). The closed graph theorem implies that the associated linear multiplication operator $M^a_{d(p),d(q)} : x \longmapsto M^a(x)$ is then necessarily continuous from $d(p)$ into $d(q)$, that is, $M^a_{d(p),d(q)} \in \mathcal{L}(d(p), d(q))$. If $p = q$, then we denote $M^a_{d(p),d(p)}$ simply by $M^a_{d(p)}$. The vector subspace of $\mathbb{C}^{\mathbb{N}}$ consisting of all the $(d(p), d(q))$-multipliers is denoted by $\mathcal{M}_{d(p),d(q)}$ (briefly $\mathcal{M}_{d(p)}$ if $p = q$). The notions of a $(\mathrm{ces}(p), \mathrm{ces}(q))$-multiplier $a \in \mathbb{C}^{\mathbb{N}}$, its associated operator $M^a_{\mathrm{ces}(p),\mathrm{ces}(q)} : \mathrm{ces}(p) \longrightarrow \mathrm{ces}(q)$ and the multiplier space $\mathcal{M}_{\mathrm{ces}(p),\mathrm{ces}(q)}$ are defined analogously. Since the Banach spaces $\mathrm{ces}(p)$ and $d(p)$ for $1 < p < \infty$ are solid and

$$|M^a(x)| = |ax| = |a| \cdot |x|, \quad a, x \in \mathbb{C}^{\mathbb{N}},$$

it follows that $\mathcal{M}_{d(p),d(q)}$ and $\mathcal{M}_{\mathrm{ces}(p),\mathrm{ces}(q)}$ are *solid* for all $1 < p, q < \infty$. For the following result we refer to table 6 on p. 69 and table 22 on p. 70 of [9].

Proposition 4.1. *Let* $a \in \mathbb{C}^{\mathbb{N}}$.

(i) *Let* $1 < p \leq q < \infty$. *Then* $a \in \mathcal{M}_{d(p),d(q)}$ *if and only if* $(a_n n^{(1/q)-(1/p)})_n \in \ell_\infty$.
(ii) *Let* $1 < q < p < \infty$. *Then* $a \in \mathcal{M}_{d(p),d(q)}$ *if and only if* $a \in d(r)$ *with* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Remark 4.2. For $1 < p \leq q < \infty$, it follows that $(\frac{1}{q} - \frac{1}{p}) \leq 0$ and so $\ell_\infty \subseteq \mathcal{M}_{d(p),d(q)}$. If $p = q$, then $\mathcal{M}_{d(p)} = \ell_\infty$. Moreover, for $a \in \ell_\infty$ it is the case that $\|M^a_{d(p)}\|_{\mathrm{op}} = \|a\|_\infty$. Indeed, (4.3) in Section 4 of [10] states that $\|M^a_{d(p)}\|_{\mathrm{op}} \leq \|a\|_\infty$. Given $n \in \mathbb{N}$, the vector $x := n^{-1/p} e_n$ satisfies $\|x\|_{d(p)} = 1$. Moreover,

$$(ax)\hat{} = n^{-1/p}\, |a_n|\, (1, 1, \ldots, 1, 0, 0, \ldots)$$

with 1 occuring n-times and so $\|(ax)\hat{}\|_p = |a_n|$, that is, $\|M^a_{d(p)}(x)\|_{d(p)} = |a_n|$. It follows that $\|M^a_{d(p)}\|_{\mathrm{op}} = \|a\|_\infty$. □

Duality will feature prominently in this section. Let $1 < p, q < \infty$ and $a \in \mathcal{M}_{d(p),d(q)}$, that is, $M^a_{d(p),d(q)} \in \mathcal{L}(d(p), d(q))$. Note that the dual spaces are given by $(d(p))' = \mathrm{ces}(p')$ and $(d(q))' = \mathrm{ces}(q')$. Moreover, $p \leq q$ if and only if $q' \leq p'$. The calculation

$$\langle M^a_{d(p),d(q)}(x), y' \rangle = \langle ax, y' \rangle = \langle x, ay' \rangle, \quad x \in d(p),\ y' \in (d(q))' = \mathrm{ces}(q'), \tag{4.2}$$

shows that the dual operator $(M^a_{d(p),d(q)})' \in \mathcal{L}((d(q))', (d(p))') = \mathcal{L}(\text{ces}(q'), \text{ces}(p'))$ of $M^a_{d(p),d(q)}$ is precisely the multiplication operator $M^a_{\text{ces}(q'),\text{ces}(p')} : u \longmapsto au$ from $\text{ces}(q')$ into $\text{ces}(p')$. Accordingly, $\mathcal{M}_{d(p),d(q)} = \mathcal{M}_{\text{ces}(q'),\text{ces}(p')}$.

Every Fréchet space has a web and is ultra-bornological; see Corollary 24.29 and Remark 24.15(c) of [25], respectively. The same is true of every (LB)-space, [25, Remark 24.36]. Accordingly, the closed graph theorem is available for closed operators between two Fréchet spaces and between two (LB)-spaces, [25, Theorem 24.31]. This observation allows us to meaningfully adapt the definitions and terminology used for Banach spaces prior to Proposition 4.1. Namely, with obvious notation, the (solid) multiplier spaces $\mathcal{M}_{\text{ces}(p+)}, \mathcal{M}_{d(p+)}, \mathcal{M}_{\text{ces}(p-)}$ and $\mathcal{M}_{d(p-)}$ exist as do the corresponding multiplication operators $M^a_{\text{ces}(p+)}, M^a_{d(p+)}, M^a_{\text{ces}(p-)}$ and $M^a_{d(p-)}$ for the appropriate elements $a \in \mathbb{C}^{\mathbb{N}}$. Our aim is to identify the spaces $\mathcal{M}_{d(p+)}, \mathcal{M}_{d(p-)}$ and to determine various properties of the multiplication operators that they generate. As alluded to above, duality will be an important aspect. In this regard, we point out that the strong dual spaces are given by

$$(d(p+))'_\beta = \text{ces}(p'-), p \in [1,\infty) \text{ and } (d(p-))'_\beta = \text{ces}(p'+), \quad 1 < p \leq \infty, \tag{4.3}$$

and by

$$(\text{ces}(p+))'_\beta = d(p'-), p \in [1,\infty) \text{ and } (\text{ces}(p-))'_\beta = d(p'+), \quad 1 < p \leq \infty; \tag{4.4}$$

see Proposition 4.3 and Remark 4.4 in [11]. Moreover, each space $d(p+), \text{ces}(p+)$, for $p \in [1,\infty)$, is a Fréchet-Schwartz space and each space $d(p-), \text{ces}(p-)$, for $1 < p \leq \infty$, is a (DFS)-space; see [3, Proposition 3.5(ii)], [7, Proposition 1(ii)] and [11, Lemma 4.2(i) & Proposition 4.3(ii)]. In particular, all spaces involved are complete Montel spaces and hence, they are also reflexive.

The following result is Proposition 6 of [5].

Proposition 4.3. *Let $p \in [1,\infty)$ and $a = (a_n)_n \in \mathbb{C}^{\mathbb{N}}$. The following conditions are equivalent.*

(i) *$a \in \mathcal{M}_{\text{ces}(p+)}$, that is, $M^a_{\text{ces}(p+)} \in \mathcal{L}(\text{ces}(p+))$.*
(ii) *For each $r > p$ there exists $s \in (p, r]$ such that $(a_n n^{(1/r)-(1/s)})_n \in \ell_\infty$.*
(iii) *For each $\eta \in (0,1)$ the element $(a_n n^{-\eta})_n \in \ell_\infty$.*

Remark 4.4. (i) Part (iii) of Proposition 4.3 implies that $\ell_\infty \subseteq \mathcal{M}_{\text{ces}(p+)}, p \in [1,\infty)$. Moreover, this containment is *proper* as $a = (\log(n+1))_n \in (\mathcal{M}_{\text{ces}(p+)} \setminus \ell_\infty)$.

(ii) Given $\eta \in (0,1)$, let $\omega_\eta := (n^{-\eta})_n \in \mathbb{C}^{\mathbb{N}}$ and write $u \in \ell_\infty(\omega_\eta)$ if and only if the coordinatewise product $u\omega_\eta \in \ell_\infty$. Then Proposition 4.3 implies that

$$\mathcal{M}_{\text{ces}(p+)} = \bigcap_{0<\eta<1} \ell_\infty(\omega_\eta), \quad p \in [1,\infty), \tag{4.5}$$

[5, Remark 2]. In particular, $\mathcal{M}_{\text{ces}(p+)}$ is *independent* of $p \in [1,\infty)$ and, via part (i), *properly* contains ℓ_∞. □

The following result, [7, Proposition 18], will also be needed .

Proposition 4.5. *Let $1 < p \leq \infty$ and $a = (a_n)_n \in \mathbb{C}^{\mathbb{N}}$. The following conditions are equivalent.*

(i) *$a \in \mathcal{M}_{\text{ces}(p-)}$, that is, $M^a_{\text{ces}(p-)} \in \mathcal{L}(\text{ces}(p-))$.*

(ii) *For each* $u \in (1,p)$ *there exists* $v \in [u,p)$ *such that* $(a_n n^{(1/v)-(1/u)})_n \in \ell_\infty$.

Remark 4.6. Proposition 4.5 can be formulated in terms of the weighted spaces $\ell_\infty(w_{v,u})$, where $w_{v,u}(n) := n^{(1/v)-(1/u)}$ for $n \in \mathbb{N}$. Namely,

$$\mathcal{M}_{\mathrm{ces}(p-)} = \bigcap_{1<u<p}(\bigcup_{u\leq v<p} \ell_\infty(w_{v,u})), \quad 1 < p \leq \infty; \tag{4.6}$$

[7, p. 14]. It is clear from (4.6) that $\ell_\infty \subseteq \mathcal{M}_{\mathrm{ces}(p-)}$ and that this containment is *proper* as $a := (\log(n+1))_n \in (\mathcal{M}_{\mathrm{ces}(p-)} \setminus \ell_\infty)$. □

Let X denote any one of the spaces $d(p+), \mathrm{ces}(p+)$, $p \in [1,\infty)$, or any one of the spaces $d(p-), \mathrm{ces}(p-)$, $1 < p \leq \infty$. Given $a \in \mathbb{C}^{\mathbb{N}}$ recall that we write $a \in \mathcal{M}_X$ if the operator $M^a \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$, as given by (4.1), satisfies $M^a(X) \subseteq X$. In this case the restriction $M^a|_X$, denoted by M^a_X, belongs to $\mathcal{L}(X)$. Define $\mathcal{M}_{\mathrm{op}}(X) := \{M^a_X : a \in \mathcal{M}_X\} \subseteq \mathcal{L}(X)$. Using the identities (4.3) and (4.4), an analogous calculation as in (4.2) reveals that the linear map $\Phi : T \longmapsto T'$ is a vector space isomorphism which satisfies

$$\Phi(\mathcal{M}_{\mathrm{op}}(d(p-))) = \mathcal{M}_{\mathrm{op}}(\mathrm{ces}(p'+)), \quad 1 < p \leq \infty, \tag{4.7}$$

and

$$\Phi(\mathcal{M}_{\mathrm{op}}(d(p+))) = \mathcal{M}_{\mathrm{op}}(\mathrm{ces}(p'-)), \quad 1 \leq p < \infty. \tag{4.8}$$

We can now characterize the multiplication operators in $\mathcal{L}(d(p-))$.

