SPECTRA AND ESSENTIAL SPECTRAL RADII OF COMPOSITION OPERATORS ON WEIGHTED BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract. We determine the spectra of composition operators acting on weighted Banach spaces H_v^∞ of analytic functions on the unit disc defined for a radial weight v, when the symbol of the operator has a fixed point in the open unit disc. We also investigate in this case the growth rate of the Koenigs eigenfunction and its relation with the essential spectral radius of the composition operator.

1. Introduction

The purpose of this paper is to determine the spectrum of a composition operator which is continuous on a weighted Banach space of analytic functions on the open unit disc $\mathbb{D}$ of type H^∞, and to investigate how the essential spectral radius of the operator determines the growth of the Koenigs eigenfunction for the symbol. We denote by $H(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$. As usual, H^∞ is the space of bounded analytic functions on $\mathbb{D}$ endowed with the norm $||f||_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|$. Given an analytic self map φ on $\mathbb{D}$, the composition operator on $H(\mathbb{D})$ is defined by $C_\varphi(f) = f \circ \varphi$. Clearly, for each positive n, $C_\varphi^n = C_{\varphi_n}$, where φ_n is the n-th iterate of φ. We refer the reader to the books of Cowen and MacCluer [9] and Shapiro [20] for a deep study of composition operators on classical spaces of holomorphic functions on the disc.

A weight $v : \mathbb{D} \to \mathbb{R}$ is a radial bounded continuous strictly positive function on the unit disc $\mathbb{D}$ of the complex plane. We consider the weighted Bergman spaces of infinite order

$$H_v^\infty = \{f \in H(\mathbb{D}) : ||f||_v := \sup_{z\in D} v(z)|f(z)| < \infty\}$$

and

$$H_v^0 = \{f \in H_v^\infty : \lim_{|z|\to 1} v(z)|f(z)| = 0\},$$

endowed with the norm $\|\cdot\|_v$. Notice that the norm topology of H_v^∞ is finer than the compact-open topology induced on the space. For more details about spaces of this type we refer the reader to [2, 3, 15] and the references therein.

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In this paper we determine the spectrum of the composition operator C_φ on both H_v^∞ and H_v^0 for more general weights v than the *standard weights* $v_p(z) = (1-|z|^2)^p$, $p > 0$, when φ have an attractive fixed point in $\mathbb{D}$; thus we extend the results obtained by Aron and Lindström in [1]. This is presented in Theorem 3.5 and Corollary 3.6. The description of the spectrum of non-compact composition operators acting on Banach spaces of analytic functions has recently been object of investigations in [10, 14, 22, 16, 18, 19] and [1].

For general radial weights v and holomorphic self maps φ having an attractive fixed point in $\mathbb{D}$ we also study how the essential spectral radius of C_φ on both H_v^∞ and H_v^0 determines whether the Koenigs eigenfunction σ of C_φ belongs to H_v^∞ and H_v^0 respectively. Every holomorphic self map φ having non-zero derivative at its Denjoy-Wolf point $w \in \mathbb{D}$ has a unique Koenigs eigenfunction $\sigma \in H(\mathbb{D})$ determined by $\sigma \circ \varphi = \varphi'(w)\sigma$, $\sigma'(w) = 1$. We refer the reader to chapters 5 and 6 of Shapiro's book [20] and to the survey [21]. Bourdon [6] proved that the Koenigs eigenfunction $\sigma \in H_{v_p}^0$ if and only if $|\varphi'(0)| > r_{e,H_{v_p}^0}(C_\varphi)$, in case φ has an attractive fixed point in $\mathbb{D}$; see [6], Theorem 4.4. Bourdon also proved that if $\sigma \in H_{v_p}^\infty$, then $|\varphi'(0)| \geq r_{e,H_{v_p}^\infty}(C_\varphi)$, and that the converse does not hold; see [6, Section 4]. Moreover, it is known that $\sigma \in H^\infty$ if and only if $r_{e,H^\infty}(C_\varphi) = 0$. Our results and examples in Section 4 extend part of Bourdon results, and show that it is not possible to extend his results to arbitrary radial weights on the unit disc.

