# On Mean Ergodic Operators

A.A. Albanese*, J. Bonet** and W.J. Ricker

**Abstract.** Aspects of the theory of mean ergodic operators and bases in Fréchet spaces were recently developed in [1]. This investigation is extended here to the class of barrelled locally convex spaces. Duality theory, also for operators, plays a prominent role.

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## 1. Introduction

Certain aspects of the theory of mean ergodic operators in Banach spaces (see, e.g., [14] and the references therein) are related to the theory of bases. This is well documented in [10] (see also the references) where it is shown, amongst other results, that a Banach space with a basis is reflexive if and only if every power bounded operator is mean ergodic. The proof is based on classical results of A.A. Pelczynski and of M. Zippin, connecting bases with reflexivity. In order to extend the results of [10] to the Fréchet space setting, it is necessary to have available the corresponding results of Pelczynski and of Zippin. These were established, and then applied to mean ergodic operators, in the recent article [1]. Since much of modern analysis also occurs in locally convex Hausdorff spaces (briefly, lcHs) which are not metrizable, there is some interest in extending the recent results of [1] beyond the Fréchet space setting. This is our aim here.

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A continuous linear operator $T$ in a lcHs $X$ (the space of all such operators is denoted by $\mathcal{L}(X)$) is called *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X, \tag{1.1}$$

exist in $X$. An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^\infty$ is an equicontinuous subset of $\mathcal{L}(X)$. Of course, for $X$ a Banach space, this means that $\sup_{m\geq 0} \|T^m\| < \infty$. A power bounded operator $T$ is mean ergodic precisely when

$$X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Im}(I-T)}, \tag{1.2}$$

where $I$ is the identity operator, Im(I − T) denotes the range of $(I-T)$ and the bar denotes the "closure in $X$". In general, the right–hand side of (1.2) is the set of all $x \in X$ for which the sequence $\{\frac{1}{n}\sum_{m=1}^n T^m x\}_{m=1}^\infty$ converges to 0 in $X$. Let us indicate some of our main results.

Technical terms concerning lcHs' $X$ and certain aspects of $\mathcal{L}(X)$ will be defined in later sections. Let us recall at this stage that if $\Gamma_X$ is a system of continuous seminorms determining the topology of $X$, then the strong operator topology $\tau_s$ in $\mathcal{L}(X)$ is determined by the family of seminorms

$$q_x(S) := q(Sx), \quad S \in \mathcal{L}(X),$$

for each $x \in X$ and $q \in \Gamma_X$ (in which case we write $\mathcal{L}_s(X)$). Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of $X$. The topology $\tau_b$ of uniform convergence on bounded sets is defined in $\mathcal{L}(X)$ via the seminorms

$$q_B(S) := \sup_{x\in B} q(Sx), \quad S \in \mathcal{L}(X),$$

for each $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$ (in which case we write $\mathcal{L}_b(X)$). For $X$ a Banach space, $\tau_b$ is the operator norm topology in $\mathcal{L}(X)$. If $\Gamma_X$ is countable and $X$ is complete, then $X$ is called a Fréchet space.

Given $T \in \mathcal{L}(X)$, let

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}, \tag{1.3}$$

denote the Cesàro means of $T$ (see also (1.1)). Then $T$ is mean ergodic precisely when $\{T_{[n]}\}_{n=1}^\infty$ is a convergent sequence in $\mathcal{L}_s(X)$. If $\{T_{[n]}\}_{n=1}^\infty$ happens to be convergent in $\mathcal{L}_b(X)$, then $T$ is called *uniformly mean ergodic.* The space $X$ itself is called mean ergodic (resp. uniformly mean ergodic) if every power bounded operator on $X$ is mean ergodic (resp. uniformly mean ergodic).

The natural setting for mean ergodic operators seems to be the class of barrelled lcHs'. We show in Section 3 that most of the results on mean ergodicity that were established in [1] for operators on Fréchet spaces carry over to barrelled spaces; see also [21, Ch. VIII, §3]. The same is also true of Propositions 2.3 and 2.4 below. Since the strong dual of a distinguished Fréchet space is barrelled, the current results can be combined with those of [1] via duality theory.

An important class of Fréchet spaces consists of the Köthe echelon spaces $\lambda_p(A)$, whose mean ergodicity properties were thoroughly investigated in [1]. Just as important is the class of Köthe co–echelon spaces $k_p(V)$, for $p \in \{0\} \cup [1, \infty]$, all barrelled, but, typically not Fréchet spaces except for very special cases. In Section 4 the general results of Sections 2 and 3 are applied to give a complete description of the ergodicity properties of these co–echelon spaces. For instance, if $1 < p < \infty$, then the reflexive space $k_p(V)$, necessarily mean ergodic, is uniformly mean ergodic iff it is Montel. The non–reflexive co–echelon spaces $k_1(V)$ and $k_\infty(V)$ are mean ergodic iff they are uniformly mean ergodic iff they are Montel. Provided it is complete, $k_0(V)$ is mean ergodic iff it is uniformly mean ergodic iff it is a Schwartz space.

## 2. Preliminary results

Given a lcHs $X$ and $T \in \mathcal{L}(X)$ we have

$$(I - T)T_{[n]} = T_{[n]}(I - T) = \frac{1}{n}(T - T^{n+1}), \quad n \in \mathbb{N}, \tag{2.1}$$

and also, with $T_{[0]} := I$, that

$$\frac{1}{n}T^n = T_{[n]} - \frac{(n-1)}{n}T_{[n-1]}, \quad n \in \mathbb{N}. \tag{2.2}$$

If $T \in \mathcal{L}(X)$ is power bounded, then

$$\overline{\mathrm{Im}(I - T)} = \{x \in X : \lim_{n\to\infty} T_{[n]}x = 0\} \tag{2.3}$$

and hence, in particular,

$$\overline{\mathrm{Im}(I - T)} \cap \mathrm{Ker}(I - T) = \{0\}, \tag{2.4}$$

[21, Ch. VIII, §3]. Moreover, such a $T$ clearly satisfies

$$\lim_{n\to\infty} \frac{1}{n}T^n = 0, \text{ in } \mathcal{L}_s(X). \tag{2.5}$$

The following fact, without a proof, occurs in [1, Proposition 2.3].

**Proposition 2.1.** *Let $X$ be a barrelled lcHs. If $T \in \mathcal{L}(X)$ satisfies (2.5) and*

$$\{T_{[n]}x\}_{n=1}^\infty \text{ is bounded in } X, \text{ for each } x \in X, \tag{2.6}$$

*then $T$ satisfies both (2.3) and (2.4).*

*Proof.* It follows from (2.1) and (2.5) that $\lim_{n\to\infty} T_{[n]}w = 0$ for $w \in \mathrm{Im(I - T)}$. Since $X$ is barrelled, condition (2.6) implies that $\{T_{[n]}\}_{n=1}^\infty$ is an equicontinuous subset of $\mathcal{L}(X)$, [13, p.137]. So, given any $q \in \Gamma_X$ there exists $p \in \Gamma_X$ such that

$$q(T_{[n]}x) \le p(x), \quad x \in X, \ n \in \mathbb{N}.$$

Fix $z \in \overline{\mathrm{Im}(I - T)}$. Given $\varepsilon > 0$ there exists $w_\varepsilon \in \mathrm{Im(I{-}T)}$ satisfying $p(z - w_\varepsilon) < \varepsilon$. Then we have

$$q(T_{[n]}z) \le \varepsilon + q(T_{[n]}w_\varepsilon), \quad n \in \mathbb{N},$$

which implies that $\limsup_n q(T_{[n]}z) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we can conclude that $T_{[n]}z \to 0$ in $X$. This establishes one containment in (2.3).

Conversely, let $x \in X$ satisfy $\lim_{n\to\infty} T_{[n]}x = 0$. It follows from (1.3) that

$$x - T_{[n]}x = (I - T)\sum_{m=1}^{n} \frac{1}{n}(I + T + \ldots + T^{m-1})x \in \operatorname{Im}(I - T),$$

for each $n \in \mathbb{N}$. Combined with $T_{[n]}x \to 0$ in $X$, it is immediate that $x \in \overline{\operatorname{Im}(I - T)}$. This establishes equality in (2.3).