Theorem 4.7. *Let* $1 < p \leq \infty$ *and* $a = (a_n)_n \in \mathbb{C}^{\mathbb{N}}$. *The following conditions are equivalent.*

(i) $a \in \mathcal{M}_{d(p-)}$, *that is,* $M^a_{d(p-)} \in \mathcal{L}(d(p-))$.
(ii) *For each* $u \in (1,p)$ *there exists* $v \in [u,p)$ *such that* $(a_n n^{(1/v)-(1/u)})_n \in \ell_\infty$.

Proof. According to (4.7) it is the case that $a \in \mathcal{M}_{d(p-)}$ if and only if $a \in \mathcal{M}_{\mathrm{ces}(p'+)}$. By Proposition 4.3 this is equivalent to the condition:

$$\text{For each } r > p' \text{ there exists } s \in (p', r] \text{ such that } (a_n n^{(1/r)-(1/s)})_n \in \ell_\infty. \tag{4.9}$$

But, (4.9) is precisely equivalent to the statement in part (ii). Indeed, fix any $u \in (1,p)$. Then $r := u'$ satisfies $r > p'$. By (4.9) there exists $s \in (p', r]$ such that $(a_n n^{(1/r)-(1/s)})_n \in \ell_\infty$. Set $v := s'$ and note that $v \in [u,p)$. Since $\frac{1}{r} - \frac{1}{s} = \frac{1}{v} - \frac{1}{u}$, we can conclude that $(a_n n^{(1/v)-(1/u)})_n \in \ell_\infty$. This argument is clearly reversible. □

A similar duality argument as in the proof of Theorem 4.7 (using now (4.8) and Proposition 4.5 in place of (4.7) and Proposition 4.3) can be applied to establish the following result.

Theorem 4.8. *Let* $p \in [1,\infty)$ *and* $a = (a_n)_n \in \mathbb{C}^{\mathbb{N}}$. *The following conditions are equivalent.*

(i) $a \in \mathcal{M}_{d(p+)}$, *that is,* $M^a_{d(p+)} \in \mathcal{L}(d(p+))$.
(ii) *For each* $r > p$ *there exists* $s \in (p,r]$ *such that* $(a_n n^{(1/r)-(1/s)})_n \in \ell_\infty$.
(iii) *For each* $\eta \in (0,1)$ *the element* $(a_n n^{-\eta})_n \in \ell_\infty$.

Remark 4.9. Proposition 4.3, Remark 4.4 and Theorem 4.8 imply that

$$\mathcal{M}_{d(p+)} = \mathcal{M}_{\mathrm{ces}(p+)} = \bigcap_{0<\eta<1} \ell_\infty(\omega_\eta), \quad p \in [1,\infty). \tag{4.10}$$

Note that the final term in (4.10) is independent of p. Also, Proposition 4.5, Remark 4.6 and Theorem 4.7 show that

$$\mathcal{M}_{d(p-)} = \mathcal{M}_{\mathrm{ces}(p-)} = \bigcap_{1<u<p}(\bigcup_{u\leq v<p} \ell_\infty(w_{v,u})), \quad 1 < p \leq \infty. \tag{4.11}$$

Let X be a lcHs and $\mathcal{A} \subseteq \mathcal{L}(X)$ be a non-empty set. The *commutant* $\mathcal{A}^c$ of $\mathcal{A}$ is defined by

$$\mathcal{A}^c := \{T \in \mathcal{L}(X) : TS = ST \text{ for all } S \in \mathcal{A}\},$$

in which case $\mathcal{A}^c$ is a unital subalgebra of $\mathcal{L}(X)$, that is, $\mathcal{A}^c$ is a vector subspace of $\mathcal{L}(X)$ with $I \in \mathcal{A}^c$ and both RS and SR belong to $\mathcal{A}^c$ whenever $R, S \in \mathcal{A}^c$. In the event that $\mathcal{A}$ is commutative it is routine to check that $\mathcal{A} \subseteq \mathcal{A}^c$. If $\mathcal{A}$ satisfies $\mathcal{A} = \mathcal{A}^c$, then $\mathcal{A}$ is called *maximal abelian*. Of course, $\mathcal{A}$ is then necessarily commutative.

Theorem 4.10. (i) *For each* $p \in [1, \infty)$ *both of the commutative, unital subalgebras* $\mathcal{M}_{\mathrm{op}}(d(p+)) \subseteq \mathcal{L}(d(p+))$ *and* $\mathcal{M}_{\mathrm{op}}(\mathrm{ces}(p+)) \subseteq \mathcal{L}(\mathrm{ces}(p+))$ *are maximal abelian.*

(ii) *For each* $1 < p \leq \infty$ *both of the commutative, unital subalgebras* $\mathcal{M}_{\mathrm{op}}(d(p-)) \subseteq \mathcal{L}(d(p-))$ *and* $\mathcal{M}_{\mathrm{op}}(\mathrm{ces}(p-)) \subseteq \mathcal{L}(\mathrm{ces}(p-))$ *are maximal abelian.*

Proof. Fix $p \in [1, \infty)$. Since the canonical vectors $\{e_n : n \in \mathbb{N}\}$ are a Schauder basis (even unconditional) in $d(p+)$ (cf. Lemma 4.1(i) in [11]) each $x \in d(p+)$ has a unique expansion $x = \sum_{k=1}^\infty x_k e_k$ and the linear map $P_n : d(p+) \longrightarrow d(p+)$ defined by

$$P_n(x) := x_n e_n, \quad x \in d(p+), \tag{4.12}$$

is a continuous rank-1 projection; see [20, Section 4.2]. Moreover, $P_n P_m = P_m P_n = \delta_{mn} P_n$, for all $m, n \in \mathbb{N}$, and $\{P_n : n \in \mathbb{N}\} \subseteq \mathcal{M}_{\mathrm{op}}(d(p+))$ as $P_m = M^{\varphi_m}_{d(p+)}$, where $\varphi_m := (\delta_{mk})_k \in \ell_\infty \subseteq \mathcal{M}_{d(p+)}$ for each $m \in \mathbb{N}$. By (4.12) it is clear that $P_n(d(p+)) \subseteq \langle e_n \rangle := \{\lambda e_n : \lambda \in \mathbb{C}\}$ for each $e \in \mathbb{N}$.

Claim. $\{P_n : n \in \mathbb{N}\}^c = (\mathcal{M}_{\mathrm{op}}(d(p+)))^c$.

Since $\{P_n : n \in \mathbb{N}\} \subseteq \mathcal{M}_{\mathrm{op}}(d(p+))$, it is clear that $(\mathcal{M}_{\mathrm{op}}(d(p+)))^c \subseteq \{P_n : n \in \mathbb{N}\}^c$.

On the other hand, suppose that $T \in \mathcal{L}(d(p+))$ satisfies $TP_n = P_n T$ for each $n \in \mathbb{N}$. Fix $a \in \mathcal{M}_{d(p+)}$. Given any $x \in d(p+)$, the series $x = \sum_{k=1}^\infty x_k e_k$ converges (unconditionally) in $d(p+)$ and so, by the continuity of $M^a_{d(p+)} \in \mathcal{L}(d(p+))$ and (4.12), it follows that

$$M^a_{d(p+)}(x) = \sum_{k=1}^\infty x_k M^a_{d(p+)}(e_k) = \sum_{k=1}^\infty x_k a_k e_k = \sum_{k=1}^\infty a_k P_k(x). \tag{4.13}$$

Applying the continuous operator T to the convergent series in (4.13) yields

$$TM^a_{d(p+)}(x) = \sum_{k=1}^\infty a_k T P_k(x). \tag{4.14}$$

Replacing x in (4.13) with $T(x)$ yields

$$M^a_{d(p+)}(T(x)) = \sum_{k=1}^\infty a_k P_k(T(x)) = \sum_{k=1}^\infty a_k T(P_k(x)).$$

Comparing this identity with (4.14) shows that $TM^a_{d(p+)} = M^a_{d(p+)} T$. Since $a \in \mathcal{M}_{d(p+)}$ is arbitrary, it follows that $T \in (\mathcal{M}_{\mathrm{op}}(d(p+)))^c$. This establishes that $\{P_n : n \in \mathbb{N}\}^c \subseteq (\mathcal{M}_{\mathrm{op}}(d(p+)))^c$ and completes the proof of the *Claim*.

The commutativity of $\mathcal{M}_{\mathrm{op}}(d(p+))$ implies that $\mathcal{M}_{\mathrm{op}}(d(p+)) \subseteq (\mathcal{M}_{\mathrm{op}}(d(p+)))^c$.

To establish the reverse containment it suffices to show, because of the above *Claim*, that $T \in \mathcal{M}_{\mathrm{op}}(d(p+))$ whenever $T \in \mathcal{L}(d(p+))$ satisfies

$$TP_n = P_n T, \quad n \in \mathbb{N}. \tag{4.15}$$

Now, applying (4.15) to each basis vector $e_n \in d(p+)$ yields

$$T(e_n) = T(P_n(e_n)) = P_n(T(e_n)) \in \langle e_n \rangle, \quad n \in \mathbb{N},$$

that is, for each $n \in \mathbb{N}$, there exists a unique scalar $a_n \in \mathbb{C}$ such that $T(e_n) = a_n e_n$. Define $a := (a_n)_n$ in which case $M^a \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$.

Fix $x \in d(p+)$. Then $\lim_{m\to\infty} \sum_{k=1}^m x_k e_k = x$ in $d(p+)$. By the linearity and the continuity of T, also

$$\lim_{m\to\infty} \sum_{k=1}^m x_k T(e_k) = \lim_{m\to\infty} \sum_{k=1}^m a_k x_k e_k = T(x)$$

with convergence in $d(p+)$. Since $d(p+) \subseteq \mathbb{C}^{\mathbb{N}}$ *continuously* (cf. (4.1) in [11, Section 4]), it follows that also

$$\lim_{m\to\infty} \sum_{k=1}^m x_k e_k = x \ \text{ and } \ \lim_{m\to\infty} \sum_{k=1}^m a_k x_k e_k = T(x)$$

with both sequences converging in the Fréchet space $\mathbb{C}^{\mathbb{N}}$. But,

$$\sum_{k=1}^m a_k x_k e_k = M^a\Big(\sum_{k=1}^m x_k e_k\Big), \quad m \in \mathbb{N},$$

and so, by the continuity of $M^a \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$, also $\lim_{m\to\infty} \sum_{k=1}^m a_k x_k e_k = M^a(x)$ in $\mathbb{C}^{\mathbb{N}}$. By the uniqueness of limits of $\mathbb{C}^{\mathbb{N}}$ we can conclude that $M^a(x) = T(x)$, that is, $M^a(x) \in d(p+)$. Since $x \in d(p+)$ is arbitrary, we have established that $M^a(d(p+)) \subseteq d(p+)$, that is, $M^a_{d(p+)} \in \mathcal{M}_{\rm op}(d(p+))$. Moreover, we have seen that M^a and T coincide on the dense subspace $\operatorname{span}(\{e_n : n \in \mathbb{N}\})$ of $d(p+)$. Accordingly, $T = M^a_{d(p+)} \in \mathcal{M}_{\rm op}(d(p+))$. This completes the proof of $\mathcal{M}_{\rm op}(d(p+))$ being maximal abelian.