2. Preliminaries

A radial weight v is called *typical* if it is non-increasing with respect to $|z|$ and satisfies $\lim_{|z|\to 1} v(z) = 0$. The associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) = (\sup\{|f(z)| : f \in H_v^\infty, ||f||_v \leq 1\})^{-1}.$$

If v is a radial weight, then also $\tilde{v}$ is a radial weight and it is nonincreasing. If we take $\tilde{v}$ instead of v and v is typical, then $\tilde{v}$ is also typical and both the spaces H_v^∞ and H_v^0 do not change when we replace v by $\tilde{v}$. Moreover, $v \leq \tilde{v}$. We say that v is an *essential weight* if there is a constant C such that $v(z) \leq \tilde{v}(z) \leq Cv(z)$ for all $z \in \mathbb{D}$. For the standard weights $v_p(z) = (1-|z|^2)^p$, $p > 0$, we have that $\tilde{v}_p = v_p$. Further, given $z \in \mathbb{D}$, the element $\delta_z \in (H_v^\infty)^*$ defined by $\delta_z(f) = f(z)$ satisfies $||\delta_z||_v = 1/\tilde{v}(z)$, and for each $z \in \mathbb{D}$ there is $f_z \in H_v^\infty$, $||f_z||_v \leq 1$, such that $|f_z(z)| = 1/\tilde{v}(z)$. More information about the associated weight $\tilde{v}$ can be found in [2, 3]. The polynomials are contained in H_v^0, and they are dense in H_v^0 if the weight v is typical. A *moderate weight* v is a smooth weight which satisfies $-\Delta \log v(z) \sim (1-|z|^2)^{-2}$ for all $z \in \mathbb{D}$, where $\Delta = \partial\overline{\partial}$ is the Laplacian (see [5], [11]). Every moderate weight is essential by a theorem of Seip; see Proposition 2 in [11]. Two weights v and w are equivalent if there are positive constants $c, C > 0$ such that $cv \le w \le Cv$ on $\mathbb{D}$. A real function f defined on $[0,1[$ is called *almost decreasing* if there is $C > 0$ such that $s < t$ implies $f(t) \le Cf(s)$.

For $a \in \mathbb{D}$, let $\varphi_a(z) = (a-z)/(1-\overline{a}z)$, so that φ_a is an automorphism of $\mathbb{D}$ that exchanges the points 0 and a. If v is a typical weight that satisfies the Lusky condition [15]

$$(*) \qquad \inf_n \frac{\tilde{v}(1-2^{-n-1})}{\tilde{v}(1-2^{-n})} > 0,$$

then Theorem 2.3 in [3] ensures that all operators C_φ are bounded on both H_v^0 and H_v^∞. Several conditions equivalent to $(*)$ can be seen in [11]. If condition $(*)$ is satisfied, then C_{φ_a} is an invertible bounded operator on both H_v^0 and H_v^∞ for every $a \in \mathbb{D}$.

In Section 4 we investigate the growth of Koenigs eigenfunction for an analytic self map φ on the unit disc satisfying $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. Our results yield immediately consequences for analytic self maps with a Denjoy-Wolff point $w \in \mathbb{D}$ such that $\varphi'(w) \ne 0$, at least if the weight v satisfies Lusky's condition $(*)$. Indeed, if $w \ne 0$, define $\psi = \varphi_w \circ \varphi \circ \varphi_w$. The Koenigs eigenfunctions σ_ψ for ψ and σ for φ are related by the simple formula $\sigma = (|w|^2 - 1)\sigma_\psi \circ \varphi_w$. Here the factor $(|w|^2 - 1)$ yields the normalization of the derivative $\sigma'(w) = 1$; see page 571 in [6]. Now

$$v(z)|\sigma(z)| = \frac{v(z)}{\tilde{v}(\varphi_w(z))}(|w|^2 - 1)\tilde{v}(\varphi_w(z))|\sigma_\psi(\varphi_w(z))|$$

is bounded or tends to 0 as $|z|$ tends to 1 if and only if $v(z)|\sigma_\psi(z)|$ does, since $\frac{v(z)}{\tilde{v}(\varphi_w(z))}$ is bounded above and bounded away from 0 by [3]. An analogous consideration holds for σ^n and σ_ψ^n. On the other hand, since the composition operators C_φ and C_ψ are similar, the spectrum, and the essential spectrum radius, defined below, of both coincide.

The essential spectrum $\sigma_{e,X}(T)$ of a bounded operator T on the Banach space X is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not Fredholm. It is known that $\sigma_{e,X}(T) = \sigma_{e,X^*}(T^*)$ [12]. The essential spectral radius of T on X is given by

$$r_{e,X}(T) = \sup\{|\lambda| : \lambda \in \sigma_{e,X}(T)\}.$$

Another way of expressing the essential spectral radius is

$$r_{e,X}(T) = \lim_n ||T^n||_{e,X}^{\frac{1}{n}},$$

where $||T||_{e,X}$ denotes the essential norm of T, i.e., the distance from the compact operators on X.