Finally, observe if $x \in \overline{\operatorname{Im}(I - T)} \cap \operatorname{Ker}(I - T)$, then $x = Tx$ and hence, via (1.3), we have $x = T_{[n]}x$ for all $n \in \mathbb{N}$. It then follows from (2.3) that $x = 0$. So, (2.4) is also valid. □

Given $T \in \mathcal{L}(X)$, its *dual operator* $T^t \colon X' \to X'$, where $X'$ is the continuous dual space of $X$, is defined by $\langle Tx, x'\rangle = \langle x, T^t x'\rangle$ for all $x \in X$, $x' \in X'$. By $X_\sigma$ we denote $X$ equipped with its weak topology $\sigma(X, X')$. A subset $A \subseteq X$ is called *relatively sequentially* $\sigma(X, X')$*–compact* if every sequence in $A$ contains a subsequence which is convergent in $X_\sigma$. Such sets belong to $\mathcal{B}(X)$, [12, §24;(1)], after recalling that every sequentially compact set in any lcHs is also relatively countably compact, [12, p.310]. The following version of the Mean Ergodic Theorem for Banach spaces occurs in [8, Ch. VIII, 5.1–5.3], [16, p.214], and for lcHs' in [1, Theorem 2.4].

**Proposition 2.2.** *Let $X$ be a barrelled lcHs and $T \in \mathcal{L}(X)$. Then $T$ is mean ergodic if and only if it satisfies* (2.5) *and*

$$\{T_{[n]}x\}_{n=1}^{\infty} \text{ is relatively sequentially } \sigma(X, X')\text{–compact, } \forall x \in X. \tag{2.7}$$

*Setting $P := \tau_s\text{-}\lim_{n\to\infty} T_{[n]}$, the operator $P$ is a projection which commutes with $T$ and satisfies* $\operatorname{Im}(P) = \operatorname{Ker}(I - T)$ *and* $\operatorname{Ker}(P) = \overline{\operatorname{Im}(I - T)}$*. Moreover, $X$ has a direct sum decomposition as given by* (1.2).

Many lcHs' $X$ have the property that all relatively $\sigma(X, X')$–compact sets are also relatively sequentially $\sigma(X, X')$–compact. This includes all Fréchet spaces (actually, all (LF)–spaces), all (DF)–spaces, and many more, [6, Theorem 11, Examples 1.2]. The following fact is an extension of [1, Corollary 2.7].

**Proposition 2.3.** *Let $X$ be a reflexive lcHs in which every relatively $\sigma(X, X')$–compact set is relatively sequentially $\sigma(X, X')$–compact. Then $X$ is mean ergodic.*

*Proof.* Let $T \in \mathcal{L}(X)$ be power bounded. Then clearly (2.5) holds as does (2.6); see [1, Remark 2.6(i)]. By definition, all reflexive spaces are barrelled and all bounded sets in a reflexive lcHs $X$ are relatively $\sigma(X, X')$–compact, [12, p.299]. By the hypotheses on $X$, all bounded sets are then relatively sequentially $\sigma(X, X')$–compact. This, together with (2.6), implies that (2.7) holds. The mean ergodicity of $T$ then follows from Proposition 2.2. □

A special case of the following fact occurs in [1, Proposition 2.8].

**Proposition 2.4.** *Let $X$ be a Montel space in which every relatively $\sigma(X, X')$–compact set is relatively sequentially $\sigma(X, X')$–compact. Then $X$ is uniformly mean ergodic.*

*Proof.* Let $T \in \mathcal{L}(X)$ be power bounded. Since $X$ is reflexive, [12, p.369], it follows from Proposition 2.3 that $T$ is mean ergodic. The proof can now be completed as in that of Proposition 2.8 of [1]. □

For Banach spaces the following result is due to M. Lin [15] and for general Fréchet spaces it occurs in [1, Proposition 2.16]. An examination of the proof given in [1] shows that the Fréchet space condition is only used to conclude that the inverse of a certain linear bijection is again continuous. So, we can replace this requirement with the property that every continuous linear surjection is an open map (i.e., the open mapping theorem is valid). Of course, (ii) $\Rightarrow$ (i) is immediate from the identities

$$I - T_{[n]} = \left[ \sum_{m=1}^{n} \frac{1}{n}(I + T + \dots T^{m-1}) \right] (I - T), \quad n \in \mathbb{N}.$$

So, we have the following result.

**Proposition 2.5.** *Let $X$ be a lcHs with the property that every continuous linear surjection from $X$ onto itself is an open map. Let $T \in \mathcal{L}(X)$ satisfy* $\mathrm{Ker}(I - T) = \{0\}$ *and $\frac{1}{n}T^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. Consider the following statements.*

(i) *$I - T_{[n]}$ is surjective for some $n \in \mathbb{N}$.*
(ii) *$I - T$ is surjective.*
(iii) *$T_{[n]} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$.*

*Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). If, in addition, $X$ is a Banach space, then also (iii) $\Rightarrow$ (i).*

The class of all lcHs' which satisfy the hypothesis of Proposition 2.5 includes all ultrabornological spaces which possess a web, [17, Theorem 24.30], and in particular, includes all (LF)–spaces, the space of distributions $\mathcal{D}'$, and many more.

It is shown in Example 2.17 of [1], that the implication (iii)$\Rightarrow$(i) of Theorem 2.5 fails for Fréchet spaces in general. It might be hoped that (iii)$\Rightarrow$(i) holds at least for (LB)–spaces. We will see in Section 4 that this is not the case.

The strong topology in a lcHs $X$ (resp. in $X'$) is denoted by $\beta(X, X')$ (resp. $\beta(X', X)$) and we write $X_\beta$ (resp. $X'_\beta$); see [12, §21.2] for the definition. The final three results are concerned with duality. The first one occurs in [2, Lemma 2.1].

**Lemma 2.6.** *Let $X$, $Y$ be lcHs' with $Y$ quasi–barrelled. Then the linear map $\Phi\colon \mathcal{L}_b(X, Y) \to \mathcal{L}_b(Y'_\beta, X'_\beta)$ defined by $\Phi(T) := T^t$, for $T \in \mathcal{L}(X, Y)$, is continuous.*

*In particular, if $X$ is quasi–barrelled and $\{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ is a sequence which satisfies $\tau_b\text{-}\lim_{n\to\infty} T_n = T$ in $\mathcal{L}_b(X)$, then also $\tau_b\text{-}\lim_{n\to\infty} T_n^t = T^t$ in $\mathcal{L}_b(X'_\beta)$.*

A useful consequence is the following observation.

**Corollary 2.7.** *Let $X$ be a lcHs and $T \in \mathcal{L}(X)$.*

(i) *If $T$ is uniformly mean ergodic, then $T^t \in \mathcal{L}(X'_\beta)$ is mean ergodic.*

(ii) *Suppose that $X$ is quasi–barrelled. If $T$ is uniformly mean ergodic, then $T^t \in \mathcal{L}(X'_\beta)$ is uniformly mean ergodic.*

(iii) *Suppose that $X$ is sequentially complete and that both $X$ and $X'_\beta$ are quasi–barrelled. If $T^t \in \mathcal{L}(X'_\beta)$ is uniformly mean ergodic, then $T$ itself is uniformly mean ergodic.*

*Proof.* (i) By assumption there is $P \in \mathcal{L}(X)$ such that $\lim_{n\to\infty} T_{[n]} = P$ in $\mathcal{L}_b(X)$. Fix $x' \in X'$ and $B \in \mathcal{B}(X)$. Since

$$W_B := \{S \in \mathcal{L}(X) : \ |\langle Sx, x'\rangle| \leq 1 \text{ for all } x \in B\}$$

is a 0–neighbourhood in $\mathcal{L}_b(X)$ there is $n(0) \in \mathbb{N}$ such that $(T_{[n]} - P) \in W_B$ for all $n \geq n(0)$, that is,

$$|\langle x, (T^t_{[n]} - P^t)x'\rangle| = |\langle (T_{[n]} - P)x, x'\rangle| \leq 1, \quad x \in B.$$

Equivalently, $(T^t_{[n]}x' - P^t x') \in B^\circ$ (the polar of $B$). Since $B \in \mathcal{B}(X)$ is arbitrary, we conclude that $\lim_{n\to\infty} T^t_{[n]}x' = P^t x'$ in $X'_\beta$ for each $x' \in X'$, i.e., $\lim_{n\to\infty} T^t_{[n]} = P^t$ in $\mathcal{L}_s(X'_\beta)$.

(ii) Since $(T^t)_{[n]} = T^t_{[n]}$ for all $n \in \mathbb{N}$ (see (1.3)), it follows from Lemma 2.6 that $T^t$ is uniformly mean ergodic in $X'_\beta$.