The proofs for the remaining three cases, namely $\mathcal{M}_{\rm op}(\operatorname{ces}(p+)), \mathcal{M}_{\rm op}(d(p-))$ and $\mathcal{M}_{\rm op}(\operatorname{ces}(p-))$, follow along the same line of that for $\mathcal{M}_{\rm op}(d(p+))$. One only requires that $\{e_n : n \in \mathbb{N}\}$ is a Schauder basis for each of the three spaces $\operatorname{ces}(p+), d(p-)$ and $\operatorname{ces}(p-)$, which is indeed the case (see, respectively, [3, Proposition 3.5(i)], [11, Theorem 4.6] and [7, Proposition 1(ii)]), and that each of these three spaces is continuously included in $\mathbb{C}^{\mathbb{N}}$, which is also the case; see, respectively, [11, Section 3], [11, Lemma 4.2(ii)] and [11, Proposition 3.5(i)]. □

Remark 4.11. (i) Let X be a lcHs and $T \in \mathcal{L}(X)$. If T is invertible, then $TT^{-1} = T^{-1}T$. Suppose that X is any one of the spaces $\operatorname{ces}(p+), d(p+)$ for $1 \le p < \infty$ or any one of the spaces $\operatorname{ces}(p-), d(p-)$ for $1 < p \le \infty$. The previous comment and Theorem 4.10 imply that $\mathcal{M}_{\rm op}(X)$ is an *inverse closed* subalgebra of $\mathcal{L}(X)$, that is, if $S \in \mathcal{M}_{\rm op}(X)$ is invertible in $\mathcal{L}(X)$, then $S^{-1} \in \mathcal{M}_{\rm op}(X)$. Indeed, $\mathcal{M}_{\rm op}(X)$ is commutative and so $S \in \mathcal{M}_{\rm op}(X)$ implies that $SA = AS$ for all $A \in \mathcal{M}_{\rm op}(X)$. Hence, $S^{-1}(SA)S^{-1} = S^{-1}(AS)S^{-1}$, that is, $AS^{-1} = S^{-1}A$ for all $A \in \mathcal{M}_{\rm op}(X)$. This shows that $S^{-1} \in (\mathcal{M}_{\rm op}(X))^c = \mathcal{M}_{\rm op}(X)$.

(ii) The subalgebra $\mathcal{M}_{\rm op}(X)$ is topologically *closed* in the lcHs $\mathcal{L}_s(X)$. Indeed, let $\{T_\alpha\}_\alpha \subseteq \mathcal{M}_{\rm op}(X)$ be any net which is τ_s-convergent to $T \in \mathcal{L}(X)$. Fix any $S \in \mathcal{M}_{\rm op}(X)$. Then, for each $x \in X$, it is the case that

$$S(T(x)) = \lim_\alpha S(T_\alpha(x)) = \lim_\alpha T_\alpha(S(x)) = T(S(x)).$$

Accordingly, $ST = TS$ and so, by Theorem 4.10, $T \in (\mathcal{M}_{\rm op}(X))^c = \mathcal{M}_{\rm op}(X)$. Since the τ_b-closure of $\mathcal{M}_{\rm op}(X)$ is contained in the τ_s-closure of $\mathcal{M}_{\rm op}(X)$, it follows that $M_{\rm op}(X)$ is closed in the lcHs $\mathcal{L}_b(X)$. □

Turning to the compactness of multiplication operators we first recall what is known for the spaces ces($p+$) and ces($p-$); see, respectively, Lemma 8, Proposition 10 of [5] and Proposition 19, Remark 20 of [7]. The main ingredient for the proofs of these results are Lemma 2.1(ii) above for ces($p+$) and Lemma 2.2(ii) above for ces($p-$).

Proposition 4.12. (i) *Let* $p \in [1,\infty)$ *and* $a \in \mathcal{M}_{\mathrm{ces}(p+)}$. *The following assertions are equivalent.*

(a) *The multiplication operator* $M^a_{\mathrm{ces}(p+)} \in \mathcal{L}(\mathrm{ces}(p+))$ *is compact.*
(b) *There exists* $q > p$ *such that for all* $r \in (p,q)$ *we have that* $a \in \mathcal{M}_{\mathrm{ces}(q),\mathrm{ces}(r)}$.
(c) $a \in d(\infty-) = \bigcup_{s>1} d(s) \subseteq c_0$.

(ii) *Let* $1 < p \leq \infty$ *and* $a \in \mathcal{M}_{\mathrm{ces}(p-)}$. *The following assertions are equivalent.*

(a) *The multiplication operator* $M^a_{\mathrm{ces}(p-)} \in \mathcal{L}(\mathrm{ces}(p-))$ *is compact.*
(b) *There exists* $t > p'$ *such that* $\hat{a} = (\sup_{k\geq n}|a_k|)_n \in \ell_t$, *that is,* $a \in d(t)$.
(c) *The element* $a \in d(\infty-) := \bigcup_{q>1} d(q) = \bigcup_{t>p'} d(t) \subseteq c_0$.

Note that the space $d(\infty-)$ in (c) of parts (i) and (ii) of Proposition 4.12 is *independent* of p. The containment $d(\infty-) \subseteq c_0$ is *proper*, [5, Remark 4(i)].

Since the spaces ces($p+$), $d(p+)$, ces($p-$) and $d(p-)$ are all Montel, there is no distinction between compact and bounded operators in these spaces.

Theorem 4.13. (i) *Let* $p \in [1,\infty)$ *and* $a \in \mathcal{M}_{d(p+)}$. *Then the multiplication operator* $M^a_{d(p+)} \in \mathcal{L}(d(p+))$ *is compact if and only if* $a \in d(\infty-)$.

(ii) *Let* $1 < p \leq \infty$ *and* $a \in \mathcal{M}_{d(p-)}$. *Then the multiplication operator* $M^a_{d(p-)} \in \mathcal{L}(d(p-))$ *is compact if and only if* $a \in d(\infty-)$.

Proof. (i) We know by (4.8) that $M^a_{d(p+)} \in \mathcal{L}(d(p+))$ if and only if $M^a_{\mathrm{ces}(p'-)} = (M^a_{d(p+)})' \in \mathcal{L}(\mathrm{ces}(p'-))$. Moreover, Lemma 2.3 implies that $M^a_{d(p+)}$ is compact if and only if $M^a_{\mathrm{ces}(p'-)}$ is compact. But, $M^a_{\mathrm{ces}(p'-)}$ is compact if and only if $a \in d(\infty-)$; see Proposition 4.12(ii).

(ii) Replacing (4.8) with (4.7) and Proposition 4.12(ii) with Proposition 4.12(i), the proof of part (i) can be modified to also apply to part (ii). □

Remark 4.14. It follows from Theorem 4.13 and Proposition 4.12 that

$$\mathcal{M}^{cpt}_{d(p+)} = \mathcal{M}^{cpt}_{\mathrm{ces}(p+)} = d(\infty-) \ \text{ for } \ p \in [1,\infty)$$

and that

$$\mathcal{M}^{cpt}_{d(p-)} = \mathcal{M}^{cpt}_{\mathrm{ces}(p-)} = d(\infty-) \ \text{ for } \ 1 < p \leq \infty,$$

where the superscript *cpt* indicates those multipliers in the given space whose corresponding multiplication operator is *compact.* □

We now consider the spectra of multiplication operators in $d(p+)$, $d(p-)$, for which we first require the following result; see Proposition 7 of [5] and Proposition 22 of [7]. Since each space ces($p+$), $1 \leq p < \infty$, is a Köthe echelon space (see [3]), part (i) of the following result can also be deduced from Lemma 2, Corollary 1 and Theorem 1 of the recent paper [26]. Let $\mathbb{1} := (1,1,\ldots) \in \ell_\infty$. If $b \in \mathbb{C}^{\mathbb{N}}$ satisfies $b_n \neq 0$ for all $n \in \mathbb{N}$, we write $b^{-1} := (1/b_n)_n$.

Proposition 4.15. (i) *Let* $p \in [1,\infty)$ *and* $a = (a_n)_n \in \mathcal{M}_{\mathrm{ces}(p+)}$.

(a) $\sigma_{pt}(M^a_{\mathrm{ces}(p+)}) = \{a_n : n \in \mathbb{N}\}$.

(b) $\overline{\sigma(M^a_{\mathrm{ces}(p+)})} = \sigma^*(M^a_{\mathrm{ces}(p+)}) = \overline{\{a_n : n \in \mathbb{N}\}}$.

(c) *For each* $\lambda \in \rho(M^a_{\mathrm{ces}(p+)})$ *the element* $(\lambda\mathbb{1} - a)^{-1} \in \mathcal{M}_{\mathrm{ces}(p+)}$ *and*

$$(\lambda I - M^a_{\mathrm{ces}(p+)})^{-1} = (M^{\lambda\mathbb{1}-a}_{\mathrm{ces}(p+)})^{-1} = M^{(\lambda\mathbb{1}-a)^{-1}}_{\mathrm{ces}(p+)}.$$

(ii) *Let* $1 < p \leq \infty$ *and* $a = (a_n)_n \in \mathcal{M}_{\mathrm{ces}(p-)}$.

(a) $\sigma_{pt}(M^a_{\mathrm{ces}(p-)}) = \{a_n := n \in \mathbb{N}\}$.

(b) $\overline{\sigma(M^a_{\mathrm{ces}(p-)})} = \sigma^*(M^a_{\mathrm{ces}(p-)}) = \overline{\{a_n : n \in \mathbb{N}\}}$.

(c) *For each* $\lambda \in \rho(M^a_{\mathrm{ces}(p-)})$ *the element* $(\lambda\mathbb{1} - a)^{-1} \in \mathcal{M}_{\mathrm{ces}(p-)}$ *and*

$$(\lambda I - M^a_{\mathrm{ces}(p-)})^{-1} = (M^{\lambda\mathbb{1}-a}_{\mathrm{ces}(p-)})^{-1} = M^{(\lambda\mathbb{1}-a)^{-1}}_{\mathrm{ces}(p-)}.$$

We are now able to treat the cases of $d(p+)$ and $d(p-)$. Noting that each space $d(p+)$, $1 \leq p < \infty$, is a Köthe echelon space, [11, Corollary 4.5(ii)], the following result can be deduced from Lemma 2, Corollary 1 and Theorem 1 of [26]. We indicate a proof based on the methods of this paper.

Theorem 4.16. *Let* $p \in [1, \infty)$ *and* $a = (a_n)_n \in \mathcal{M}_{d(p+)}$.

(i) $\sigma_{pt}(M^a_{d(p+)}) = \{a_n : n \in \mathbb{N}\}$.

(ii) $\overline{\sigma(M^a_{d(p+)})} = \sigma^*(M^a_{d(p+)}) = \overline{\{a_n : n \in \mathbb{N}\}}$.

(iii) *For each* $\lambda \in \rho(M^a_{d(p+)})$ *the element* $(\lambda\mathbb{1} - a)^{-1} \in \mathcal{M}_{d(p+)}$ *and*

$$(\lambda I - M^a_{d(p+)})^{-1} = (M^{\lambda\mathbb{1}-a}_{d(p+)})^{-1} = M^{(\lambda\mathbb{1}-a)^{-1}}_{d(p+)}.$$

Proof. (i) Since $M^a_{d(p+)}(e_n) = a_n e_n$ for each $n \in \mathbb{N}$, it is clear that $\{a_n : n \in \mathbb{N}\} \subseteq \sigma_{pt}(M^a_{d(p+)})$. On the other hand, if $\lambda \in \mathbb{C}$ satisfies $M^a_{d(p+)}(x) = \lambda x$ (i.e., $a_n x_n = \lambda x_n$ for all $n \in \mathbb{N}$) for some *non-zero* $x \in d(p+)$, then $\lambda \in \{a_n : n \in \mathbb{N}\}$.