Let $\lambda \in \sigma_X(T)$ be such that $|\lambda| > r_{e,X}(T)$. Then λ lies in the unbounded component of $\mathbb{C} \setminus \sigma_{e,X}(T)$. Now the Fredholm theory gives that λ is an isolated point of $\sigma_X(T)$ which also is an eigenvalue of finite multiplicity. Let us state this well-known result as a lemma; see [12].

Lemma 2.1. *Let $T : X \to X$ be a bounded operator. If $\lambda \in \sigma_X(T)$ is such that $|\lambda| > r_{e,X}(T)$, then λ is an isolated eigenvalue of finite multiplicity.*

If the weight v is typical, then the following formula of the essential norm of C_φ on H_v^∞ in terms of the weight has been obtained in [17] or [8], in [8] with the extra assumption that H_v^0 is isomorphic to c_0; see also [4],

$$||C_\varphi||_{e,H_v^\infty} = \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{v(z)}{\tilde{v}(\varphi(z))}.$$

If C_φ is also bounded on H_v^0, then

$$||C_\varphi||_{e,H_v^0} = \lim_{r\to 1} \sup_{|z|>r} \frac{v(z)}{\tilde{v}(\varphi(z))}.$$

By Theorem 2.1 in [3], for any typical weight v, $C_\varphi : H_v^\infty \to H_v^\infty$ is bounded if and only if C_φ is bounded on H_v^0. Actually, in this case, $C_\varphi : H_v^\infty \to H_v^\infty$ is the bitranspose map of $C_\varphi : H_v^0 \to H_v^0$. Therefore,

$$r_{e,H_v^0}(C_\varphi) = r_{e,H_v^\infty}(C_\varphi) \text{ and also } \sigma_{H_v^0}(C_\varphi) = \sigma_{H_v^\infty}(C_\varphi).$$

for every typical weight v such that C_φ is bounded on H_v^0 or H_v^∞. In particular, we have

$$r_{e,H_v^0}(C_\varphi) = r_{e,H_v^\infty}(C_\varphi) = \lim_n \left(\limsup_{|z|\to 1} \frac{v(z)}{\tilde{v}(\varphi_n(z))} \right)^{1/n}.$$

If $\varphi(0) = 0$ and $0 \neq |\varphi'(0)| < 1$, then Koenigs Theorem (see 6.1 in [20]) states that the sequence of functions

$$\sigma_k(z) := \frac{\varphi_k(z)}{\varphi'(0)^k}$$

converges uniformly on compact subsets of $\mathbb{D}$ to a non-constant function σ, which is called *Koenigs* function, that satisfies $\sigma \circ \varphi = \varphi'(0)\sigma$. More generally, if f and λ solve $f \circ \varphi = \lambda f$, then there is a positive integer n such that $\lambda = \varphi'(0)^n$ and f is a constant multiple of σ^n. Note also that $|\varphi'(0)| < 1$ when $\varphi(0) = 0$ and φ is not an automorphism.

3. The spectrum of C_φ. Main result.

In this section we assume that $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$ and that the weight v is typical. The assumption $\varphi(0) = 0$ implies that C_φ is bounded both on H_v^0 and H_v^∞.

Lemma 3.1. *Let v be a typical weight and assume that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Both $\sigma_{H_v^0}(C_\varphi)$ and $\sigma_{H_v^\infty}(C_\varphi)$ contain $\varphi'(0)^n$ for all non-negative integers n.*

Proof. We use an argument of [13] to show that $z^n \in H_v^0 \subset H_v^\infty$, $n > 0$, is not in the range of $C_\varphi - \varphi'(0)^n I$ on H_v^∞. Assume first that $f \in H_v^\infty$ and $f(\varphi(z)) - \varphi'(0)f(z) = z$. Then $f'(\varphi(z))\varphi'(z) - \varphi'(0)f'(z) = 1$ and with $z = 0$ we get the contradiction $0 = 1$. For $n > 1$, suppose $f \in H_v^\infty$ and $f(\varphi(z)) - \varphi'(0)^n f(z) = z^n$. By repeated differentiation on both sides we get, for all $k < n$, that $f^{(k)}(0) = 0$. Then for $k = n$ and $z = 0$, we obtain the contradiction $0 = n!$. This means that $\varphi'(0)^n \in \sigma_{H_v^\infty}(C_\varphi)$ for all $n > 0$. If $n = 0$, then $\varphi'(0)^n = 1$ is an eigenvalue for C_φ. $\square$

In the next section we discuss when $\varphi'(0)^n$ belongs to the point spectrum of C_φ.

