Dynamics of the differentiation operator on weighted spaces of entire functions

José Bonet

Author's address:
Instituto Universitario de Matemática Pura y Aplicada IUMPA,
Edificio IDI5 (8E), Cubo F, Cuarta Planta
Universidad Politécnica de Valencia,
E-46022 Valencia, Spain
E-mail: jbonet@mat.upv.es
http://personales.upv.es/jbonet/

Abstract

The continuity of the differentiation operator on weighted Banach spaces of entire functions with sup-norm has been characterized recently by Harutyunyan and Lusky. We give necessary and sufficient conditions to ensure that the differentiation operator on these weighted Banach spaces of entire functions is hypercyclic or chaotic, when it is continuous.

1 Introduction and Notation

Our purpose is to study the dynamics of the differentiation operator $Dh = h'$ on weighted Banach spaces of holomorphic functions with sup-norm on which it is well defined and continuous. More precisely, we characterize when D is hypercyclic, when it has a dense set of periodic points and when D is topologically mixing. The continuity of the differentiation operator on weighted Banach spaces of entire functions has been characterized recently by Harutyunyan and Lusky [24]. A continuous and linear operator T from a Banach space E into itself is called *hypercyclic* if there is a vector x (which is called hypercyclic vector) in E such that its orbit $(x, Tx, T^2x, ...)$ is dense in E. An operator T on a separable Banach space E is hypercyclic if and only if it is *topologically transitive* in the sense of dynamical systems, i.e. for every pair of non-empty open subsets U and V of E there is

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$n \in \mathbb{N}$ such that $T^n(U)$ meets V. A stronger condition is the following: the operator T on E is called *topologically mixing* if for every pair of non-empty open subsets U and V of E there is $N \in \mathbb{N}$ such that $T^n(U)$ meets V for each $n \geq N$. According to Devaney [14], a continuous map on a metric space is called *chaotic* if it is topologically transitive and has a dense set of periodic points. For motivation, examples and background about linear dynamics we refer the reader to the article by Godefroy and Shapiro [16] and to the surveys of Grosse-Erdmann [19, 22]. MacLane [28] proved that the differentiation operator D is hypercyclic on the space of entire functions $H(\mathbb{C})$ endowed with the compact open topology. The behaviour of the differentiation operators on Hörmander radial algebras of entire functions (which are not metrizable locally convex spaces) was investigated by the author in [10].

A *weight* v on $\mathbb{C}$ is a strictly positive continuous function on $\mathbb{C}$ which is radial, i.e. $v(z) = v(|z|), z \in \mathbb{C}$, such that $v(r)$ is non-increasing on $[0, \infty[$ and satisfies $\lim_{r\to\infty} r^m v(r) = 0$ for each $m \in \mathbb{N}$. For such a weight, the *weighted Banach spaces of entire functions* are defined by

$$Hv := \{f \in H(\mathbb{C}) \mid ||f||_v = \sup_{z\in\mathbb{C}} v(z)|f(z)| < +\infty\},$$
$$Hv_0 := \{f \in H(\mathbb{C}) \mid \lim_{|z|\to\infty} v(z)|f(z)| = 0\},$$

endowed with the norm $\|f\|_v := \sup_{z\in\mathbb{C}} v(z)|f(z)|$. Clearly Hv_0 is a closed subspace of Hv which contains the polynomials. We denote by B_v and C_v the unit balls of Hv and Hv_0 respectively. Spaces of this type appear in the study of growth conditions of analytic functions and have been investigated in various articles, see e.g. [7, 8, 25, 26, 27] and the references therein. Composition operators on these type of spaces have been also thoroughly studied [12].

Harutyunyan and Lusky [24] characterized the continuity of the differentiation operator $Dh = h'$ on the space Hv under the assumption that the space Hv is isomorphic to the Banach space ℓ_∞. Lusky had already characterized when Hv is isomorphic to ℓ_∞ in terms of a condition on the weight v in [27]. According to [24, Theorem 4.1], *if Hv is isomorphic to ℓ_∞, the differentiation operator $D : Hv \to Hv$ is continuous if and only if there are $\beta > 0$ and $r_0 > 0$ such that $v(r)e^{\beta r}$ is increasing on $[r_0, \infty[$.* By [24, Theorems 4.1 and 4.2], the differentiation operator D is bounded and surjective on Hv_α for the weight $v_\alpha(r) = e^{-\alpha r}, \alpha > 0$, D is bounded but not surjective on Hv for $v(r) = \exp(-\log^2 r)$, and it is not continuous for $v(r) = \exp(-e^r)$.

If a Banach space E admits a hypercyclic operator, then E must be separable. This is why we investigate the differentiation operator on the space Hv_0, and we need the following complement to the results of Harutyunyan and Lusky.

Proposition 1.1 *Let v be a weight. The differentiation operator $D : Hv \to Hv$ is continuous if and only if $D : Hv_0 \to Hv_0$ is continuous.*

Proof. Assume first that $D : Hv \to Hv$ is continuous and fix $g \in Hv_0$. By Bierstedt, Bonet, Galbis [7, Proposition 1.2.(e)] the Cesàro means $(p_n)_n$ of the partial sums of the Taylor series of g converge to g in Hv_0. By assumption, the sequence of polynomials $(p_n')_n$ converges to g' in Hv. As each polynomial is contained in Hv_0, we conclude $g' \in Hv_0$. Thus $D(Hv_0) \subset Hv_0$ and $D : Hv_0 \to Hv_0$ is continuous.