(ii) Since $(\mathrm{ces}(p'-))'_\beta = d(p+)$ and $M^a_{d(p+)} = (M^a_{\mathrm{ces}(p'-)})'$, it follows from Proposition 2.6(i) that $\sigma(M^a_{d(p+)}) = \sigma(M^a_{\mathrm{ces}(p'-)})$. Then Proposition 4.15(ii), with p' in place of p, implies that

$$\overline{\sigma(M^a_{d(p+)})} = \overline{\sigma(M^a_{\mathrm{ces}(p'-)})}) = \sigma^*(M^a_{\mathrm{ces}(p'-)}) = \overline{\{a_n : n \in \mathbb{N}\}}.$$

This is precisely the statement of (ii) as $\sigma^*(M^a_{\mathrm{ces}(p'-)}) = \sigma^*(M^a_{d(p+)})$ by Proposition 2.6(ii).

(iii) Let $\lambda \in \rho(M^a_{d(p+)})$, in which case $\lambda \in \rho(M^a_{\mathrm{ces}(p'-)})$ by Proposition 2.6(i). According to Proposition 4.15(ii)(c) we have that $(\lambda\mathbb{1} - a)^{-1} \in \mathcal{M}_{\mathrm{ces}(p'-)}$ and $(M^{\lambda\mathbb{1}-a}_{\mathrm{ces}(p'-)})^{-1} = M^{(\lambda\mathbb{1}-a)^{-1}}_{\mathrm{ces}(p'-)}$. Then (4.8) shows that $(\lambda\mathbb{1} - a)^{-1} \in \mathcal{M}_{d(p+)}$ and so

$$M^{(\lambda\mathbb{1}-a)^{-1}}_{d(p+)} = (M^{(\lambda\mathbb{1}-a)^{-1}}_{\mathrm{ces}(p'-)})' = ((\lambda I - M^a_{\mathrm{ces}(p'-)})^{-1})' = (\lambda I - M^a_{d(p+)})^{-1},$$

where the last equality is a consequence of (2.1). □

Theorem 4.17. *Let* $1 < p \leq \infty$ *and* $a = (a_n)_n \in \mathcal{M}_{d(p-)}$.

(i) $\sigma_{pt}(M^a_{d(p-)}) = \{a_n : n \in \mathbb{N}\}$.

(ii) $\overline{\sigma(M^a_{d(p-)})} = \sigma^*(M^a_{d(p-)}) = \overline{\{a_n : n \in \mathbb{N}\}}$.

(iii) *For each* $\lambda \in \rho(M^a_{d(p-)})$ *the element* $(\lambda\mathbb{1} - a)^{-1} \in \mathcal{M}_{d(p-)}$ *and*

$$(\lambda I - M^a_{d(p-)})^{-1} = (M^{\lambda\mathbb{1}-a}_{d(p-)})^{-1} = M^{(\lambda\mathbb{1}-a)^{-1}}_{d(p-)}.$$

Proof. Noting that $d(p-) = (\mathrm{ces}(p'+))'_\beta$ and $M^a_{d(p-)} = (M^a_{\mathrm{ces}(p'+)})'$, one can argue analogously to the proof of Theorem 4.16. One only needs to replace (4.8) and Proposition 4.15(ii) used there by (4.7) and Proposition 4.15(i) in the current setting. □

We now turn our attention to the *mean ergodic* properties of multiplication operators in $d(p+)$ and $d(p-)$. We first record the known results for $\mathrm{ces}(p+)$ and $\mathrm{ces}(p-)$; see [5, Proposition 21], [26, Section 3], for $\mathrm{ces}(p+)$ and [7, Proposition 21] for $\mathrm{ces}(p-)$. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

Proposition 4.18. (i) *Let $p \in [1, \infty)$ and $a \in \mathcal{M}_{\mathrm{ces}(p+)}$. The following conditions are equivalent.*

- (a) *$M^a_{\mathrm{ces}(p+)} \in \mathcal{L}(\mathrm{ces}(p+))$ is power bounded.*
- (b) *$M^a_{\mathrm{ces}(p+)} \in \mathcal{L}(\mathrm{ces}(p+))$ is mean ergodic.*
- (c) *$M^a_{\mathrm{ces}(p+)} \in \mathcal{L}(\mathrm{ces}(p+))$ is uniformly mean ergodic.*
- (d) *$a \in \ell_\infty$ and $\|a\|_\infty \leq 1$.*
- (e) *$\sigma(M^a_{\mathrm{ces}(p+)}) \subseteq \overline{\mathbb{D}}$.*

(ii) *Let $1 < p \leq \infty$ and $a \in \mathcal{M}_{\mathrm{ces}(p-)}$. The following conditions are equivalent.*

- (a) *$M^a_{\mathrm{ces}(p-)} \in \mathcal{L}(\mathrm{ces}(p-))$ is power bounded.*
- (b) *$M^a_{\mathrm{ces}(p-)} \in \mathcal{L}(\mathrm{ces}(p-))$ is mean ergodic.*
- (c) *$M^a_{\mathrm{ces}(p-)} \in \mathcal{L}(\mathrm{ces}(p-))$ is uniformly mean ergodic.*
- (d) *$a \in \ell_\infty$ and $\|a\|_\infty \leq 1$.*
- (e) *$\sigma(M^a_{\mathrm{ces}(p-)}) \subseteq \overline{\mathbb{D}}$.*

Remark 4.19. The condition (e) of part (ii) in Proposition 4.18 does not actually occur in Proposition 21 of [7]. However, in view of Proposition 4.15(ii)(b), it is clear that (d) and (e) in part (ii) of Proposition 4.18 *are equivalent*. □

Concerning the proof of Proposition 4.18(i) above, as given in [5, Proposition 21], some relevant comments are in order. An important ingredient is that the Fréchet space $\mathrm{ces}(p+), 1 \leq p < \infty$, is Montel (and hence, also reflexive). Being a Fréchet-Schwartz space, the *same* property is true for $d(p+), 1 \leq p < \infty$. Also important is that the norm $\|\cdot\|_{\mathrm{ces}(q)}$ in each Banach space $\mathrm{ces}(q), 1 < q < \infty$, is a *lattice norm*, that is, if $x, y \in \mathrm{ces}(q)$ satisfy $|x| \leq |y|$, then $\|x\|_{\mathrm{ces}(q)} \leq \|y\|_{\mathrm{ces}(q)}$. The same is true for the norm $\|\cdot\|_{d(s)}$ in each Banach space $d(s), 1 < s < \infty$. Indeed, if $x, y \in d(s)$ satisfy $|x| \leq |y|$, then $0 \leq \hat{x} \leq \hat{y}$ in ℓ_s and so $\|\hat{x}\|_s \leq \|\hat{y}\|_s$, that is, $\|x\|_{d(s)} \leq \|y\|_{d(s)}$. Finally, the proof of $(d) \iff (e)$ above (i.e., of (iv) $\iff$ (v) in [5]) relies on part (b) of Proposition 4.15(i). These identities also hold in $d(p+)$; see Theorem 4.16(ii). The previous observations show that the proof of Proposition 21 (for ($\mathrm{ces}(p+)$)) given in [5] can be easily adapted to establish the following result. Or, one can apply the results of [26, Section 3] to the Köthe echelon spaces $d(p+)$.

Theorem 4.20. *Let $p \in [1, \infty)$ and $a \in \mathcal{M}_{d(p+)}$. The following conditions are equivalent.*

- (i) *$M^a_{d(p+)} \in \mathcal{L}(d(p+))$ is power bounded.*
- (ii) *$M^a_{d(p+)} \in \mathcal{L}(d(p+))$ is mean ergodic.*
- (iii) *$M^a_{d(p+)} \in \mathcal{L}(d(p+))$ is uniformly mean ergodic.*
- (iv) *$a \in \ell_\infty$ and $\|a\|_\infty \leq 1$.*
- (v) *$\sigma(M^a_{d(p+)}) \subseteq \overline{\mathbb{D}}$.*

The proof of Proposition 4.18(ii) above, as given in [7, Proposition 21], relies on the following ingredients. First, that ces($p-$) is barrelled and Montel, which is also the case for $d(p-)$ as it is a (DFS)-space. Second, that the operator norm of M^a, for $a \in \ell_\infty$, when acting in the Banach space ces(q), $1 < q < \infty$, satisfies $\|M^a_{\mathrm{ces}(q)}\|_{\mathrm{op}} \leq \|a\|_\infty$. The same is true in the Banach spaces $d(s)$, $1 < s < \infty$, that is, $\|M^a_{d(s)}\|_{\mathrm{op}} \leq \|a\|_\infty$; see Remark 4.2. Finally, it is used in [7] that $\mathrm{ces}(p-) \subseteq \mathbb{C}^{\mathbb{N}}$ with a continuous inclusion; the analogous continuous inclusion $d(p-) \subseteq \mathbb{C}^{\mathbb{N}}$ is also valid, [11, Lemma 4.2(ii)]. Accordingly, the proof of parts (i)–(iv) in the following result follow by appropriately modifying the proof of Proposition 21 in [7]. The proof of the equivalence (iv) $\Leftrightarrow$ (v) in the following result is immediate from Theorem 4.17(ii) above.

Theorem 4.21. *Let $1 < p \leq \infty$ and $a \in \mathcal{M}_{d(p-)}$. The following conditions are equivalent.*

(i) *$M^a_{d(p-)} \in \mathcal{L}(d(p-))$ is power bounded.*
(ii) *$M^a_{d(p-)} \in \mathcal{L}(d(p-))$ is mean ergodic.*
(iii) *$M^a_{d(p-)} \in \mathcal{L}(d(p-))$ is uniformly mean ergodic.*
(iv) *$a \in \ell_\infty$ and $\|a\|_\infty \leq 1$.*
(v) *$\sigma(M^a_{d(p-)}) \subseteq \overline{\mathbb{D}}$.*

Let $X \in \{d(p+), \mathrm{ces}(p+) : p \in [1, \infty)\} \cup \{d(p-); \mathrm{ces}(p-) : 1 < p \leq \infty\}$, in which case the notation $a \in \mathcal{M}_X$ with $M^a_X \in \mathcal{L}(X)$ is clear.

Theorem 4.22. *Let $a \in \mathcal{M}_X$. Then $M^a_X \in \mathcal{L}(X)$ is not supercyclic.*

Proof. Suppose that $a \in \mathcal{M}_X$ is *not* constant. Then $(M^a_X)' = M^a_{X'_\beta}$ has at least two linearly independent eigenvectors; see parts (i)(a) and (ii)(a) of Proposition 4.15 and part (i) of each of Theorem 4.16 and Theorem 4.17. Since supercyclic is the same as being 1-supercyclic in the sense of [12], it follows from Theorem 2.1 of [12] that M^a_X is *not* supercyclic.