For a positive integer m and an arbitrary weight v, let $H_{v,m}^\infty$ denote the closed subspace of H_v^∞ given by

$$H_{v,m}^\infty := \{f \in H_v^\infty : f \text{ has a zero of at least order m at } 0\}.$$

We denote by $||.||_{m,v}$ the induced norm on $H_{v,m}^\infty$. The following result and proof, which we restate, are taken from [1].

Lemma 3.2. *Let v be a weight and $w \in \mathbb{D}$ such that $|w| \geq 1/2$. Then*

$$||\delta_w||_{v,m} \leq ||\delta_w||_v \leq 2^m \ ||\delta_w||_{v,m}.$$

Proof. Since $H_{v,m}^\infty \subset H_v^\infty$, it follows that $||\delta_w||_{v,m} \leq ||\delta_w||_v$. For fixed $w \in \mathbb{D}$ there is a $f_w \in H_v^\infty$, $||f_w||_v \leq 1$, such that $|f_w(w)| = 1/\tilde{v}(w)$. If $g_w(z) := z^m f_w(z)$, then $||g_w||_v \leq 1$ and

$$||\delta_w||_{v,m} \geq |g_w(w)| = |w|^m |f_w(w)| \geq 1/2^m \ 1/\tilde{v}(w) = 1/2^m \ ||\delta_w||_v.$$

$\square$

Now we want to estimate the norm of the evaluation map acting on the subspaces $H_{v,m}^\infty$ of H_v^∞ for a general weight v. For the standard weight v_p this result has been obtained in [1] with a different proof.

Proposition 3.3. *Let $m \in \mathbb{N}$ and v be an arbitrary weight. Then there is a constant $M_m > 0$ such that*

$$|f(w)| \leq M_m \ \frac{1}{\tilde{v}(w)} \ ||f||_v \ |w|^m$$

for all $f \in H_{v,m}^\infty$ and $w \in \mathbb{D}$.

Proof. It is easy to see that $H_{v,m}^\infty = z^m H_v^\infty$. Consequently, we can apply the closed graph theorem to get that the map $f \mapsto f/z^m$ is well-defined, linear and continuous from $H_{v,m}^\infty$ into H_v^∞. Therefore, there is $M_m > 0$ such that $||f/z^m||_v \leq M_m ||f||_{m,v}$ for each $f \in H_{v,m}^\infty$. If $w \in \mathbb{D}$ and $f \in H_{v,m}^\infty$, we have

$$|f(w)| = |w|^m |\frac{f(w)}{w^m}| \leq |w|^m ||\delta_w||_v ||f/z^m||_v \leq M_m |w|^m ||f||_{m,v} \frac{1}{\tilde{v}(w)}.$$

$\square$

Recall that (z_k) is an *iteration sequence* for φ if $\varphi(z_k) = z_{k+1}$ for all k. We need the following crucial lemma due to Cowen and MacCluer [9].

Lemma 3.4. *If φ is not an automorphism and $\varphi(0) = 0$, then given $0 < r < 1$, there exists $1 \leq M < \infty$ such that if $(z_k)_{k=-K}^\infty$ is an iteration sequence with $|z_n| \geq r$ for some non-negative integer n and $(w_k)_{k=-K}^n$ are arbitrary numbers, then there exists $f \in H^\infty$ with $f(z_k) = w_k$, $-K \leq k \leq n$ and*

$||f||_\infty \leq M \sup\{|w_k| : -K \leq k \leq n\}$. *Further there exists* $b < 1$ *such that for any iteration sequence* (z_k) *we have* $|z_{k+1}|/|z_k| \leq b$ *whenever* $|z_k| \leq 1/2$.

Theorem 3.5. *Let* v *be a typical weight. Suppose* φ*, not an automorphism, has fixed point* $0 \in \mathbb{D}$*. Then*

$$\sigma_{H_v^\infty}(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,H_v^\infty}(C_\varphi)\} \cup \{\varphi'(0)^n\}_{n=0}^\infty.$$

Proof. By Lemmas 2.1 and 3.1 it remains to show that

$$\{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,H_v^\infty}(C_\varphi)\} \subset \sigma_{H_v^\infty}(C_\varphi).$$