(iii) Let $T^t_{[n]} \to Q$ in $\mathcal{L}_b(X'_\beta)$. By Lemma 2.6 applied to $T^t$ in $X'_\beta$ we have that $(T^{tt})_{[n]} \to Q^t$ in $\mathcal{L}_b(X''_\beta)$ as $n \to \infty$. Observe that the restriction $(T^{tt})_{[n]}|_X = T_{[n]}$, for $n \in \mathbb{N}$. Interpreting any given element $x \in X$ (and then also $T_{[n]}x$) as an element of $X''_\beta$ we have that $Q^t x = \lim_{n\to\infty} T_{[n]}x$ in $X''_\beta$. Since $X$ is quasi–barrelled, $X''_\beta$ induces the original topology on $X$, [12, p.301], that is, $\{T_{[n]}x\}_{n=1}^\infty$ is Cauchy in $X$ and hence, by sequential completeness, converges in $X$. It follows that the limit must be $Q^t x$, that is, $Q^t x \in X$. Hence, $Q^t(X) \subseteq X$ and, since the topology of $X$ is that induced by $X''_\beta$, it follows that $P := Q^t|_X$ belongs to $\mathcal{L}(X)$. Moreover, $(T^{tt})_{[n]} \to Q^t$ in $\mathcal{L}_b(X''_\beta)$ implies that $T_{[n]} \to P$ in $\mathcal{L}_b(X)$ as $n \to \infty$. □

*Remark* 2.8. Every reflexive lcHs $X$ satisfies the hypotheses of (iii) in Corollary 2.7. So does every distinguished Fréchet space (such spaces are not necessarily reflexive). Also, every sequentially complete, quasi–barrelled (DF)–space $X$ has the required properties (as $X'_\beta$, being a Fréchet space, is surely quasi–barrelled). Of course, every sequentially complete, quasi–barrelled space is actually barrelled, [12, p.368].

If $X$ is a quasi–barrelled lcHs, then the general theory of such spaces ensures that $T \in \mathcal{L}(X)$ is power bounded if and only if $T^t \in \mathcal{L}(X'_\beta)$ is power bounded, [13, (6), p.138]. Combining this with Corollary 2.7(iii) gives the following result; see also Proposition 2.4.

**Corollary 2.9.** *Let $X$ be a sequentially complete, barrelled lcHs with $X'_\beta$ quasi–barrelled. Then $X$ is uniformly mean ergodic if and only if $X'_\beta$ is uniformly mean ergodic.*

We mention that if $T \in \mathcal{L}(X)$ is mean ergodic, then so is $T^t \in \mathcal{L}(X'_\sigma)$, where $X'_\sigma$ denotes $X'$ equipped with its weak–star topology $\sigma(X', X)$. Actually, it suffices for $T$ to be mean ergodic in $X_\sigma$.

As an application of Corollary 2.9 we have some examples.

*Example.* (i) The separable (LB)–spaces $L_{p^+} := \cup_{r>p} L^r([0,1])$, for $1 < p < \infty$, are all reflexive. The corresponding strong duals $(L_{p^+})'_\beta = \cap_{1 \leq r < p'} L^r([0,1]) =: L_{p'^-}$, with $p'$ the conjugate exponent of $p$, are reflexive Fréchet spaces, equipped with the seminorms

$$q_{p',\beta(m)}(f) := \left( \int_0^1 |f(t)|^{\beta(m)} dt \right)^{1/\beta(m)}, \quad f \in L_{p'^-},$$

for any increasing sequence $1 \leq \beta(m) \uparrow p'$ as $m \to \infty$. These Fréchet spaces have been studied in [5]. By [1, Proposition 2.11] each Fréchet space $L_{p'^-}$, for $1 < p' < \infty$, fails to be uniformly mean ergodic. So, each (LB)–space $L_{p^+}$, for $1 < p < \infty$, fails to be uniformly mean ergodic; see Corollary 2.9.

(ii) For each $1 < p < \infty$, the sequence space $\ell^{p^-} := \cup_{1 \leq r < p} \ell^r$ is a separable, reflexive (LB)–space. The corresponding strong dual $(\ell^{p^-})'_\beta = \cap_{r > p'} \ell^r =: \ell^{p'^+}$, with $p'$ the conjugate exponent of $p$, is a reflexive Fréchet space, equipped with the seminorms

$$q_{p',n}(x) := \left( \sum_{i=1}^\infty |x_i|^{\beta(n)} \right)^{1/\beta(n)}, \quad x \in \ell^{p'^+},$$

where $\beta(n) := p' + \frac{1}{n}$ for $n \in \mathbb{N}$. This family of Fréchet spaces was studied in [18]. By [1, Proposition 2.15] the Fréchet space $\ell^{p'^+}$, for $1 < p' < \infty$, is not uniformly mean ergodic. So, again by Corollary 2.9, the (LB)–space $\ell^{p^-}$, for $1 < p < \infty$, is not uniformly mean ergodic.

## 3. Mean ergodic results

A sequence $(P_n)_{n=1}^\infty \subseteq \mathcal{L}(X)$ is a *Schauder decomposition* of $X$ if it satisfies:

(S1) $P_n P_m = P_{\min\{m,n\}}$ for all $m, n \in \mathbb{N}$,

(S2) $P_n \to I$ in $\mathcal{L}_s(X)$ as $n \to \infty$, and

(S3) $P_n \neq P_m$ whenever $n \neq m$.

By setting $Q_1 := P_1$ and $Q_n := P_n - P_{n-1}$ for $n \geq 2$ we arrive at a sequence of pairwise orthogonal projections (i.e. $Q_n Q_m = 0$ if $n \neq m$) satisfying $\sum_{n=1}^\infty Q_n = I$, with the series converging in $\mathcal{L}_s(X)$. If the series is unconditionally convergent in $\mathcal{L}_s(X)$, then $\{P_n\}_{n=1}^\infty$ is called an *unconditional Schauder decomposition*, [19]. Such decompositions are intimately associated with (non–trivial) spectral measures; see (the proof of) [4, Proposition 4.3] and [19, Lemma 5 and Theorem 6]. If $X$ is barrelled, then (S2) implies that $\{P_n\}_{n=1}^\infty$ is an equicontinuous subset of $\mathcal{L}(X)$. According to (S1) each $P_n$ and $Q_n$, for $n \in \mathbb{N}$, is a projection and $Q_n \to 0$ in $\mathcal{L}_s(X)$ as $n \to \infty$. Condition (S3) ensures that $Q_n \neq 0$ for each $n \in \mathbb{N}$.

Let $\{P_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ be a Schauder decomposition of $X$. Then the dual projections $\{P_n^t\}_{n=1}^\infty \subseteq \mathcal{L}(X'_\sigma)$ always form a Schauder decomposition of $X'_\sigma$, [11, p.378]. If, in addition, $\{P_n^t\}_{n=1}^\infty \subseteq \mathcal{L}(X'_\beta)$ is a Schauder decomposition of $X'_\beta$, then the original sequence $\{P_n\}_{n=1}^\infty$ is called *shrinking*, [11, p.379]. Since (S1) and (S3) clearly hold for $\{P_n^t\}_{n=1}^\infty$, this means precisely that $P_n^t \to I$ in $\mathcal{L}_s(X'_\beta)$; see (S2).

In dealing with the uniform mean ergodicity of operators the following notion, due to J.C. Díaz and M.A. Miñarro, [7, p.194], is rather useful. A Schauder decomposition $\{P_n\}_{n=1}^\infty$ in a lcHs $X$ is said to have *property* $(M)$ if $P_n \to I$ in $\mathcal{L}_b(X)$ as $n \to \infty$. Since every non-zero projection $P$ in a Banach space satisfies $\|P\| \geq 1$, it is clear that no Schauder decomposition in any Banach space can have property $(M)$. For non-normable spaces the situation is quite different. For instance, if $X$ is a Fréchet Montel space (resp. Fréchet GDP–space, which is a larger class of spaces; see [4]), then *every* Schauder decomposition in $X$ has property $(M)$; see [7] (resp. [4, Proposition 4.2]). The following two technical results will be needed latter.