If $a = \alpha\mathbb{1}$, for some $\alpha \in \mathbb{C}$, then it follows for all choices of X that $M^a_X = \alpha I_X$ and so $(M^a_X)^n = \alpha^n I_X$ for all $n \in \mathbb{N}_0$. It follows that $\{\lambda(M^a_X)^n(x) : \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\} \subseteq \mathrm{span}(\{x\})$ for each $x \in X$. So, M^a_X is surely not supercyclic. □

5. Inclusion maps and Cesàro operators

Consider any pair $1 \leq p, q < \infty$. Let X_{p+} denote any one of the spaces $d(p+), \mathrm{ces}(p+)$ or ℓ_{p+} and let Y_{q+} denote any one of the spaces $d(q+), \mathrm{ces}(q+)$ or ℓ_{q+}. If $\mathbb{1} \in \mathcal{M}_{X_{p+}, Y_{q+}}$, then the multiplication operator $M^{\mathbb{1}}_{X_{p+}, Y_{q+}}$ is precisely the natural (identity) *inclusion map* of X_{p+} into Y_{q+}, which we denote by $i_{X_{p+}, Y_{q+}}$. In this case we say that $i_{X_{p+}, Y_{q+}}$ *exists*; it is then necessarily continuous by the closed graph theorem.

For any pair $1 < p, q \leq \infty$, with X_{p-} denoting any one of the spaces $d(p-), \mathrm{ces}(p-)$ or ℓ_{p-} and Y_{q-} denoting any one of the spaces $d(q-), \mathrm{ces}(q-)$ or ℓ_{q-}, the analogous notation $i_{X_{p-}, Y_{q-}}$ is adopted for $M^{\mathbb{1}}_{X_{p-}, Y_{q-}}$ whenever $\mathbb{1} \in \mathcal{M}_{X_{p-}, Y_{q-}}$.

Except for the pairs of spaces $d(p+), d(q+)$ and $d(p-), d(q-)$ it is known precisely when the maps $i_{X_{p+}, Y_{q+}}$ and $i_{X_{p-}, X_{q-}}$ exist, as recorded in the following result.

Proposition 5.1. (i) *Let $1 \leq p, q < \infty$ be an arbitrary pair.*

(a) *The inclusion map $i_{\ell_{p+}, \ell_{q+}} : \ell_{p+} \longrightarrow \ell_{q+}$ exists if and only if $p \leq q$.*

(b) *The inclusion map* $i_{\ell_{p+},\mathrm{ces}(q+)} : \ell_{p+} \longrightarrow \mathrm{ces}(q+)$ *exists if and only if* $p \leq q$.
(c) *The inclusion map* $i_{\mathrm{ces}(p+),\mathrm{ces}(q+)} : \mathrm{ces}(p+) \longrightarrow \mathrm{ces}(q+)$ *exists if and only if* $p \leq q$.
(d) $\mathrm{ces}(p+) \nsubseteq \ell_{q+}$ *for all choices of* $1 \leq p, q < \infty$.

(ii) *Let* $1 < p, q \leq \infty$ *be an arbitrary pair.*

(a) *The inclusion map* $i_{\ell_{p-},\ell_{q-}} : \ell_{p-} \longrightarrow \ell_{q-}$ *exists if and only if* $p \leq q$.
(b) *The inclusion map* $i_{\ell_{p-},\mathrm{ces}(q-)} : \ell_{p-} \longrightarrow \mathrm{ces}(q-)$ *exists if and only if* $p \leq q$.
(c) *The inclusion map* $i_{\mathrm{ces}(p-),\mathrm{ces}(q-)} : \mathrm{ces}(p-) \longrightarrow \mathrm{ces}(q-)$ *exists if and only if* $p \leq q$.
(d) $\mathrm{ces}(p-) \nsubseteq \ell_{q-}$ *for all choices of* $1 < p, q \leq \infty$.

For part (i) of Proposition 5.1 we refer to [5, Proposition 27] and for part (ii) of Proposition 5.1 see [7, Proposition 25].

We will require the following facts.

Lemma 5.2. (i) $\ell_{1+} \nsubseteq d(t)$ *for each* $1 < t < \infty$.
(ii) $\ell_{1+} \nsubseteq d(q+)$ *for each* $q \in [1, \infty)$.
(iii) $\ell_1 \nsubseteq d(\infty-)$.
(iv) $\ell_{p-} \nsubseteq d(\infty-)$ *for each* $1 < p \leq \infty$.

Proof. (i) Fix $t > 1$. By the proof of Remark 2.8(i) in [10] there exists $0 \leq x \in \ell_1$ such that $x \notin d(t)$. Since $\ell_1 \subseteq \ell_{1+}$, it follows that $x \in \ell_{1+}$ but $x \notin d(t)$.

(ii) Given $q \in [1, \infty)$ select any $t > q$. By part (i) there exists $x \in \ell_{1+}$ such that $x \notin d(t)$. Since $d(q+) \subseteq d(t)$, also $x \notin d(q+)$.

(iii) Recall that $d(\infty-) = \bigcup_{n=1}^{\infty} d(n)$. For each $n \in \mathbb{N}$, it follows from the proof of part (i) that there exists $0 \leq x^{[n]} \in \ell_1$ such that $x^{[n]} \notin d(n)$. Dividing by $\|x^{[n]}\|_1$, if necessary, we can assume that $\|x^{[n]}\|_1 = 1$ for each $n \in \mathbb{N}$. Define $x := \sum_{n=1}^{\infty} 2^{-n} x^{[n]}$, in which case $0 \leq x \in \ell_1$. Moreover, $x \notin d(n)$ for every $n \in \mathbb{N}$. Indeed, if $x \in d(n_0)$ for some $n_0 \in \mathbb{N}$, then $0 \leq x^{[n_0]} 2^{-n_0} \leq x$ implies that $x^{[n_0]} 2^{-n_0} \in d(n_0)$ as $d(n_0)$ is *solid*. Then also $x^{[n_0]} \in d(n_0)$ which is a contradiction. So, $x \notin d(\infty-)$.

(iv) Fix $1 < p \leq \infty$. By part (iii) there exists $x \in \ell_1$ such that $x \notin d(\infty-)$. Since $\ell_1 \subseteq \ell_{p-} = \bigcup_{1<r<p} \ell_r$, it follows that $x \in \ell_{p-}$. □

As in the previous section, duality is again relevant. Due to the reflexivity of all the spaces involved we note that $i_{X_{p+},Y_{q+}} : X_{p+} \longrightarrow Y_{q+}$ exists if and only if its dual operator $(i_{X_{p+},Y_{q+}})'$ exists, that is, if and only if $i_{Y_{q'-},X_{p'-}} : Y_{q'-} \longrightarrow X_{p'-}$ exists. The following result consists of the cases not covered by Proposition 5.1.

Theorem 5.3. (i) *Let* $1 \leq p, q < \infty$ *be an arbitrary pair.*

(a) *The inclusion map* $i_{d(p+),d(q+)} : d(p+) \longrightarrow d(q+)$ *exists if and only if* $p \leq q$.
(b) *The inclusion map* $i_{d(p+),\ell_{q+}} : d(p+) \longrightarrow \ell_{q+}$ *exists if and only if* $p \leq q$.
(c) *The inclusion map* $i_{d(p+),\mathrm{ces}(q+)} : d(p+) \longrightarrow \mathrm{ces}(q+)$ *exists if and only if* $p \leq q$.
(d) $\ell_{p+} \nsubseteq d(q+)$ *and* $\mathrm{ces}(p+) \nsubseteq d(q+)$ *for all choices of* $1 \leq p, q < \infty$.

(ii) *Let* $1 < p, q \leq \infty$ *be an arbitrary pair.*

(a) *The inclusion map* $i_{d(p-),d(q-)} : d(p-) \longrightarrow d(q-)$ *exists if and only if* $p \leq q$.
(b) *The inclusion map* $i_{d(p-),\ell_{q-}} : d(p-) \longrightarrow \ell_{q-}$ *exists if and only if* $p \leq q$.
(c) *The inclusion map* $i_{d(p-),\mathrm{ces}(q-)} : d(p-) \longrightarrow \mathrm{ces}(q-)$ *exists if and only if* $p \leq q$.
(d) $\ell_{p-} \nsubseteq d(q-)$ *and* $\mathrm{ces}(p-) \nsubseteq d(q-)$ *for all choices of* $1 < p, q, \leq \infty$.

Proof. (i) (a) By the discussion prior to Theorem 5.3 and the fact that $d(p+) = (\mathrm{ces}(p'-))'_\beta$ and $d(q+) = (\mathrm{ces}(q'-))'_\beta$, it follows that $i_{d(p+),d(q+)}$ exists if and only if $i_{\mathrm{ces}(q'-),\mathrm{ces}(p'-)}$ exists which, by Proposition 5.1(ii)(c) is the case if and only if $q' \le p'$, that is, if and only if $p \le q$.

(b) Since $d(p+) = (\mathrm{ces}(p'-))'_\beta$ and $\ell_{q+} = (\ell_{q'-})'_\beta$, an analogous argument as in (a), but now using Proposition 5.1(ii)(b), establishes that $i_{d(p+),\ell_{q+}}$ exists if and only if $p \le q$.

(c) Suppose that $p \le q$. By part (b) the inclusion map $i_{d(p+),\ell_{q+}}$ exists and by Proposition 5.1(i)(b) the inclusion map $i_{\ell_{q+},\mathrm{ces}(q+)}$ exists. Hence, the composition $i_{d(p+),\mathrm{ces}(q+)} = i_{\ell_{q+},\mathrm{ces}(q+)} \circ i_{d(p+),\ell_{q+}}$ exists.

If $p > q$, select any $r \in (p, q)$. Then Proposition 2.1(ii)(f) of [11] implies that $d(p) \not\subseteq \mathrm{ces}(r)$. Since $d(p) \subseteq d(p+)$ and $\mathrm{ces}(q+) \subseteq \mathrm{ces}(r)$, it follows that $d(p+) \not\subseteq \mathrm{ces}(q+)$, that is, $d(p+)$ is *not* contained in $\mathrm{ces}(q+)$.

(d) Suppose there exists a pair $1 \le p, q < \infty$ such that $\ell_{p+} \subseteq d(q+)$. Since $\ell_{1+} \subseteq \ell_{p+}$, it follows that $\ell_{1+} \subseteq d(q+)$ which contradicts Lemma 5.2(ii). So, $\ell_{p+} \not\subseteq d(q+)$ for all $1 \le p, q < \infty$. Moreover, as $\ell_{p+} \subseteq \mathrm{ces}(p+)$ for all $1 \le p < \infty$ (cf. Proposition 5.1(b)), it follows that also $\mathrm{ces}(p+) \not\subseteq d(q+)$ for all $1 \le p, q < \infty$.

(ii) (a) By the discussion prior to Theorem 5.3 and the fact that $d(p-) = (\mathrm{ces}(p'+))'_\beta$ and $d(q-) = (\mathrm{ces}(q'+))'_\beta$, it follows that $i_{d(p-),d(q-)}$ exists if and only if $i_{\mathrm{ces}(q'+),\mathrm{ces}(p'+)}$ exists. By Proposition 5.1(i)(c) this is the case if and only if $q' \le p'$, that is, if and only if $p \le q$.

(b) Since $d(p-) = (\mathrm{ces}(p'+))'_\beta$ and $\ell_{q-} = (\ell_{q'+})'_\beta$, a similar argument as in (a), but now using Proposition 5.1(i)(b), implies that $i_{d(p-),\ell_{q-}}$ exists if and only if $p \le q$.

(c) Since $d(p-) = (\mathrm{ces}(p'+))'_\beta$ and $\mathrm{ces}(q-) = (d(q'+))'_\beta$, a similar duality argument as in (a), but now using part (i)(c) of this theorem, shows that $i_{d(p-),\mathrm{ces}(q-)}$ exists if and only if $p \le q$.