If $r_{e,H_v^\infty}(C_\varphi) = 0$, we are done since $0 \in \sigma_{H_v^\infty}(C_\varphi)$ when φ is not an automorphism. So we assume that $\rho := r_{e,H_v^\infty}(C_\varphi) > 0$. Since $\varphi(0) = 0$, we have that $\varphi(z) = z\psi(z)$, with $\psi \in H^\infty$. Hence $H_{v,m}^\infty$ is an invariant subspace under C_φ. Further, $H_{v,m}^\infty$ has finite codimension in H_v^∞. Now Lemma 7.17 in [9], which is also valid for Banach spaces, gives that $\sigma_{H_{v,m}^\infty}(C_\varphi) \subset \sigma_{H_v^\infty}(C_\varphi)$. So it is enough to show that any λ with $0 < |\lambda| < \rho$ belongs to $\sigma_{H_{v,m}^\infty}(C_\varphi)$ for some m to be found. Let C_m denote the restriction of C_φ to the invariant closed subspace $H_{v,m}^\infty$. Since $C_m - \lambda I$ is not invertible if $(C_m - \lambda I)^*$ is not bounded from below, we just need to find m with $(C_m - \lambda I)^*$ not bounded from below.

Let $1 \leq M < \infty$ be the constant in Lemma 3.4 for $r = 1/4$. Iteration sequences will be denoted by $\zeta = (z_k)_{k=-K}^\infty$ with $K > 0$ and $|z_0| \geq 1/2$. Let $n := \max\{k : |z_k| \geq 1/4\}$. Then $n \geq 0$ and $|z_k| < 1/4$ for $k > n$. By Lemma 3.4 there is $b < 1$ with $|z_{k+1}/z_k| \leq b$ for all $k \geq n$. We may assume that $1/2 \leq b < 1$. This implies

$$|z_k| \leq b^{k-n}|z_n| \text{ for } k \geq n \ . \tag{3.1}$$

We now choose m so large that

$$\frac{b^m}{|\lambda|} < 1 \tag{3.2}$$

Given any such iteration sequence $\zeta = (z_k)_{k=-K}^\infty$ let us define the linear functional L_ζ on $H_{v,m}^\infty$ by

$$L_\zeta(f) = \sum_{k=-K}^\infty \lambda^{-k} f(z_k).$$

Indeed L_ζ is bounded: Proposition 3.3 yields

$$|\sum_{k=-K}^\infty \lambda^{-k} f(z_k)| \leq M_m \, ||f||_v \{\sum_{k=-K}^n |\lambda|^{-k}|z_k|^m \tilde{v}(z_k)^{-1} + \sum_{k=n+1}^\infty |\lambda|^{-k}|z_k|^m \tilde{v}(z_k)^{-1}\}.$$

Moreover, since $|z_k| < 1/4$ for $k > n$ and $\tilde{v}(z)$ is continuous, there is a constant $C > 0$ such that $\tilde{v}(z_k)^{-1} \leq C$ for $k > n$. Further, applying (3.1) and (3.2) we get

$$\sum_{k=n+1}^\infty |\lambda|^{-k}|z_k|^m \tilde{v}(z_k)^{-1} \leq \sum_{k=n+1}^\infty |\lambda|^{-k}|z_k|^m C \leq C\frac{|z_n|^m}{|\lambda|^n} \sum_{k=n+1}^\infty (\frac{b^m}{|\lambda|})^{k-n} < \infty.$$

Therefore, L_ζ is bounded. Let us find next a lower bound for $||L_\zeta||_{v,m}$.

There exists $f_{z_0} \in H_v^\infty$ with $||f_{z_0}||_v \le 1$, so that $|f_{z_0}(z_0)| = 1/\tilde{v}(z_0)$. By Lemma 3.4, there is $f_1 \in H^\infty$ with $||f_1||_\infty \le M$, satisfying $|f_1(z_0)| = 1$, $z_0^m f_1(z_0) f_{z_0}(z_0) > 0$ and $f_1(z_k) = 0$ for $-K \le k \le n,\ k \ne 0$.

Now, the function $g(z) := z^m f_1(z) f_{z_0}(z)$ belongs to $H_{v,m}^\infty$ and $\|g\|_v \le M$. Further,

$$L_\zeta(g) = z_0^m f_1(z_0) f_{z_0}(z_0) + \sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m f_1(z_k) f_{z_0}(z_k). \tag{3.3}$$

Since $\tilde{v}$ is non-increasing, we get using again (4.1) and (4.2)

$$|\sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m f_1(z_k) f_{z_0}(z_k)| \le M \frac{|z_n|^m}{|\lambda|^n} \sum_{k=n+1}^{\infty} (\frac{b^m}{|\lambda|})^{k-n} \frac{1}{\tilde{v}(z_n)} \le M \frac{|z_n|^m}{|\lambda|^n} \frac{1}{\tilde{v}(z_n)} \frac{b^m}{|\lambda| - b^m}.$$