**Lemma 3.1.** *Let $X$ be a barrelled lcHs which admits a non–shrinking Schauder decomposition. Then there exists a Schauder decomposition $\{P_j\}_{j=1}^\infty \subset \mathcal{L}(X)$ of $X$, a functional $\xi \in X'$ and a bounded sequence $\{z_j\}_{j=1}^\infty \subset X$ with $z_j \in (P_{j+1} - P_j)(X)$ such that $|\langle z_j, \xi\rangle| > \frac{1}{2}$ for all $j \in \mathbb{N}$.*

*Proof.* Adapt the proof of Lemma 4.4 in [1]. □

**Lemma 3.2.** *Let $X$ be a barrelled lcHs which admits a Schauder decomposition without property (M). Then there exists a Schauder decomposition $\{P_j\}_{j=1}^\infty \subseteq \mathcal{L}((X)$ of $X$, a seminorm $q \in \Gamma_X$ and a bounded sequence $\{z_j\}_{j=1}^\infty \subset X$ with $z_j \in (P_{j+1} - P_j)(X)$ such that $q(z_j) > \frac{1}{2}$ for all $j \in \mathbb{N}$.*

*Proof.* The proof of Lemma 4.5 in [1] also applies here. □

*Remark* 3.3. Let $\{P_j\}_{j=1}^\infty$ be any Schauder decomposition in the complete barrelled lcHs $X$ with $\Gamma_X$ a system of continuous seminorms generating the topology of $X$. Then $\{P_j\}_{j=1}^\infty$ is an equicontinuous sequence. Hence, for every $p \in \Gamma_X$ there exist $q \in \Gamma_X$ and $M_p > 0$ such that

$$p(P_j x) \leq M_p q(x), \quad x \in X,$$

for all $j \in \mathbb{N}$. By setting $\tilde{r}(x) := \sup_{j\in\mathbb{N}} r(P_j x)$, for every $r \in \Gamma_X$, we obtain

$$p(x) \leq \tilde{p}(x) \leq M_p q(x) \leq M_p \tilde{q}(x), \quad x \in X.$$

Accordingly, $\tilde{\Gamma}_X := \{\tilde{p} : \ p \in \Gamma_X\}$ is also a system of continuous seminorms generating the topology of $X$ and satisfies

$$\tilde{p}(P_j x) \leq \tilde{p}(x), \quad x \in X, \, j \in \mathbb{N}. \tag{3.1}$$

The proof of the next result and Theorems 3.6 and 3.8 below follow those given in [1] for the corresponding result in Fréchet spaces. We include the essential parts of these proofs to illustrate certain differences in the current setting and for the sake of self containment.

**Theorem 3.4.** *Let $X$ be a complete barrelled lcHs which admits a non–shrinking Schauder decomposition. Then there exists a power bounded operator on $X$ which is not mean ergodic.*

*Proof.* The proof is similar to that of Theorem 1.5 of [1]. For the sake of completeness, we include the proof.

Let $(P_j)_j \subset \mathcal{L}(X)$ denote a Schauder decomposition as given by Lemma 3.1 and define projections $Q_j := P_j - P_{j-1}$ ($P_0 := 0$) and closed subspaces $X_j := Q_j(X)$, $j \in \mathbb{N}$.

By Lemma 3.1 there exist a bounded sequence $\{z_j\}_{j=1}^{\infty} \subset X$ with $z_j \in X_{j+1}$, and $\xi \in X'$ such that $|\langle z_j, \xi\rangle| > \frac{1}{2}$ for all $j \in \mathbb{N}$. Set $e_j := z_j/\langle z_j, \xi\rangle \in X_{j+1}$. Then $\{e_j\}_{j=1}^{\infty}$ is a bounded sequence of $X$ and $\langle e_j, \xi\rangle = 1$ for all $j \in \mathbb{N}$.

By Remark 3.3 there exists a system $\Gamma_X$ of continuous seminorms generating the topology of $X$ such that

$$p(P_j x) \leq p(x), \quad x \in X, \tag{3.2}$$

for all $p \in \Gamma_X$ and $j \in \mathbb{N}$. Moreover, since $\xi \in X'$, there exists $p_0 \in \Gamma_X$ such that $|\langle x, \xi\rangle| \leq p_0(x)$ for all $x \in X$.

As in [10, p.150], take a sequence $a = \{a_j\}_{j=1}^{\infty} \subseteq \mathbb{R}$ with $\sum_{j=1}^{\infty} a_j = 1$, $a_j > 0$, and set $A_n := \sum_{j=1}^{n} a_j$. For $x \in X$ and integers $m > n \geq 2$ we have

$$\sum_{k=n}^{m} A_k Q_k x = \left(\sum_{j=1}^{n-1} a_j\right)\left(\sum_{k=n}^{m} Q_k x\right) + \sum_{j=n}^{m} a_j \left(\sum_{k=j}^{m} Q_k x\right).$$

Since $\sum_{k=1}^{\infty} Q_k x$ sums to $x$ in $X$, we see that $\{\sum_{k=1}^{m} A_k Q_k x\}_{m=1}^{\infty}$ is a Cauchy sequence and hence, converges in $X$. Moreover, for each $p \in \Gamma_X$, by (3.2) we have

$$\begin{aligned} p\left(\sum_{k=1}^{m} A_k Q_k x\right) &= p\left(\sum_{j=1}^{m} a_j (P_m - P_{j-1})x\right) \\ &\leq \sum_{j=1}^{m} a_j (p(P_m x) + p(P_{j-1}x)) \leq 2p(x), \end{aligned} \tag{3.3}$$

for each $m \in \mathbb{N}$. Define a linear map $T_a\colon X \to X$ by

$$T_a x := \sum_{k=1}^{\infty} A_k Q_k x + \sum_{j=2}^{\infty} \langle P_{j-1}x, \xi\rangle a_j e_j, \quad x \in X. \tag{3.4}$$

From (3.3) we obtain, for each $p \in \Gamma_X$ with $p \geq p_0$, that

$$p(T_a x) \leq p(\sum_{k=1}^{\infty} A_k Q_k x) + \sum_{j=2}^{\infty} |\langle P_{j-1}x, \xi\rangle| a_j p(e_j)$$
$$\leq 2p(x) + \sum_{j=2}^{\infty} a_j p_0(P_{j-1}x) p(e_j).$$

Note that $M_p := \sup_{j\in\mathbb{N}} p(e_j) < \infty$, because $\{e_j\}_{j=1}^{\infty}$ is bounded in $X$. Moreover, by (3.2) we have $p_0(P_{j-1}x) \leq p_0(x) \leq p(x)$ for all $x \in X$. Hence,

$$p(T_a x) \leq (2 + M_p)p(x)$$

for all $x \in X$, where $2 + M_p$ depends only on $p$.

To show that $T_a$ is power bounded, it suffices to show that for arbitrary sequences $a = \{a_j\}_{j=1}^{\infty}$ and $b = \{b_j\}_{j=1}^{\infty}$ of positive numbers with $\sum_{j=1}^{\infty} a_j = 1 = \sum_{j=1}^{\infty} b_j$ we have $T_a T_b = T_c$, with $c$ a sequence of the same type. This is the claim in p. 150 of [10] which is purely algebraic and is proved on p. 151 of [10].

Finally, proceeding as in the final part of the proof of Theorem 1.5 of [1] one shows that $\mathrm{Ker}(I - T_a) = \{0\}$ and $\xi \in \mathrm{Ker}(I - T_a^t)$, i.e., $\mathrm{Ker}(I - T_a^t) \neq \{0\}$. Thus, we can apply Theorem 2.12 of [1] to conclude that $T_a$ is not mean ergodic. □

Recall that a sequence $\{x_n\}_{n=1}^{\infty}$ in a lcHs $X$ is a *basis* if, for every $x \in X$, there is a unique sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ such that the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges to $x$ in $X$. By setting $f_n(x) := \alpha_n$ we obtain a linear form $f_n : X \to \mathbb{C}$ which is called the *n-th coefficient functional associated to* $\{x_n\}_{n=1}^{\infty}$. The functionals $f_n$, $n \in \mathbb{N}$, are uniquely determined by $\{x_n\}_{n=1}^{\infty}$ and $\{(x_n, f_n)\}_{n=1}^{\infty}$ is a biorthogonal sequence (i.e. $\langle x_n, f_m\rangle = \delta_{mn}$ for $m, n \in \mathbb{N}$). For each $n \in \mathbb{N}$, the map $P_n : X \to X$ defined by

$$P_n : x \mapsto \sum_{i=1}^{n} f_i(x)x_i = \sum_{i=1}^{n} \langle x, f_i\rangle x_i, \quad x \in X, \tag{3.5}$$

is a linear projection with range equal to the finite dimensional space $\mathrm{span}(x_i)_{i=1}^{n}$. If, in addition, $\{f_n\}_{n=1}^{\infty} \subseteq X'$, then the basis $\{x_n\}_{n=1}^{\infty}$ is called a *Schauder basis* for $X$. In this case, $\{P_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X)$ is clearly a Schauder decomposition of $X$ and each dual operator

$$P_n^t : x' \mapsto \sum_{i=1}^{n} \langle x_i, x'\rangle f_i, \qquad x' \in X', \tag{3.6}$$

for $n \in \mathbb{N}$, is a projection with range equal to $\mathrm{span}(f_i)_{i=1}^{n}$. Moreover, for every $x' \in X'$ the series $\sum_{i=1}^{\infty} \langle x_i, x'\rangle f_i$ converges to $f$ in $X'_\sigma$. For this reason, $\{f_n\}_{n=1}^{\infty}$ is also referred to as the *dual basis* of the Schauder basis $\{x_n\}_{n=1}^{\infty}$. The terminology "$X$ has a Schauder basis" will also be abbreviated simply to "$X$ has a basis".