(d) Fix any pair $1 < p, q \le \infty$. According to Lemma 5.2(iv) there exists $x \in \ell_{p-}$ such that $x \notin d(\infty-)$. Since $d(q-) \subseteq d(\infty-)$, it follows that $x \notin d(q-)$. □

We now turn our attention to the boundedness/compactness of various inclusion maps. For part (i) of the following result see [5, Proposition 27] and for part (ii) see [7, Proposition 26]. Recall that the spaces $\mathrm{ces}(p+), \mathrm{ces}(p-)$ are Montel whereas the spaces ℓ_{p+}, ℓ_{p-} are *not* Montel.

Proposition 5.4. (i) *Let* $1 \le p \le q < \infty$ *be an arbitrary pair.*

(a) *The inclusion map* $i_{\ell_{p+},\ell_{q+}} : \ell_{p+} \longrightarrow \ell_{q+}$ *is bounded if and only if* $p < q$. *However,* $i_{\ell_{p+},\ell_{q+}}$ *is never compact.*

(b) *The inclusion map* $i_{\ell_{p+},\mathrm{ces}(q+)} : \ell_{p+} \longrightarrow \mathrm{ces}(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.

(c) *The inclusion map* $i_{\mathrm{ces}(p+),\mathrm{ces}(q+)} : \mathrm{ces}(p+) \longrightarrow \mathrm{ces}(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.

(d) *The inclusion map* $i_{\ell_{p+},\mathrm{ces}(p+)} : \ell_{p+} \longrightarrow \mathrm{ces}(p+)$ *is not bounded.*

(ii) *Let* $1 < p \le q \le \infty$ *be an arbitrary pair.*

(a) *The inclusion map* $i_{\ell_{p-},\ell_{q-}} : \ell_{p-} \longrightarrow \ell_{q-}$ *is bounded if and only if* $p < q$. *However,* $i_{\ell_{p-},\ell_{q-}}$ *is never compact.*

(b) *The inclusion map* $i_{\ell_{p-},\mathrm{ces}(q-)} : \ell_{p-} \longrightarrow \mathrm{ces}(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.

(c) *The inclusion map* $i_{\mathrm{ces}(p-),\mathrm{ces}(q-)} : \mathrm{ces}(p-) \longrightarrow \mathrm{ces}(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.

Remark 5.5. Concerning part (i)(b) of Proposition 5.4, it *only* follows from Proposition 27(iii) of [5] that $i_{\ell_{p+},\mathrm{ces}(q+)}$ is bounded whenever $p < q$. However, for $p > q$ there is no inclusion of ℓ_{p+} into $\mathrm{ces}(q+)$; see Proposition 5.1(i)(b). For the case of $p = q$, the inclusion map $i_{\ell_{p+},\mathrm{ces}(p+)}$ is *not* bounded. If so, by Lemma 2.1(ii), for $T := i_{\ell_{p+},\mathrm{ces}(p+)}$ with $X_k := \ell_{p+(1/k)}, k \in \mathbb{N}$, and $Y_m := \mathrm{ces}(p + \frac{1}{m}), m \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that the natural inclusion $\ell_{p+(1/k_0)} \subseteq \mathrm{ces}(p + \frac{1}{m})$ is continuous for all $m \in \mathbb{N}$. But, for $m := 1 + k_0$ we have that $(p+\frac{1}{k_0}) > (p+\frac{1}{m})$ which yields a contradiction to [11, Proposition 2.1(ii)(d)]. Accordingly, the inclusion $\ell_{p+} \subseteq \mathrm{ces}(p+)$ is *not* bounded and hence, also *not* compact. □

We now present an analogue of Proposition 5.4 which admits the Montel spaces $d(p+), p \in [1,\infty)$, and $d(p-), 1 < p \leq \infty$.

Theorem 5.6. (i) *Let* $1 \leq p \leq q < \infty$ *be an arbitrary pair.*

(a) *The inclusion map* $i_{d(p+),d(q+)} : d(p+) \longrightarrow d(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.

(b) *The inclusion map* $i_{d(p+),\ell_{q+}} : d(p+) \longrightarrow \ell_{q+}$ *is compact if and only if* $p < q$. *Moreover,* $i_{d(p+),\ell_{q+}}$ *is bounded if and only if* $p < q$.

(c) *The inclusion map* $i_{d(p+),\mathrm{ces}(q+)} : d(p+) \longrightarrow \mathrm{ces}(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.

(ii) *Let* $1 < p \leq q \leq \infty$ *be an arbitrary pair.*

(a) *The inclusion map* $i_{d(p-),d(q-)} : d(p-) \longrightarrow d(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.

(b) *The inclusion map* $i_{d(p-),\ell_{q-}} : d(p-) \longrightarrow \ell_{q-}$ *is compact if and only if* $p < q$. *Moreover,* $i_{d(p-),\ell_{q-}}$ *is bounded if and only if* $p < q$.

(c) *The inclusion map* $i_{d(p-),\mathrm{ces}(q-)} : d(p-) \longrightarrow \mathrm{ces}(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.

Proof. (i) (a) Since both $d(p+), d(q+)$ are Montel spaces, there is no distinction between bounded and compact maps. By the discussion prior to Theorem 5.3 and the fact that $d(p+) = (\mathrm{ces}(p'-))'_\beta$ and $d(q+) = (\mathrm{ces}(q'-))'_\beta$, it follows from Lemma 2.3 that $i_{d(p+),d(q+)}$ is compact if and only if its dual operator $i_{\mathrm{ces}(q'-),\mathrm{ces}(p'-)}$ is compact. By Proposition 5.4(ii)(c) this is the case if and only if $q' < p'$, that is, if and only if $p < q$.

(b) If $p < q$, then it follows from the continuous factorization $i_{d(p+),\ell_{q+}} = i_{d(q+),\ell_{q+}} \circ i_{d(p+),d(q+)}$ (cf. parts (a) and (b) of Theorem 5.3(i)) and the compactness of $i_{d(p+),d(q+)}$ (see part (a) above), that $i_{d(p+),\ell_{q+}}$ is compact.

If $p > q$, then the inclusion of $d(p+)$ in ℓ_{q+} does not exist; see Theorem 5.3(i)(b).

Consider now $p = q$ and suppose that the inclusion map $i_{d(p+),\ell_{p+}}$ is compact. Then it is also bounded. By an argument analogous to that in Remark 5.5, based on Lemma 2.1(ii), but now for $T := i_{d(p+),\ell_{q+}}$ with $X_k := d(p + \frac{1}{k}), k \in \mathbb{N}$, and $Y_m := \ell_{q+(1/m)}, m \in \mathbb{N}$, we arrive at a contradiction via [11, Proposition 2.1(ii)(e)].

So, we have established that $i_{d(p+),\ell_{q+}}$ is compact if and only if $p < q$.

Since every compact operator is also bounded, it follows that $i_{d(p+),\ell_{q+}}$ is bounded whenever $p < q$. Moreover, it was already noted that the inclusion of $d(p+)$ in ℓ_{q+} does not exist if $p > q$. So, it remains to show that $i_{d(p+),\ell_{q+}}$ is not bounded if $p = q$; this was just established above.

(c) Since both $d(p+), \mathrm{ces}(q+)$ are Montel spaces, the boundedness of $i_{d(p+),\mathrm{ces}(q+)}$ is equivalent to its compactness.

Suppose that $p < q$. Then the continuous factorization $i_{d(p+),\mathrm{ces}(q+)} = i_{\ell_{q+},\mathrm{ces}(q+)} \circ i_{d(p+),\ell_{q+}}$ (cf. Proposition 5.1(i)(b) and Theorem 5.3(i)(b)), with $i_{d(p+),\ell_{q+}}$ bounded (by part (b) above), implies that $i_{d(p+),\mathrm{ces}(q+)}$ is bounded.

If $p > q$, then the inclusion of $d(p+)$ in $\mathrm{ces}(q+)$ does not exist; see Theorem 5.3(i)(c).

Consider now $p = q$ and suppose that $i_{d(p+),\mathrm{ces}(p+)}$ is bounded. By a similar argument to that in Remark 5.5, based on Lemma 2.1(ii), but now for $T := i_{d(p+),\mathrm{ces}(p+)}$ with $X_k := d(p + \frac{1}{k}), k \in \mathbb{N}$, and $Y_m, m \in \mathbb{N}$, as in Remark 5.5, we arrive at a contradiction via [11, Proposition 2.1(ii)(f)].

(ii) (a) The argument given in part (i)(a) can be adapted to also apply here. Indeed, the dual operator of $i_{d(p-),d(q-)}$ is precisely $i_{\mathrm{ces}(q'+),\mathrm{ces}(p'+)}$ and so we can invoke here Proposition 5.4(i)(c) (in place of Proposition 5.4(ii)(c) used in part (i)(a)).

(c) We argue as in part (a). Indeed, the dual operator of $i_{d(p-),\mathrm{ces}(q-)}$ is $i_{d(q'+),\mathrm{ces}(p'+)}$ and so we can apply part (i)(c) above to $i_{d(q'+),\mathrm{ces}(p'+)}$ to conclude that $i_{d(p-),\mathrm{ces}(q-)}$ is compact (equivalently bounded) if and only if $i_{d(q'+),\mathrm{ces}(p'+)}$ is compact (equivalently bounded) which is the case if and only if $q' < p'$ (i.e., $p < q$).

(b) If $p < q$, then it follows from the continuous factorization $i_{d(p-),\ell_{q-}} = i_{d(q-),\ell_{q-}} \circ i_{d(p-),d(q-)}$ (see parts (a), (b) of Theorem 5.3(ii)) and the compactness of $i_{d(p-),d(q-)}$ (cf. part (a) above) that $i_{d(p-),\ell_{q-}}$ is compact.

For $p > q$ the inclusion of $d(p-)$ in ℓ_{q-} does not exist; see Theorem 5.3(ii)(b).

Let $p = q$ and suppose that $i_{d(p-),\ell_{p-}}$ is compact, in which case it is also bounded. By Lemma 2.2(ii) there exists $m \in \mathbb{N}$ such that for all $k \geq m$ we have $d(p + \frac{1}{k}) \subseteq \ell_{p+(1/m)}$ with a continuous inclusion. Since $k := m + 1$ satisfies $(p + \frac{1}{k}) > (p + \frac{1}{m})$, this is impossible by [11, Proposition 2.1(ii)(e)]. So, $i_{d(p-),\ell_{p-}}$ is *not* bounded and, in particular, also *not* compact.

So, we have verified that $i_{d(p-),\ell_{q-}}$ is compact if and only $p < q$.

That $i_{d(p-),\ell_{q-}}$ is bounded if and only if $p < q$ can be argued as in the last paragraph of the proof of part (i)(b) above. □

Adopting the notation introduced prior to Proposition 5.1 for the Cesàro operator it is clear what is meant by the operators $C_{X_{p+},Y_{q+}} : X_{p+} \longrightarrow Y_{q+}$ and $C_{X_{p-},Y_{q-}} : X_{p-} \longrightarrow Y_{q-}$, whenever they exist. Their continuity is then a consequence of the closed graph theorem. We begin by recalling the following result; for part (i) see [5, Proposition 28] and for part (ii) see [7, Proposition 27].