If, in addition, we choose m so that

$$M \frac{1}{|\lambda|^n} \frac{1}{\tilde{v}(z_n)} \frac{b^m}{|\lambda| - b^m} < \frac{1}{2\tilde{v}(z_0)},$$

then

$$|\sum_{k=n+1}^{\infty} \lambda^{-k} g(z_k)| \le \frac{|z_n|^m}{2\tilde{v}(z_0)} \le \frac{|z_0|^m}{2\tilde{v}(z_0)}.$$

Hence by (3.3),

$$|L_\zeta(g)| \ge \frac{|z_0^m|}{\tilde{v}(z_0)} - |\sum_{k=n+1}^{\infty} \lambda^{-k} g(z_k)| \ge \frac{|z_0|^m}{2\tilde{v}(z_0)}.$$

Therefore, using Proposition 3.3, we obtain the desired lower bound

$$||L_\zeta||_{v,m} \ge \frac{|z_0|^m}{2M\tilde{v}(z_0)} \ge \frac{1}{2MM_m} ||\delta_{z_0}||_{v,m}.$$

The final step is to estimate $||(C_m^* - \lambda I)L_\zeta||_{v,m}$ for a suitable iteration sequence ζ. First observe that

$$(C_m^* - \lambda I)L_\zeta = -\lambda^{K+1} \delta_{z_{-K}}.$$

For the bounded composition operator $C_\varphi^l : H_v^\infty \to H_v^\infty$ we have that

$$||C_\varphi^l||_{e,H_v^\infty} = \lim_{r\to 1} \sup_{|\varphi_l(w)|>r} \frac{v(w)}{\tilde{v}(\varphi_l(w))}.$$

Pick μ so that $|\lambda| < \mu < \rho$. Since ρ is the essential spectral radius, there is n_0 so that for every $l \ge n_0$,

$$||C_\varphi^l||_{e,H_v^\infty} > \mu^l.$$

Hence for any $l \ge n_0$ we can find a $w \in \mathbb{D}$ so that

$$\frac{v(w)}{\tilde{v}(\varphi_l(w))} \ge \mu^l > 0, \text{ and } \quad |\varphi_l(w)| \ge 1/2.$$

Thus, we apply Lemma 3.2 to get

$$\frac{||\delta_{\varphi_l(w)}||_{v,m}}{||\delta_w||_{v,m}} \geq \frac{1}{2^m}\frac{||\delta_{\varphi_l(w)}||_v}{||\delta_w||_v} = \frac{1}{2^m}\frac{\tilde{v}(w)}{\tilde{v}(\varphi_l(w))} \geq \frac{\mu^l}{2^m}.$$

This means that for every $K \geq n_0$ with the above choice of $w \in \mathbb{D}$ we can form an iteration sequence $(z_k)_{k=-K}^{\infty}$ by letting $z_{-K} = w$ and $z_{k+1} = \varphi(z_k)$ for $k \geq -K$. Then $|z_0| = |\varphi_K(w)| \geq 1/2$.

Finally,

$$\frac{||(C_m^* - \lambda I)L_\zeta||_{v,m}}{||L_\zeta||_{v,m}} \leq \frac{2MM_m}{||\delta_{\varphi_K(w)}||_{v,m}}\ |\lambda|^{K+1}||\delta_w||_{v,m} \leq |\lambda| MM_m 2^{m+1}\ (\frac{|\lambda|}{\mu})^K.$$

By choosing $K \geq n_0$ big enough, it follows that $C_m^* - \lambda I$ is not bounded from below. □

Corollary 3.6. *Let v be a typical weight. Suppose that φ, not an automorphism, has fixed point $0 \in \mathbb{D}$. Then*

$$\sigma_{H_v^0}(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,H_v^0}(C_\varphi)\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

Proof. We can apply Theorem 3.5 since $\sigma_{H_v^0}(C_\varphi) = \sigma_{H_v^\infty}(C_\varphi)$ and $r_{e,H_v^0}(C_\varphi) = r_{e,H_v^\infty}(C_\varphi)$. □

4. Remarks and examples about the Koenigs function and the essential spectral radius of C_φ

In this section we investigate when $\varphi'(0)^n$, $n \in \mathbb{N}$, belongs to the point spectrum of C_φ on H_v^∞ and H_v^0 respectively.