**Theorem 3.5.** *Let $X$ be a complete barrelled lcHs with a Schauder basis and in which every relatively $\sigma(X, X')$–compact subset of $X$ is relatively sequentially*

*$\sigma(X, X')$–compact. Then $X$ is reflexive if and only if every power bounded operator on $X$ is mean ergodic.*

*Proof.* If $X$ is reflexive, then $X$ is mean ergodic by Proposition 2.3. Conversely, if $X$ is not reflexive, then Theorem 1.2 of [1] shows that $X$ admits a non–shrinking Schauder basis. By Theorem 3.4, $X$ is not mean ergodic. □

**Theorem 3.6.** *Let $X$ be a complete barrelled lcHs which admits a Schauder decomposition without property (M). Then there exists a power bounded, mean ergodic operator $T \in \mathcal{L}(X)$ which is not uniformly mean ergodic.*

*Proof.* Let $\{P_j\}_{j=1}^{\infty} \subseteq \mathcal{L}(X)$ denote a Schauder decomposition as given by Lemma 3.2 and define projections $Q_j := P_j - P_{j-1}$ ($P_0 := 0$) and closed subspaces $X_j := Q_j(X)$ for all $j \in \mathbb{N}$. By Lemma 3.2 there exist a bounded sequence $\{z_j\}_{j=1}^{\infty} \subseteq X$ and a continuous seminorm $q$ on $X$ with $z_j \in X_{j+1}$ and $q(z_j) > 1/2$ for all $j \in \mathbb{N}$.

Since $\{P_j\}_{j=1}^{\infty}$ is an equicontinuous sequence (because $X$ is barrelled), we can apply Remark 3.3 to choose a system $\Gamma_X$ of continuous seminorms generating the topology of $X$ such that

$$p(P_j x) \le p(x), \quad x \in X, \tag{3.7}$$

for all $p \in \Gamma_X$ and $j \in \mathbb{N}$. Clearly, there also exists $p_0 \in \Gamma_X$ such that $p_0 \ge q$ on $X$. Hence, $p_0(z_j) > 1/2$ for all $j \in \mathbb{N}$.

For any sequence $a = \{a_j\}_{j=1}^{\infty}$ of positive numbers with $\sum_{j=1}^{\infty} a_j = 1$ we set $A_n := \sum_{j=1}^{n} a_j$ and define a linear map $T_a \colon X \to X$ by

$$T_a x := \sum_{k=1}^{\infty} A_k Q_k x, \quad x \in X.$$

As in the proof of Theorem 3.4 one shows that $T_a$ is well defined, satisfies

$$p(T_a x) \le 2p(x), \quad x \in X, \tag{3.8}$$

for all $p \in \Gamma_X$, and is power bounded.

Proceeding as in the proof of Theorem 5.2 of [1], one shows that $\mathrm{Ker}(I - T_a) = \{0\}$ and $\mathrm{Ker}(I - T_a^t) = \{0\}$ and hence, by Theorem 2.12 of [1], $T_a$ is mean ergodic.

It remains to show that $T := T_a$ is not uniformly mean ergodic for the choice $a_j := 2^{-j}$. In this case, $A_k = 1 - 2^{-k}$ for all $k \in \mathbb{N}$. Moreover, from $Q_j Q_k = 0$ whenever $j \ne k$ and $Q_k^2 = Q_k$ it follows that

$$T^m x = \sum_{k=1}^{\infty} A_k^m Q_k x, \quad x \in X,$$

for all $m \in \mathbb{N}$. Hence,

$$T_{[n]} x = \frac{1}{n} \sum_{k=1}^{\infty} \frac{A_k}{1 - A_k} \cdot (1 - A_k^n) Q_k x, \quad x \in X,\ n \in \mathbb{N}. \tag{3.9}$$

Since $T$ is mean ergodic, there exists $P \in \mathcal{L}(X)$ with $T_{[n]} \to P$ in $\mathcal{L}_s(X)$.

Next, if $x \in X_j$ for a fixed $j \in \mathbb{N}$, by (3.9) we have that

$$T_{[n]}x = \frac{1}{n}\frac{A_j}{1-A_j} \cdot (1-A_j^n)x,$$

for all $n \in \mathbb{N}$, as $Q_j Q_k = 0$ whenever $j \neq k$ and $Q_j^2 = Q_j$. Since $0 < (1-A_j^n) < 1$, it follows that

$$p(T_{[n]}x) \leq \frac{1}{n}\frac{A_j}{1-A_j}p(x)$$

for all $p \in \Gamma_X$ and $n \in \mathbb{N}$. Therefore, $p(T_{[n]}x) \to 0$ as $n \to \infty$ for all $p \in \Gamma_X$. Since $T_{[n]}x \to Px$ as $n \to \infty$, we see that $Px = 0$. That is, $Py = 0$ for all $y \in \cup_{j=1}^{\infty} X_j$. Since $\cup_{j=1}^{\infty} X_j$ is dense in $X$ and $P \in \mathcal{L}(X)$, we obtain that $P = 0$ on $X$, that is, $T_{[n]} \to 0$ in $\mathcal{L}_s(X)$.

Suppose that $T$ is uniformly mean ergodic. Then $T_{[n]} \to 0$ in $\mathcal{L}_b(X)$. In particular, since $\{z_j\}_{j=1}^{\infty}$ is a bounded sequence in $X$, we have

$$\lim_{n\to\infty} \sup_{j\in\mathbb{N}} p(T_{[n]}z_j) = 0 \tag{3.10}$$

for all $p \in \Gamma_X$. But, for all $j \in \mathbb{N}$,

$$p_0(T_{[2^j]}z_j) > \frac{1}{4}[1-(1-2^{-j})^{2^j}],$$

with $\lim_{j\to\infty}(1-2^{-j})^{2^j} = e^{-1}$. This is in contradiction with (3.10). □

For Fréchet spaces, the following result occurs in [1, Theorem 1.3].

**Theorem 3.7.** *Let $X$ be a complete barrelled lcHs with a Schauder basis and in which every relatively $\sigma(X, X')$–compact subset of $X$ is relatively sequentially $\sigma(X, X')$–compact. Then $X$ is Montel if and only if every power bounded operator on $X$ is uniformly mean ergodic, that is, if and only if $X$ is uniformly mean ergodic.*

*Proof.* Suppose that $X$ is Montel. Then Proposition 2.4 implies that $X$ is uniformly mean ergodic. Conversely, suppose that $X$ is not Montel. Observe that the Schauder decomposition $\{P_n\}_{n=1}^{\infty} \subset \mathcal{L}(X)$ induced by the basis of $X$ has the property that each space $Q_n(X) := (P_n - P_{n-1})(X)$, $n \in \mathbb{N}$, is Montel because $\dim Q_n(X) = 1$ for all $n \in \mathbb{N}$. By [2, Proposition 3.8(iii)], the Schauder decomposition $\{P_n\}_{n=1}^{\infty}$ does not satisfy property $(M)$ and hence, Theorem 3.6 guarantees the existence of a power bounded, mean ergodic operator in $\mathcal{L}(X)$ which fails to be uniformly mean ergodic. □

**Theorem 3.8.** *Let $X$ be a sequentially complete lcHs which contains an isomorphic copy of the Banach space $c_0$. Then there exists a power bounded operator on $X$ which is not mean ergodic.*

*Proof.* Suppose that $J$ is a topological isomorphism from $c_0$ into $X$. Let $\{e_n\}_{n=1}^{\infty}$ be the canonical basis of $c_0$. Then the elements $y_n := Je_n$ form a Schauder basis of $Y := J(c_0)$.