Proposition 5.7. (i) *Let* $1 \leq p, q < \infty$ *be an arbitrary pair.*

(a) *The operator* $C_{\ell_{p+},\ell_{q+}} : \ell_{p+} \longrightarrow \ell_{q+}$ *exists if and only if* $p \leq q$.
(b) *The operator* $C_{\ell_{p+},\mathrm{ces}(q+)} : \ell_{p+} \longrightarrow \mathrm{ces}(q+)$ *exists if and only if* $p \leq q$.
(c) *The operator* $C_{\mathrm{ces}(p+),\mathrm{ces}(q+)} : \mathrm{ces}(p+) \longrightarrow \mathrm{ces}(q+)$ *exists if and only if* $p \leq q$.
(d) *The operator* $C_{\mathrm{ces}(p+),\ell_{q+}} : \mathrm{ces}(p+) \longrightarrow \ell_{q+}$ *exists if and only if* $p \leq q$.

(ii) *Let* $1 < p, q \leq \infty$ *be an arbitrary pair.*

(a) *The operator* $C_{\ell_{p-},\ell_{q-}} : \ell_{p-} \longrightarrow \ell_{q-}$ *exists if and only if* $p \leq q$.

(b) *The operator* $C_{\ell_{p-},\mathrm{ces}(q-)} : \ell_{p-} \longrightarrow \mathrm{ces}(q-)$ *exists if and only if* $p \le q$.
(c) *The operator* $C_{\mathrm{ces}(p-),\mathrm{ces}(q-)} : \mathrm{ces}(p-) \longrightarrow \mathrm{ces}(q-)$ *exists if and only if* $p \le q$.
(d) *The operator* $C_{\mathrm{ces}(p-),\ell_{q-}} : \mathrm{ces}(p-) \longrightarrow \ell_{q-}$ *exists if and only if* $p \le q$.

The following result, an analogue of Proposition 5.7(i), admits the Montel spaces $d(p+)$, for $p \in [1, \infty)$.

Theorem 5.8. *Let* $1 \le p, q < \infty$ *be an arbitrary pair.*

(i) *The operator* $C_{d(p+),d(q+)} : d(p+) \longrightarrow d(q+)$ *exists if and only if* $p \le q$.
(ii) *The operator* $C_{d(p+),\ell_{q+}} : d(p+) \longrightarrow \ell_{q+}$ *exists if and only if* $p \le q$.
(iii) *The operator* $C_{d(p+),\mathrm{ces}(q+)} : d(p+) \longrightarrow \mathrm{ces}(q+)$ *exists if and only if* $p \le q$.
(iv) *The operator* $C_{\ell_{p+},d(q+)} : \ell_{p+} \longrightarrow d(q+)$ *exists if and only if* $p \le q$.
(v) *The operator* $C_{\mathrm{ces}(p+),d(q+)} : \mathrm{ces}(p+) \longrightarrow d(q+)$ *exists if and only if* $p \le q$.

Proof. (i) If $p \le q$, then $p_k \le q_k$ for all $k \in \mathbb{N}$, where $p_k := p + \frac{1}{k}$ and $q_k := q + \frac{1}{k}$. According to [10, Proposition 5.3(iii)] the Banach space operator $C_{d(p_k),d(q_k)} : d(p_k) \longrightarrow d(q_k)$ is continuous for each $k \in \mathbb{N}$. Hence, the continuity of $C_{d(p+),d(q+)}$ follows from Lemma 2.1(i).

Suppose that $p > q$. Choose $m \in \mathbb{N}$ such that $q_m < p$. If $C_{d(p+),d(q+)}$ exists (in which case it is necessarily continuous), then Lemma 2.1(i) ensures the existence of $k_0 \in \mathbb{N}$ such that $C_{d(p_{k_0}),d(q_m)} : d(p_{k_0}) \longrightarrow d(q_m)$ is continuous. Noting that $p_{k_0} > q_m$ yields a contradiction to [10, Proposition 5.3(iii)].

(ii) An analogous argument applies as in part (i) by replacing the use of Proposition 5.3(iii) in [10] with Proposition 5.3(ii) in [10].

(iii) Again argue as in part (i) by replacing the use of Proposition 5.3(iii) in [10] with Proposition 5.3(i) in [10].

(iv) Adapt the proof of part (i) by now using Proposition 5.3(iv) of [10] in place of Proposition 5.3(iii) in [10].

(v) Again argue as in part (i) by replacing the use of Proposition 5.3(iii) in [10] with Proposition 5.3(v) in [10]. □

The analogue of Proposition 5.7(ii) above, now involving the Montel spaces $d(p-)$, for $1 < p \le \infty$, is as follows.

Theorem 5.9. *Let* $1 < p, q \le \infty$ *be an arbitrary pair.*

(i) *The operator* $C_{d(p-),d(q-)} : d(p-) \longrightarrow d(q-)$ *exists if and only if* $p \le q$.
(ii) *The operator* $C_{d(p-),\ell_{q-}} : d(p-) \longrightarrow \ell_{q-}$ *exists if and only if* $p \le q$.
(iii) *The operator* $C_{d(p-),\mathrm{ces}(q-)} : d(p-) \longrightarrow \mathrm{ces}(q-)$ *exists if and only if* $p \le q$.
(iv) *The operator* $C_{\ell_{p-},d(q-)} : \ell_{p-} \longrightarrow d(q-)$ *exists if and only if* $p \le q$.
(v) *The operator* $C_{\mathrm{ces}(p-),d(q-)} : \mathrm{ces}(p-) \longrightarrow d(q-)$ *exists if and only if* $p \le q$.

Proof. Suppose first that $1 < p \le q \le \infty$. When forming the inductive limits $d(p-) = \mathrm{ind}_k\, d(p_k)$, and $\ell_{p-} = \mathrm{ind}_k\, \ell_{p_k}$ and $\mathrm{ces}(p-) = \mathrm{ind}_k\, \mathrm{ces}(p_k)$ with $1 < p_k \uparrow p$ and the inductive limits $d(q-) = \mathrm{ind}_k\, d(q_k)$, and $\ell_{q-} = \mathrm{ind}_k\, \ell_{q_k}$ and $\mathrm{ces}(q-) = \mathrm{ind}_k\, \mathrm{ces}(q_k)$ with $1 < q_k \uparrow q$, we can select $p_k \le q_k$ for each $k \in \mathbb{N}$. For this choice of $\{p_k\}_{k=1}^{\infty}$ and $\{q_k\}_{k=1}^{\infty}$ the continuity of the Cesàro operator in each of parts (i) - (v) follows from Lemma 2.2(i) above and [10, Proposition 5.3].

Suppose now that $p > q$. For case (i) choose $k \in \mathbb{N}$ such that $p_k \in (q, p)$. *If* $C_{d(p-),d(q-)}$ is continuous, then Lemma 2.2(i) ensures the existence of $m \in \mathbb{N}$ such that the Banach space operator $C_{d(p_k),d(q_m)}$ is continuous. But, $p_k > q_m$ and so we have a contradiction to Proposition 5.3(iii) of [10].

The remaining cases (ii)-(v) can be established in a similar way. In each case, $k \in \mathbb{N}$ is chosen to satisfy $p_k \in (q, p)$. The contradiction, for each of (ii)-(v), is then a consequence of the relevant part of Proposition 5.3 in [10]. □

Concerning the boundedness/compactness of the Cesàro operator we will require the following result. For part (i) see Proposition 29 in [5] and for part (ii) see Proposition 28 in [7]. Unlike for $\mathrm{ces}(p+), \mathrm{ces}(q-)$, the spaces ℓ_{p+}, ℓ_{q-} are *not* Montel.

Proposition 5.10. (i) *Let* $1 \leq p \leq q < \infty$ *be an arbitrary pair.*

- (a) *The operator* $C_{\ell_{p+},\ell_{q+}} : \ell_{p+} \longrightarrow \ell_{q+}$ *is bounded if and only if it is compact if and only if* $p < q$.
- (b) *The operator* $C_{\ell_{p+},\mathrm{ces}(q+)} : \ell_{p+} \longrightarrow \mathrm{ces}(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.
- (c) *The operator* $C_{\mathrm{ces}(p+),\mathrm{ces}(q+)} : \mathrm{ces}(p+) \longrightarrow \mathrm{ces}(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.
- (d) *The operator* $C_{\mathrm{ces}(p+),\ell_{q+}} : \mathrm{ces}(p+) \longrightarrow \ell_{q+}$ *is bounded if and only if it is compact if and only if* $p < q$.

(ii) *Let* $1 < p \leq q \leq \infty$ *be an arbitrary pair.*

- (a) *The operator* $C_{\ell_{p-},\ell_{q-}} : \ell_{p-} \longrightarrow \ell_{q-}$ *is bounded if and only if it is compact if and only if* $p < q$.
- (b) *The operator* $C_{\ell_{p-},\mathrm{ces}(q-)} : \ell_{p-} \longrightarrow \mathrm{ces}(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.
- (c) *The operator* $C_{\mathrm{ces}(p-),\mathrm{ces}(q-)} : \mathrm{ces}(p-) \longrightarrow \mathrm{ces}(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.
- (d) *The operator* $C_{\mathrm{ces}(p-),\ell_{q-}} : \mathrm{ces}(p-) \longrightarrow \ell_{q-}$ *is bounded if and only if* $p < q$.

The following result is a version of Proposition 5.10(i) which involves the Montel spaces $d(p+)$, for $p \in [1, \infty)$.

Theorem 5.11. *Let* $1 \leq p \leq q < \infty$ *be an arbitrary pair.*

- (i) *The operator* $C_{d(p+),d(q+)} : d(p+) \longrightarrow d(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.
- (ii) *The operator* $C_{d(p+),\ell_{q+}} : d(p+) \longrightarrow \ell_{q+}$ *is bounded if and only if it is compact if and only if* $p < q$.
- (iii) *The operator* $C_{d(p+),\mathrm{ces}(q+)} : d(p+) \longrightarrow \mathrm{ces}(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.
- (iv) *The operator* $C_{\ell_{p+},d(q+)} : \ell_{p+} \longrightarrow d(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.
- (v) *The operator* $C_{\mathrm{ces}(p+),d(q+)} : \mathrm{ces}(p+) \longrightarrow d(q+)$ *is bounded (equivalently compact) if and only if* $p < q$.

Proof. For the case $p > q$, none of the operators in (i)-(v) exist (cf. Theorem 5.8) and so $p \leq q$ is a necessary condition.

(i) Consider first $p = q$ and suppose that $C_{d(p+),d(p+)} : d(p+) \longrightarrow d(p+)$ is bounded. By Lemma 2.1(ii), for $T := C_{d(p+),d(p+)}$ with $X_k := d(p+\frac{1}{k}), k \in \mathbb{N}$, and $Y_m := d(p+\frac{1}{m}), m \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that the Banach space operator $C_{d(p+(1/k_0)),d(p+(1/m))} : d(p+\frac{1}{k_0}) \longrightarrow d(p+\frac{1}{m})$ is continuous for all $m \in \mathbb{N}$. Choose $m := k_0+1$, in which case $(p+\frac{1}{k_0}) > (p+\frac{1}{m})$ with $C_{d(p+(1/k_0)),d(p+(1/m))}$ continuous; this contradicts [10, Proposition 5.3(iii)].

Suppose now that $p < q$. Fix any $r \in (p, q)$. Then the operator $C_{d(p+),d(q+)} : d(p+) \longrightarrow d(q+)$ is compact because it factorizes continuously as $C_{d(p+),d(q+)} = C_{d(r+),d(q+)} \circ i_{d(p+),d(r+)}$ (cf. Theorem 5.3(i)(a) and Theorem 5.8(i)) with $i_{d(p+),d(r+)}$ compact; see Theorem 5.6(i)(a).