Theorem 4.1. *Let $n \in \mathbb{N}$ and v be a typical weight. Suppose $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. If $|\varphi'(0)|^n > r_{e,H_v^\infty}(C_\varphi) = r_{e,H_v^0}(C_\varphi)$, then the Koenigs eigenfunction σ^n belongs to H_v^0 with eigenvalue $\varphi'(0)^n$.*

Proof. Since $\varphi'(0)^n \in \sigma_{H_v^0}(C_\varphi)$ and $|\varphi'(0)|^n > r_{e,H_v^0}(C_\varphi)$, it follows from Lemma 2.1 that $\varphi'(0)^n$ is an eigenvalue of finite multiplicity. Further, by the work of Koenigs, only constant multiples of σ^n can be corresponding eigenfunctions, and consequently $\sigma^n \in H_v^0$. □

We investigate conditions to obtain the converse of Theorem 4.1.

Theorem 4.2. *Let $n \in \mathbb{N}$ and v be a typical weight which is essential. Suppose that $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$ and let $0 < q \leq p$. Assume the following two conditions:*

(a) *There is $D > 0$ such that $v_p \leq Dv$ on $\mathbb{D}$,*
(b) *The function v/v_q is almost decreasing with respect to $|z|$.*

If the eigenfunction σ^n for C_φ belongs to H_v^∞, then $|\varphi'(0)|^n \geq r_{e,H_v^\infty}(C_\varphi)^{p/q}$.

Proof. First of all, by (a), $H_v^\infty \subset H_{v_p}^\infty$, hence $\sigma^n \in H_{v_p}^\infty$. By Bourdon's results in [6], it follows that $|\varphi'(0)|^n \geq r_{e,H_{v_p}^\infty}(C_\varphi)$.

On the other hand, by our general assumptions on φ, $|\varphi_n(z)| \le |z|$ for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. We can apply (b) to get $C > 0$ with

$$\frac{v(z)}{v(\varphi_n(z))} \le C \, \frac{v_q(z)}{v_q(\varphi_n(z))} \text{ for all } \ z \in \mathbb{D} \text{ and } n \in \mathbb{N}.$$

Since

$$r_{e,H_v^0}(C_\varphi) = \lim_n \left(\limsup_{|z|\to 1} \frac{v(z)}{\tilde{v}(\varphi_n(z))} \right)^{1/n}$$

and v is essential, we get $r_{e,H_v^0}(C_\varphi) \le r_{e,H_{v_q}^0}(C_\varphi)$. It is easy to check that $r_{e,H_{v_p}^\infty}(C_\varphi)^q = r_{e,H_{v_q}^\infty}(C_\varphi)^p$. This implies $|\varphi'(0)|^n \ge r_{e,H_{v_p}^\infty}(C_\varphi) = r_{e,H_{v_q}^\infty}(C_\varphi)^{p/q} \ge r_{e,H_v^\infty}(C_\varphi)^{p/q}$. □

Corollary 4.3. *Suppose that $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. Let v be a typical weight which is essential and such that, for some $q > 0$, $v(z)/v_q(z)$ is almost decreasing with respect to $|z|$, and, further, there is $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ there is $C(\varepsilon) > 0$ with $(1-|z|^2)^\varepsilon \frac{v_q(z)}{v(z)} \le C(\varepsilon)$ for every $z \in \mathbb{D}$. If for some $n \in \mathbb{N}$ the eigenfunction σ^n for C_φ belongs to H_v^∞, then $|\varphi'(0)|^n \ge r_{e,H_v^0}(C_\varphi) = r_{e,H_v^\infty}(C_\varphi)$.*

Proof. Fix $0 < \varepsilon < \varepsilon_0$. We show that v satisfies the assumptions (a) and (b) of Theorem 4.2 for $q < p = q + \varepsilon$. Indeed, (b) is trivial, as v/v_q is assumed to be almost decreasing with respect to $|z|$. On the other hand

$$\frac{v_p(z)}{v(z)} = (1-|z|^2)^\varepsilon \frac{v_q(z)}{v(z)} \le C(\varepsilon).$$

We apply Theorem 4.2 to conclude $|\varphi'(0)|^n \ge r_{e,H_v^\infty}(C_\varphi)^{(q+\varepsilon)/q}$. Since this holds for each $0 < \varepsilon < \varepsilon_0$, the conclusion follows. □

Observe that the second assumption in Corollary 4.3 holds if for each $0 < \varepsilon < \varepsilon_0$, $\lim_{r\to 1} \frac{v(r)}{v_q(r)(1-r^2)^\varepsilon} > 0$.