Denote by $||\ ||_{c_0}$ the norm in $c_0$ and by $\Gamma_X$ a system of continuous seminorms generating the topology of $X$. Then, for all $p \in \Gamma_X$, there exists $M_p > 0$ such that

$$p(Jx) \leq M_p ||x||_{c_0}, \quad x \in c_0.$$

There also exist $p_0 \in \Gamma_X$ and $K > 0$ such that

$$||x||_{c_0} \leq K p_0(Jx), \quad x \in c_0.$$

Therefore, we have that

$$\sup_{j\in\mathbb{N}} |x_j| \leq K p_0(\sum_{j=1}^{\infty} x_j y_j) \quad \text{and} \quad p(\sum_{j=1}^{\infty} x_j y_j) \leq M_p \sup_{j\in\mathbb{N}} |x_j| \tag{3.11}$$

for all $x = (x_j)_{j=1}^{\infty} \in c_0$ and $p \in \Gamma_X$.

Let $\{e'_n\}_{n=1}^{\infty} \subset \ell^1$ denote the dual basis of $\{e_n\}_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, define $y'_n \in Y'$ by $y'_n := e'_n \circ J^{-1}$, in which case $\{y'_n\}_{n=1}^{\infty}$ is the dual basis of $\{y_n\}_{n=1}^{\infty}$ and

$$|\langle y, y'_n \rangle| \leq K p_0(y), \quad y \in Y,$$

as $y = Jx$ for some $x \in c_0$. By the Hahn–Banach theorem, for each $n \in \mathbb{N}$ we can find $f_n \in X'$ such that $f_n|_Y = y'_n$ and

$$|\langle x, f_n \rangle| \leq K p_0(x), \quad x \in X. \tag{3.12}$$

Define $x_n := \sum_{i=1}^{n} y_i$ and $g_n := f_n - f_{n+1}$, for each $n \in \mathbb{N}$, and observe that

$$\langle x_k, g_n \rangle = \langle x_k, f_n \rangle - \langle x_k, f_{n+1} \rangle = \delta_{kn}$$

for all $k, n \in \mathbb{N}$. We can then define projections $P_n \colon X \to X$ via

$$P_n x := \sum_{k=1}^{n} \langle x, g_k \rangle x_k, \quad x \in X,$$

so that $P_n(X) = \operatorname{span}\{x_j\}_{j=1}^{n} = \operatorname{span}\{y_j\}_{j=1}^{n}$ and $P_n P_m = P_{\min\{n,m\}}$.

Set $h := f_1$ and observe that

$$\langle x_n, h \rangle = \langle \sum_{j=1}^{n} y_j, f_1 \rangle = 1, \quad n \in \mathbb{N}.$$

On the other hand, $x_n \in (P_n - P_{n-1})(X)$ (with $P_0 := 0$) for all $n \in \mathbb{N}$, and $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $X$ because $x_n = J(\sum_{j=1}^{n} e_j)$, $n \in \mathbb{N}$. Since $\| \sum_{j=1}^{n} e_j \|_{c_0} = 1$, we have

$$p(x_n) \leq M_p, \quad n \in \mathbb{N}, \tag{3.13}$$

for all $p \in \Gamma_X$. In particular,

$$p_0(x_n) \geq \frac{1}{K} \| \sum_{j=1}^{n} e_j \|_{c_0} = \frac{1}{K}, \quad n \in \mathbb{N}.$$

Moreover, the identities

$$P_n x = \sum_{k=1}^{n} (\langle x, f_k \rangle - \langle x, f_{n+1} \rangle) y_k, \quad x \in X, \ n \in \mathbb{N},$$

together with (3.11) and (3.12) imply that

$$p(P_n x) \leq M_p \sup_{1\leq k\leq n} |\langle x, f_k\rangle - \langle x, f_{n+1}\rangle| \leq 2M_p K p_0(x) \tag{3.14}$$

for all $p \in \Gamma_X$, $n \in \mathbb{N}$ and $x \in X$. Accordingly, $\{P_n\}_{n=1}^{\infty} \subset \mathcal{L}(X)$ is equicontinuous.

Let $a = \{a_j\}_{j=1}^{\infty}$ be any sequence of positive numbers with $\sum_{j=1}^{\infty} a_j = 1$ and set $A_n := \sum_{j=1}^{n} a_j$ for $n \in \mathbb{N}$. As in the statement of Theorem 3 of [10], we define

$$S_a x := x - \sum_{n=2}^{\infty} a_n P_{n-1} x + \sum_{n=2}^{\infty} \langle P_{n-1}x, h\rangle x_n, \quad x \in X.$$

Then by (3.13), (3.12) and (3.11) we have, for each $x \in X$, that

$$\begin{aligned} p(S_a x) &\leq p(x) + 2M_p K p_0(x) + M_p \sup_{n\geq 2} |\langle P_{n-1}x, h\rangle| \\ &\leq p(x) + 2M_p K p_0(x) + M_p K p_0(P_{n-1}x) \\ &\leq p(x) + 2M_p K p_0(x) + M_p K^2 M_{p_0} p_0(x) \\ &= (1 + 2M_p K + M_p K^2 M_{p_0}) p(x) \end{aligned}$$

for all $p \in \Gamma_X$ with $p \geq p_0$. So, $S_a \in \mathcal{L}(X)$.

The fact that $S_a$ is power bounded follows from the Claim on p. 156 of [10], stating that $S_a S_b = S_c$ for an appropriate $c$.

It remains to show that $S_a$ is not mean ergodic. For this, we can now proceed exactly as in the final part of the proof of Theorem 1.6 of [1]. □

## 4. Mean ergodicity of co-echelon spaces

We wish to give an application of the previous results to Köthe co-echelon spaces.

Let $I$ be a countable index set. A Köthe matrix $A = (a_n)_{n=1}^{\infty}$ is an increasing sequence of strictly positive functions on $I$. Let $V = (v_n)_{n=1}^{\infty}$ denote the associated decreasing sequence of functions $v_n := 1/a_n$, $n \in \mathbb{N}$. Define the inductive limits

$$k_p(V) = k_p(I, V) = \operatorname*{ind}_n \ell_p(v_n),\ 1 \leq p \leq \infty, \text{ and } k_0(V) = k_0(I, V) = \operatorname*{ind}_n c_0(v_n),$$

generated by the (weighted) Banach spaces

$$\ell_p(v_n) = \{x = (x_i)_{i\in I} \in \mathbb{C}^I : q_{p,n}(x) = \left(\sum_{i\in I} (v_n(i)|x_i|)^p\right)^{1/p} < \infty\}, \text{ if } 1 \leq p < \infty,$$

and

$$\ell_\infty(v_n) = \{x = (x_i)_{i\in I} \in \mathbb{C}^I : q_{\infty,n}(x) = \sup_{i\in I} v_n(i)|x_i| < \infty\},$$

$$c_0(v_n) = \{x = (x_i)_{i\in I} \in \mathbb{C}^I : (v_n(i)|x_i|)_{i\in I} \text{ converges uniformly to 0 in } I\}.$$

That is, $k_p(V)$ is the increasing union of the Banach spaces $\ell_p(v_n)$, respectively $c_0(v_n)$, for $n \in \mathbb{N}$, endowed with the strongest locally convex topology under which the inclusion of each of these Banach spaces is continuous, i.e., $k_p(V)$ is an

(LB)–space and so a barrelled, ultrabornological (DF)–space. The spaces $k_p(V)$ are called *co–echelon spaces* of order $p$.