(ii) For compactness, which implies boundedness, the case $p = q$ follows by a contrapositive argument analogous to that in part (i), now with $Y_m := \ell_{p+(1/m)}, m \in \mathbb{N}$, but the same spaces $X_k, k \in \mathbb{N}$. The contradiction is now achieved via [10, Proposition 5.3(ii)].

For $p < q$ fix any $r \in (p, q)$. Then the continuous factorization $C_{d(p+),\ell_{q+}} = C_{d(r+),\ell_{q+}} \circ i_{d(p+),d(r+)}$ (cf. Theorem 5.3(i)(a) and Theorem 5.8(ii)) with $i_{d(p+),d(r+)}$ compact (by Theorem 5.6(i)(a)) implies that $C_{d(p+),\ell_{q+}}$ is a compact operator. Hence, it is also bounded.

(iii) The case $p = q$ follows via a similar contrapositive argument to that in part (i), now with $Y_m := \mathrm{ces}(p+\frac{1}{m}), m \in \mathbb{N}$, but the same spaces $X_k, k \in \mathbb{N}$. The contradiction is now achieved by [10, Proposition 5.3(i)].

For $p < q$ fix any $r \in (p, q)$. Then the continuous factorization $C_{d(p+),\mathrm{ces}(q+)} = C_{d(r+),\mathrm{ces}(q+)} \circ i_{d(p+),d(r+)}$ (cf. Theorem 5.3(i)(a) and Theorem 5.8(iii)) with $i_{d(p+),d(r+)}$ compact (see part (ii)) implies that $C_{d(p+),\mathrm{ces}(q+)}$ is a compact operator.

(iv) Suppose that $p = q$. *If* $C_{\ell_{p+},d(p+)}$ is compact, then the continuous factorization $C_{d(p+),d(p+)} = C_{\ell_{p+},d(p+)} \circ i_{d(p+),\ell_{p+}}$ (cf. Theorem 5.3(i)(b) and Theorem 5.8(iv)) shows that $C_{d(p+),d(p+)}$ would be compact. This contradicts part (i).

For $p < q$ fix any $r \in (p, q)$. Then the continuous factorization $C_{\ell_{p+},d(q+)} = C_{\ell_{r+},d(q+)} \circ i_{\ell_{p+},\ell_{r+}}$ (cf. Proposition 5.1(i)(a) and Theorem 5.8(iv)) with $i_{\ell_{p+},\ell_{r+}}$ bounded (by Proposition 5.4(i)(a)) implies that $C_{\ell_{p+},d(q+)}$ is bounded. Since $d(q+)$ is Montel, $C_{\ell_{p+},d(q+)}$ is also compact.

(v) Suppose that $p = q$ and that $C_{\mathrm{ces}(p+),d(p+)}$ is compact. Then the continuous factorization $C_{d(p+),d(q+)} = C_{\mathrm{ces}(p+),d(p+)} \circ i_{d(p+),\mathrm{ces}(p+)}$ (cf. Theorem 5.3(i)(c) and Theorem 5.8(v)) implies that $C_{d(p+),d(q+)}$ is compact. This contradicts part (i).

For $p < q$ fix $r \in (p, q)$. Then the continuous factorization $C_{\mathrm{ces}(p+),d(q+)} = C_{\mathrm{ces}(r+),d(q+)} \circ i_{\mathrm{ces}(p+),\mathrm{ces}(r+)}$ (cf. Proposition 5.1(i)(c) and Theorem 5.8(v)) with $i_{\mathrm{ces}(p+),\mathrm{ces}(r+)}$ compact (by Proposition 5.4(i)(c)) shows that $C_{\mathrm{ces}(p+),d(q+)}$ is compact. □

Our final result, which admits the Montel spaces $d(p-)$, $1 < p \leq \infty$, is akin to Proposition 5.10(ii).

Theorem 5.12. *Let* $1 < p \leq q \leq \infty$ *be an arbitrary pair.*

(i) *The operator* $C_{d(p-),d(q-)} : d(p-) \longrightarrow d(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.

(ii) *The operator* $C_{d(p-),\ell_{q-}} : d(p-) \longrightarrow \ell_{q-}$ *is bounded if and only if it is compact if and only if* $p < q$.

(iii) *The operator* $C_{d(p-),\mathrm{ces}(q-)} : d(p-) \longrightarrow \mathrm{ces}(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.

(iv) *The operator* $C_{\ell_{p-},d(q-)} : \ell_{p-} \longrightarrow d(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.

(v) *The operator* $C_{\mathrm{ces}(p-),d(q-)} : \mathrm{ces}(p-) \longrightarrow d(q-)$ *is bounded (equivalently compact) if and only if* $p < q$.

Proof. For the case $p > q$, none of the Cesàro operators in (i)-(v) exist (cf. Theorem 5.9) and so $p \leq q$ is a necessary condition.

(i) Suppose that $p = q$ and that $C_{d(p-),d(p-)} : d(p-) \longrightarrow d(p-)$ is a bounded operator. By Lemma 2.2(ii), for $T := C_{d(p-),d(p-)}$ with $X_k := d(p_k)$ and $1 < p_k \uparrow p$, for $k \in \mathbb{N}$, and with $Y_m := d(p_m)$ for $m \in \mathbb{N}$, there exists $m_0 \in \mathbb{N}$ such that the Banach space operator $C_{d(p_k),d(p_{m_0})} : d(p_k) \longrightarrow d(p_{m_0})$ is continuous for all $k > m_0$. Set $k := m_0 + 1$ (i.e., $p_k > p_{m_0}$) gives a contradiction to [10, Proposition 5.3(iii)].

Assume that $p < q$. Choose any pair r, s such that $p < r < s < q$. Then we have the continuous factorization $C_{d(p-),d(q-)} = i_{d(s),d(q-)} \circ C_{d(r),d(s)} \circ i_{d(p-),d(r)}$. Indeed, the continuity of $i_{d(p-),d(r)}$ follows from [25, Proposition 24.7], the continuity of $i_{d(s),d(q-)}$ is clear from the definition of the inductive limit topology in $d(q-)$, [25, p. 280, Definition], and the continuity of the Banach space operator $C_{d(r),d(s)}$ follows from [10, Proposition 5.3(iii)]. Since $C_{d(r),d(s)}$ is actually compact, [10, Proposition 5.4(iii)], it follows that $C_{d(p-),d(q-)}$ is compact.

(ii) Assume that $p = q$ and that $C_{d(p-),\ell_{p-}} : d(p-) \longrightarrow \ell_{p-}$ is bounded. Arguing via Lemma 2.2(ii) as in the proof of part (i), now with $T = C_{d(p-),\ell_{p-}}$ and $Y_m := \ell_{p_m}$ for $m \in \mathbb{N}$ (the spaces $X_k, k \in \mathbb{N}$, are as in part (i)), leads to a contradiction of Proposition 5.3(ii) in [10]. In particular, $C_{d(p-),\ell_{p-}}$ also fails to be compact.

Suppose that $p < q$. Choose any pair r, s satisfying $p < r < s < q$. Then we have the continuous factorization $C_{d(p-),\ell_{q-}} = i_{\ell_s,\ell_{q-}} \circ i_{d(s),\ell_s} \circ C_{d(r),d(s)} \circ i_{d(p-),d(r)}$. Indeed, $i_{d(s),\ell_s)}$ is continuous by [10, Proposition 5.1(ii)] and the continuity of both $C_{d(r),d(s)}$ and $i_{d(p-),d(r)}$ were established in part (i). Finally, $i_{\ell_s,\ell_{q-}}$ is continuous by the definition of the inductive limit topology in ℓ_{q-}. As noted in the proof of part (i), the operator $C_{d(r),d(s)}$ is compact and hence, so is $C_{d(p-),\ell_{q-}}$. In particular, $C_{d(p-),\ell_{q-}}$ is also bounded.

(iii) Assume that $p = q$ and that $C_{d(p-),\mathrm{ces}(p-)} : d(p-) \longrightarrow \mathrm{ces}(p-)$ is bounded. Via Lemma 2.2(ii) we can argue as in the proof of part (i), now with $T := C_{d(p-),\mathrm{ces}(p-)}$ and $Y_m := \mathrm{ces}(p_m)$ for $m \in \mathbb{N}$ (the spaces $X_k, k \in \mathbb{N}$, are as in part (i)), to produce a contradiction to Proposition 5.3(i) in [10].

Suppose that $p < q$. Choose any pair r, s satisfying $p < r < s < q$. Then we have the continuous factorization $C_{d(p-),\mathrm{ces}(q-)} = i_{\mathrm{ces}(s),\mathrm{ces}(q-)} \circ i_{d(s),\mathrm{ces}(s)} \circ C_{d(r),d(s)} \circ i_{d(p-),d(r)}$. Indeed, $i_{d(s),\mathrm{ces}(s)}$ is continuous by [10, Proposition 5.1(i)] and the continuity of $C_{d(r),d(s)}$ and $i_{d(p-),d(r)}$ were established in the proof of part (i). Finally, $i_{\mathrm{ces}(s),\mathrm{ces}(q-)}$ is continuous by definition of the inductive limit topology in $\mathrm{ces}(q-)$. Since $C_{d(r),d(s)}$ is a compact operator, so is $C_{d(p-),\mathrm{ces}(q-)}$.

(iv) Suppose that $p = q$ and that $C_{\ell_{p-},d(p-)} : \ell_{p-} \longrightarrow d(p-)$ is bounded. Arguing via Lemma 2.2(ii) as in the proof of part (i), now for $T := C_{\ell_{p-},d(p-)}$ with $X_k := \ell_{p_k}$ and $1 < p_k \uparrow p$, for $k \in \mathbb{N}$ (the spaces $Y_m, m \in \mathbb{N}$, are as in part (i)), leads to a contradiction of Proposition 5.3(iv) in [10].

If $p < q$ choose any $r \in (p, q)$. Then we have the continuous factorization $C_{\ell_{p-},d(q-)} = C_{\ell_{r-},d(q-)} \circ i_{\ell_{p-},\ell_{r-}}$ (cf. Proposition 5.1(ii)(a) and Theorem 5.9(iv)) with $i_{\ell_{p-},\ell_{r-}}$ bounded (by

Proposition 5.4(ii)(a)). So, $C_{\ell_{p-},d(q-)}$ is bounded which is equivalent to compactness as $d(q-)$ is a Montel space.

(v) Suppose that $p = q$ and that $C_{\mathrm{ces}(p-),d(p-)} : \mathrm{ces}(p-) \longrightarrow d(p-)$ is bounded. Arguing via Lemma 2.2(ii) as in the proof of part (i), now for $T := C_{\mathrm{ces}(p-),d(p-)}$ with $X_k := \mathrm{ces}(p_k)$ and $1 < p_k \uparrow p$, for $k \in \mathbb{N}$ (the spaces $Y_m, m \in \mathbb{N}$, are as in part (i)), yields a contradiction to Proposition 5.3(v) in [10].

If $p < q$ choose any $r \in (p, q)$. Then the continuous factorization $C_{\mathrm{ces}(p-),d(q-)} = C_{\mathrm{ces}(r-),d(q-)} \circ i_{\mathrm{ces}(p-),\mathrm{ces}(r-)}$ (cf. Proposition 5.1(ii)(c) and Theorem 5.9(v)) with $i_{\mathrm{ces}(p-),\mathrm{ces}(r-)}$ compact (by Proposition 5.4(ii)(c)) shows that $C_{\mathrm{ces}(p-),d(q-)}$ is a compact operator. □

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