The weights $v(z) = (1-|z|^2)^p(1-\log(1-|z|^2))^{-\alpha}$, $\alpha > 0$ and $0 < p < \infty$, are moderate, hence essential by a theorem of Seip (see Proposition 2 in [11]), and satisfy the assumptions of Corollary 4.3, as a calculation shows. In fact $(1-\log(1-|z|^2))^{-\alpha}$ is even decreasing. The following example yields that for these weights we cannot obtain a characterization like Bourdon's Theorem 4.4 in [6] "$\sigma \in H_{v_p}^0$ if and only if $|\varphi'(0)| > r_{e,H_{v_p}^0}(C_\varphi)$". In particular, the converse of Theorem 4.1 does not hold for general weights v.

Example 1. The function $\sigma(z) = \frac{z}{1-z}$ is the Koenigs function for $\varphi(z) = \frac{z}{2-z}$. Clearly $\sigma \in H_v^0$ for the weight $v(z) = (1-|z|^2)(1-\log(1-|z|^2))^{-1}$ and $\varphi'(0) = 1/2$. It follows from Corollary 4.3 that $r_{e,H_v^0}(C_\varphi) \le \frac{1}{2}$. Notice that

$$\varphi_n(z) = \frac{z}{2^n - (2^n-1)z} \qquad \text{and} \qquad \lim_{r\to 1} \frac{1-r^2}{1-|\varphi_n(r)|^2} = 2^{-n} \ .$$

Then using

$$\frac{v(z)}{v(\varphi_n(z))} = \frac{1-|z|^2}{1-|\varphi_n(z)|^2}\left(1 - \frac{\log(\frac{1-|\varphi_n(z)|^2}{1-|z|^2})}{1-\log(1-|z|^2)}\right),$$

we conclude that

$$\lim_n \left(\limsup_{|z|\to 1} \frac{v(z)}{v(\varphi_n(z))}\right)^{1/n} = 1/2.$$

Hence $r_{e,H_v^0}(C_\varphi) = 1/2 = \varphi'(0)$. This shows that "greater or equal" cannot be replaced by "strictly greater" in Corollary 4.3.

Example 2. By Theorem 4.1 and Corollary 4.3 we have

$$|\varphi'(0)| > r_{e,H_v^0}(C_\varphi) \Rightarrow \sigma \in H_v^0 \Rightarrow |\varphi'(0)| \geq r_{e,H_v^0}(C_\varphi).$$

Example 1 shows that the first arrow cannot be reversed. We exhibit an example of a Koenigs eigenfunction $\sigma \in H_v^\infty \setminus H_v^0$ for a self map φ such that $|\varphi'(0)| = r_{e,H_v^0}(C_\varphi) = r_{e,H_v^\infty}(C_\varphi)$, thus showing that the second arrow cannot be reversed either. To see this, take the weight $v(z) = (1-|z|)(1-\log(1-|z|))^{-1}$, which is equivalent to the one in Example 1, and $\sigma(z) := (1/(1-z))\log(1/(1-z))$, as in the first example in page 578 of [6]. Then σ is the Koenigs eigenfunction of $\varphi = \sigma^{-1} \circ \sigma/2$, so $\varphi(0) = 0$ and $\varphi'(0) = 1/2$. For the weight $v_1(z) = 1-|z|$, it is shown in [6] that $|\varphi'(0)| = r_{e,H_{v_1}^0}(C_\varphi)$. Clearly $\sigma \in H_v^\infty \setminus H_v^0$, and we can apply Corollary 4.3 to get $|\varphi'(0)| = r_{e,H_{v_1}^0}(C_\varphi) \geq r_{e,H_v^0}(C_\varphi)$. Thus the second implication cannot be reversed. To see the equality of the two radii, observe that

$$\limsup_{|z|\to 1} \frac{v(z)}{v(\varphi_n(z))} = \limsup_{|z|\to 1} \frac{v_1(z)}{v_1(\varphi_n(z))}\left(\frac{1-\log(1-|\varphi_n(z)|)}{1-\log(1-|z|)}\right).$$

Now, for each n there is a sequence $(z_k^n)_k$ in $\mathbb{D}$ with $|z_k^n| \to 1$ as $k \to \infty$, such that the sequence $((1-|z_k^n|)/(1-|\varphi_n(z_k^n)|))_k$ converges. This implies that

$$\lim_k \left(\frac{1-\log(1-|\varphi_n(z_k^n)|)}{1-\log(1-|z_k^n|)}\right) = 1,$$

and we see that the inequality $r_{e,H_{v_1}^0}(C_\varphi) \leq r_{e,H_v^0}(C_\varphi)$ also holds.

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