Given a decreasing sequence $V = (v_n)_{n=1}^{\infty}$ of strictly positive functions on $I$, set

$$\bar{V} = \{\bar{v} = (\bar{v}(i))_{i\in I} \in \mathbb{R}_+^I : \ \forall n \in \mathbb{N} \ \sup_{i\in I} \frac{\bar{v}(i)}{v_n(i)} < \infty\}$$

and associate to $\bar{V}$ the following projective limit spaces

$$K_p(\bar{V}) = K_p(I, \bar{V}) = \operatorname*{proj}_{\bar{v}\in\bar{V}} \ell_p(\bar{v}), \text{ if } 1 \le p \le \infty; \ K_0(\bar{V}) = K_0(I, \bar{V}) = \operatorname*{proj}_{\bar{v}\in\bar{V}} c_0(\bar{v})\,.$$

These spaces are equipped with the complete locally convex topology given by the seminorms $(q_{p,\bar{v}})_{\bar{v}\in\bar{V}}$, where

$$q_{p,\bar{v}}(x) = \left(\sum_{i\in I} (\bar{v}(i)|x_i|)^p\right)^{1/p}, \ 1 \le p < \infty, \text{ and } q_{\infty,\bar{v}}(x) = \sup_{i\in I} \bar{v}(i)|x_i|\,.$$

Then $k_p(V)$ is continuously embedded in $K_p(\bar{V})$ for $1 \le p \le \infty$ or $p = 0$, with $k_p(V) = K_p(\bar{V})$ for $1 \le p \le \infty$. More precisely, $k_p(V) = K_p(\bar{V})$ algebraically and topologically for $1 \le p < \infty$ and $k_\infty(V) = K_\infty(\bar{V})$ algebraically. Moreover, $k_0(V)$ is, in general, a proper topological subspace of the barrelled (DF)-space $K_0(\bar{V})$ such that its completion is equal to $K_0(\bar{V})$. The (LB)–space $k_p(V)$ is complete for $1 \le p \le \infty$, and reflexive for $1 < p < \infty$. In particular, the vectors $e_j = (\delta_{ij})_{i\in I}$ form a Schauder basis for $k_p(V)$ if $1 \le p < \infty$ or $p = 0$. For all these facts we refer to [3].

**Proposition 4.1.** *Let $V = (v_n)_{n=1}^{\infty}$ be a decreasing sequence of strictly positive functions on $I$ and $1 < p < \infty$. Then the reflexive (LB)–space $k_p(V)$ ($= K_p(\bar{V})$ algebraically and topologically) is uniformly mean ergodic if and only if it is a Montel space (hence, a (DFM)–space).*

*Proof.* Since the complete barrelled space $k_p(V)$ admits a Schauder basis and its bounded sets are relatively sequentially $\sigma(k_p(V), (k_p(V))')$–compact, [6, Theorem 11, Examples 1,2], the result follows from Theorem 3.7. □

**Proposition 4.2.** *Let $V = (v_n)_{n=1}^{\infty}$ be a decreasing sequence of strictly positive functions on $I$. Then the following assertions are equivalent.*

(i) $k_1(V)$ *is mean ergodic.*
(ii) $k_1(V)$ *is uniformly mean ergodic.*
(iii) $k_1(V)$ *is a Montel space (hence, a (DFM)–space).*
(iv) $k_1(V)$ *does not contain an isomorphic copy of* $\ell_1$.

*Proof.* The complete barrelled (LB)-space $k_1(V)$ admits a Schauder basis and every relatively $\sigma(k_1(V), (k_1(V))')$–compact subset of $k_1(V)$ is relatively sequentially $\sigma(k_1(V), (k_1(V))')$–compact, [6, Theorem 11, Examples 1, 2]. So, by Theorem 3.7 we have (ii) ⇔ (iii), and by [3, Theorem 4.7] we have (iii) ⇔ $k_1(V)$ is reflexive. On the other hand, $k_1(V)$ is reflexive ⇔ (i); see Theorem 3.5.

Next, (iii) $\Rightarrow$ (iv) is obvious, because a Montel space cannot contain an isomorphic copy of any infinite dimensional Banach space.

(iv) $\Rightarrow$ (iii): Suppose that $k_1(V)$ is not a Montel space. Then there exist an infinite set $I_0 \subset I$ and $n \in \mathbb{N}$ such that

$$\inf_{i \in I_0} \frac{v_m(i)}{v_n(i)} = c_m > 0\,, \quad \forall m \geq n,$$

[3, Theorem 4.7]. Then, in the sectional subspace $E_0$ of $k_1(V)$ defined by

$$E_0 := \{x \in k_1(V) : \; x_j = 0 \text{ for all } j \in I \setminus I_0\},$$

the topology of $\ell_1(v_m)$ coincides with that of $\ell_1(v_n)$ for all $m \geq n$. Indeed, for every $x \in E_0$ and $m \geq n$ we have

$$q_{1,m}(x) \leq q_{1,n}(x) = \sum_{i \in I_0} v_n(i)|x_i| \leq c_m^{-1} \sum_{i \in I_0} v_m(i)|x_i| = c_m^{-1} q_{1,m}(x)\,.$$

Consequently, the topology of $k_1(V)$ also coincides with that of $\ell_1(v_n)$ in $E_0$. Hence, $k_1(V)$ contains an isomorphic copy of $\ell_1$, which is a contradiction. □

**Proposition 4.3.** *Let $V = (v_n)_{n=1}^\infty$ be a decreasing sequence of strictly positive functions on $I$. Then the following assertions are equivalent.*

(i) *$k_\infty(V)$ is mean ergodic.*
(ii) *$k_\infty(V)$ is uniformly mean ergodic.*
(iii) *$k_\infty(V)$ is a Montel space (hence, a (DFM)-space).*
(iv) *$k_\infty(V)$ does not contain an isomorphic copy of $\ell_\infty$.*
(v) *$K_0(\bar{V}) = K_\infty(\bar{V}) = k_\infty(V)$ algebraically and topologically.*

*Proof.* By the discussion just prior to Proposition 2.3, together with Proposition 2.4, it is clear that (iii) $\Rightarrow$ (ii). That (ii) $\Rightarrow$ (i) is obvious. Since $\ell_\infty$ is an infinite dimensional Banach space (i.e., its closed unit ball is not compact), it is clear that (iii) $\Rightarrow$ (iv). Moreover, (iii) $\Leftrightarrow$ (v) by [3, Theorem 4.7].

(iv) $\Rightarrow$ (iii): Suppose that $k_\infty(V)$ is not a Montel space. Then there exist an infinite set $I_0 \subset I$ and $n \in \mathbb{N}$ such that

$$\inf_{i \in I_0} \frac{v_m(i)}{v_n(i)} = c_m > 0\,, \quad \forall m \geq n,$$

[3, Theorem 4.7]. Then, in the sectional subspace $E_0$ of $k_\infty(V)$ defined by

$$E_0 := \{x \in k_\infty(V) : \; x_j = 0 \text{ for all } j \in I \setminus I_0\},$$

the topology of $\ell_\infty(v_m)$ coincides with that of $\ell_\infty(v_n)$ for all $m \geq n$. Indeed, for every $x \in E_0$ and $m \geq n$ we have

$$q_{\infty,m}(x) \leq q_{\infty,n}(x) = \sup_{i \in I_0} v_n(i)|x_i| \leq c_m^{-1} \sup_{i \in I_0} v_m(i)|x_i| = c_m^{-1} q_{\infty,m}(x)\,.$$

Consequently, the topology of $k_\infty(V)$ also coincides with that of $\ell_\infty(v_n)$ in $E_0$. Hence, $k_\infty(V)$ contains an isomorphic copy of $\ell_\infty$. This is a contradiction.

(i) $\Leftrightarrow$ (iv): Suppose that $k_\infty(V)$ contains an isomorphic copy of $\ell_\infty$. This implies that $k_\infty(V)$ is not mean ergodic by [1, Remark 2.14(i)] . □

**Proposition 4.4.** *Let $V = (v_n)_{n=1}^{\infty}$ be a decreasing sequence of strictly positive functions on $I$. Suppose that the (LB)–space $k_0(V)$ is complete. Then the following assertions are equivalent.*

(i) *$k_0(V)$ is mean ergodic.*
(ii) *$k_0(V)$ is uniformly mean ergodic.*
(iii) *$k_0(V)$ is a Schwartz space (hence, a (DFS)-space).*
(iv) *$k_0(V)$ does not contain an isomorphic copy of $c_0$.*
(v) *$k_0(V) = k_\infty(V)$ algebraically and topologically.*

*Proof.* The complete, barrelled (LB)–space $k_0(V)$ admits a Schauder basis and every relatively $\sigma(k_0(V), (k_0(V))')$–compact subset of $k_0(V)$ is relatively sequentially $\sigma(k_0(V), (k_0(V))')$–compact, [6, Theorem 11, Examples 1, 2].

So, by Theorem 3.7 above and [3, Theorem 4.9] we have (ii) $\Leftrightarrow$ $k_0(V)$ is a Montel space $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v).

Next, (ii) $\Rightarrow$ (i) is obvious. Also, (iii) $\Rightarrow$ (iv) is obvious, because a Schwartz space cannot contain an isomorphic copy of any infinite dimensional Banach space.

(iv) $\Rightarrow$ (iii): Suppose that $k_0(V)$ is not a Schwartz space. Since $k_0(V)$ is complete, there exist an infinite set $I_0 \subset I$ and $n \in \mathbb{N}$ such that

$$\inf_{i \in I_0} \frac{v_m(i)}{v_n(i)} = c_m > 0\,, \quad \forall m \geq n,$$

[3, Theorems 3.7, 4.9]. Then, in the sectional subspace $E_0$ of $k_0(V)$ defined by

$$E_0 := \{x \in k_0(V) :\ x_j = 0 \text{ for all } j \in I \setminus I_0\},$$

the topology of $c_0(v_m)$ coincides with that of $c_0(v_n)$ for all $m \geq n$. Indeed, for every $x \in E_0$ and $m \geq n$ we have

$$q_{\infty,m}(x) \leq q_{\infty,n}(x) = \sup_{i \in I_0} v_n(i)|x_i| \leq c_m^{-1} \sup_{i \in I_0} v_m(i)|x_i| = c_m^{-1} q_{\infty,m}(x)\,.$$

Consequently, the topology of $k_0(V)$ also coincides with that of $c_0(v_n)$ in $E_0$. Hence, $k_0(V)$ contains an isomorphic copy of $c_0$. This is a contradiction.

(i) $\Rightarrow$ (ii): By Theorem 3.5 we have (i) $\Leftrightarrow$ $k_0(V)$ is reflexive. Since $k_0(V)$ is complete, it is then also Montel by [3, Theorems 3.7, 4.7], thereby implying that (ii) holds via Theorem 3.7. □

*Example.* Every (LF)–space $X$ (hence, every (LB)–space) satisfies the hypothesis of Proposition 2.5, because every linear continuous surjective map between two (LF)–spaces is necessarily open. But, in this setting, condition (iii) of Proposition 2.5 does not imply the condition (ii). Hence, also (iii) does not imply condition (i) as the following example illustrates.

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers satisfying $1 < a_{n+1} < a_n < a$ for some $a \in \mathbb{R}$ and for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ set $v_n := (a_n^i)_{i=1}^{\infty}$ and $V := (v_n)_{n=1}^{\infty}$, where $i \in I := \mathbb{N}$. Consider the co–echelon space $k_1(V)$ which is a Montel space (hence, a (DFM)–space) because, for all $n, m \in \mathbb{N}$ with $m > n$, we have $\frac{v_m(i)}{v_n(i)} = \left(\frac{a_m}{a_n}\right)^i \to 0$ as $i \to \infty$, [3, Theorem 4.7]. In particular, its (strong) topological dual is the Köthe echelon Fréchet space $\lambda_\infty(A) = \lambda_0(A)$, with $A := (v_n^{-1})_{n=1}^\infty$, [3, Theorem 4.7].

Define $T \in \mathcal{L}(k_1(V))$ by

$$Tx := ((1 - a^{-i})x_i)_{i=1}^\infty\,, \quad x \in k_1(V)\,.$$

It is easy to verify that $\mathrm{Ker}(I - T) = \{0\}$ and that $y := (a^{-i})_{i=1}^\infty \in k_1(V)$ does not belong to $\mathrm{Im}(I - T)$, i.e., $I - T$ is not surjective. So, condition (ii) of Theorem 2.5 does not hold.

Since $T^m x = ((1-a^{-i})^m x_i)_{i=1}^\infty$ for $x \in k_1(V)$ and for all $m \in \mathbb{N}$, the sequence $\{T^m x\}_{m=1}^\infty$ is bounded for all $x \in k_1(V)$. Indeed, given any $x \in k_1(V)$ there is $n \in \mathbb{N}$ such that $x \in \ell_1(v_n)$, thereby implying that

$$q_{1,n}(T^m x) = \sum_{i\in\mathbb{N}} |(1 - a^{-i})^m| \cdot |x_i| v_n(i) \le \sum_{i\in\mathbb{N}} |x_i| v_n(i) = q_{1,n}(x)$$

for all $m \in \mathbb{N}$. So, the barrelledness of $k_1(V)$ implies that the sequence $\{T^m\}_{m=1}^\infty \subset \mathcal{L}(k_1(V))$ is equicontinuous, i.e., for every $p \in \Gamma_{k_1(V)}$ the exists $q \in \Gamma_{k_1(V)}$ for which

$$p(T^m x) \le q(x)$$

for all $m \in \mathbb{N}$ and $x \in k_1(V)$. In particular, $T$ is power bounded.

Since $k_1(V)$ is a complete (LB)-space and hence, a regular (LB)–space, given any bounded set $B \subset k_1(V)$ there exist $k, n \in \mathbb{N}$ such that $B \subset kB_n$ ($B_n$ denotes the unit ball of $\ell_1(v_n)$). On the other hand, since the inclusion map $\ell_1(v_n) \hookrightarrow k_1(V)$ is continuous, given any $p \in \Gamma_{k_1(V)}$ there exists $c > 0$ such that

$$p(x) \le c q_{1,n}(x), \quad x \in \ell_1(v_n).$$

Therefore, for every $m \in \mathbb{N}$ we have

$$\sup_{x\in B} p\left(\frac{1}{m} T^m x\right) \le c\frac{1}{m} \sup_{x\in kB_n} q_{1,n}(T^m x) \le c\frac{1}{m} \sup_{x\in kB_n} q_{1,n}(x) \le ck\frac{1}{m}$$

and hence, $\sup_{x\in B} p(\frac{1}{m}T^m x) \to 0$ as $m \to \infty$. This shows that $\frac{1}{m}T^m \to 0$ in $\mathcal{L}_b(k_1(V))$.

It remains to establish condition (iii) of Proposition 2.5. For this, we observe that

$$T^t \xi = ((1 - a^{-i})\xi_i)_{i=1}^\infty\,, \quad \xi \in \lambda_\infty(A)\,,$$

so that $\mathrm{Ker}(I - T^t) = \{0\}$. Since $T$ is power bounded and both $\mathrm{Ker}(I - T) = \{0\}$ and $\mathrm{Ker}(I - T^t) = \{0\}$, we can apply [1, Theorem 2.12] to conclude that $T$ is mean ergodic. But, $k_1(V)$ is a Montel space whose relatively $\sigma(k_1(V), (k_1(V))')$–compact subsets are relatively sequentially $\sigma(k_1(V), (k_1(V))')$–compact (see the discussion prior to Proposition 2.3). So, by Proposition 2.4, $T$ is also uniformly mean ergodic. Hence, there is $P \in \mathcal{L}(k_1(V))$ such that $T_{[n]} \to P$ in $\mathcal{L}_b(k_1(V))$.

For each $r \in \mathbb{N}$, let $e_r$ be the element of $k_1(V)$ with 1 in the $r$–th coordinate and 0's elsewhere (we point out that $\{e_r\}_{r=1}^\infty$ is a Schauder basis for $k_1(V)$). Then, for all $r \in \mathbb{N}$,

$$T^m e_r = (1 - a^{-r})^m e_r \to 0 \quad \text{as} \quad m \to \infty\,,$$

so that

$$T_{[n]}e_r = \frac{\mu(1-\mu)}{n(1-\mu)}e_r \to 0 \quad \text{as} \quad m \to \infty\,,$$

with $\mu := (1-a^{-r})$. This implies that $P = 0$ and hence, that $T_{[n]} \to 0$ in $\mathcal{L}_b(k_1(V))$, i.e., condition (iii) is satisfied.

We remark that the operator $Tx := ((1-2^{-i})x_i)_{i=1}^\infty$, for $x = (x_i)_{i=1}^\infty \in s'$ (here $s'$ is the strong dual of the Fréchet space $s$ of all rapidly decreasing sequences, so that $s'$ is an (LB)–space), also satisfies condition (iii) of Proposition 2.5, but, fails condition (ii); the proof is similar to the previous one for $T$ in $k_1(V)$.

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A.A. Albanese*
Dipartimento di Matematica "E. De Giorgi"
Università del Salento
Via Provinciale per Arnesano, P.O. Box 193
I 73100 Lecce
Italy
e-mail: angela.albanese@unile.it

J. Bonet**
Instituto Universitaro de Matemática Pura y Aplicada IUMPA
Edificio ID15 (8E), Cubo F, Cuarta Planta
Universidad Politécnica de Valencia
E 46071 Valencia
Spain
e-mail: jbonet@mat.upv.es

W.J. Ricker
Math.–Geogr. Fakultät
Katholische Universität Eichstätt–Ingolstadt
D 85072 Eichstätt
Germany
e-mail: werner.ricker@ku-eichstaett.de