Now suppose that $D : Hv_0 \to Hv_0$ is continuous. There is $C > 0$ such that $||g'||_v \leq C||g||_v$ for each $g \in Hv_0$. Fix $f \in Hv, ||f||_v \leq 1$. We apply [7, Proposition 1.2.(b)] to find a sequence of polynomials $(q_n)_n$ converging to f for the compact open topology and such that $||q_n||_v \leq 1$ for each $n \in \mathbb{N}$. Since the differentiation operator is continuous for the compact open topology, the sequence of derivatives $(q_n')_n$ converges to f' for the compact open topology and

$$||q_n'||_v \leq C||q_n||_v \leq C, \quad n \in \mathbb{N}.$$

Therefore the sequence $(C^{-1}q_n')_n$ is contained in the unit ball B_v of Hv, which is closed in $H(\mathbb{C})$ for the compact open topology. Hence $f' \in CB_v \subset Hv$, and $D(B_v) \subset CB_v$, which implies that $D : Hv \to Hv$ is continuous. $\square$

Our results below are related to the study of the rate of growth of entire functions which are hypercyclic for the differentiation operator. This question has been investigated by several authors since the paper of Grosse-Erdmann [18]. See for example [5, 18, 20] and the historical remarks in the introduction of [9]. As we show below, the differentiation operator behaves very similarly to a weighted backward shift. However, it is important to point out that Lusky [26, Theorem 2.3] proved that the monomials are not a basis of the space Hv_0 for $v(r) = \exp(-r)$. Therefore our theorems cannot be directly deduced from the results on weighted backward shifts due to Grosse-Erdmann [21] and to Martínez-Giménez and Peris [29].

2 Main Results

Theorem 2.1 *Assume that the differentiation operator $D : Hv_0 \to Hv_0$ is continuous. The following conditions are equivalent:*

(1) D has a dense set of periodic points.

(2) D has a periodic point different from 0.

(3) $\lim_{r\to\infty} v(r)e^r = 0$.

Proof. Clearly condition (1) implies condition (2). Assume that (2) holds. There is $g \in Hv_0, g \neq 0$, and there is $n \in \mathbb{N}$ with $g^{(n)} = g$. We can find $c_1, ...c_n \in \mathbb{C}$, not all equal to 0, and $\theta_1, ..., \theta_n \in \mathbb{C}$, with $|\theta_j| = 1,\ \theta_j^n = 1$, such that $g(z) = c_1e^{\theta_1 z} + ... + c_ne^{\theta_n z} \in Hv_0$. We assume $c_1 \neq 0$ and $|c_1| \geq |c_j|, j = 2, ..., n$. The function

$$f(z) := e^{\theta_1 z}\Big(1 + \frac{c_2}{c_1}e^{(\theta_2-\theta_1)z} + ... + \frac{c_n}{c_1}e^{(\theta_n-\theta_1)z}\Big),\ z \in \mathbb{C},$$

belongs to Hv_0. Therefore $\lim_{r\to\infty}\varphi(r)=0$ for $\varphi(r):=v(r)\max_{|z|=r}|f(z)|$.

Fix $z\in\mathbb{C}, z\neq 0$, and find $\theta_z\in\mathbb{C}$ with $|\theta_z|=1$ and $\theta_z z=|z|$. Since $e^{|z|}=e^{\theta_1\overline{\theta_1}\theta_z z}$, we get

$$\varphi(|z|)\geq v(\overline{\theta_1}\theta_z z)|f(\overline{\theta_1}\theta_z z)|=$$
$$v(|z|)\Big|e^{|z|}\Big(1+\frac{c_2}{c_1}e^{(\theta_2-\theta_1)\overline{\theta_1}\theta_z z}+...+\frac{c_n}{c_1}e^{(\theta_n-\theta_1)\overline{\theta_1}\theta_z z}\Big)\Big|\geq$$
$$v(|z|)e^{|z|}\Big(1-\Big|e^{(\theta_2\overline{\theta_1}-1)|z|}\Big|-...-\Big|e^{(\theta_n\overline{\theta_1}-1)|z|}\Big|\Big).$$

However, for each $j=2,...,n$,

$$\Big|e^{(\theta_j\overline{\theta_1}-1)|z|}\Big|=\exp((\mathrm{Re}(\theta_j\overline{\theta_1}-1)|z|)),$$

and, since $|\theta_j\overline{\theta_1}|=1$ and $\theta_j\neq\theta_1$, we have $\mathrm{Re}(\theta_j\overline{\theta_1}-1)<0$, and we can find $R_0>0$ such that $\exp(((\mathrm{Re}(\theta_j\overline{\theta_1}-1)|z|)<1/(2n)$ for each $j=2,...,n$ and each $|z|>R_0$. This implies, for each $|z|>R_0$, $v(z)e^{|z|}\leq 2\varphi(|z|)$, and $\lim_{r\to\infty}v(r)e^r=0$, which proves condition (3).

Now suppose that condition (3) holds and denote by P the linear span of the functions $h_\theta(z):=e^{\theta z}$, $\theta\in\mathbb{C},\theta^n=1$ for some $n\in\mathbb{N}$. Clearly every element of P is a periodic point of the operator D and condition (3) implies that $P\subset Hv_0$. It remains to show that P is dense in Hv_0. To do this, we define $H:\overline{\mathbb{D}}\to Hv_0$ by $H(\zeta)(z):=e^{\zeta z}, z\in\overline{\mathbb{D}}$. The function H is well defined and bounded, since $||H(\zeta)||_v\leq\sup_{r\geq 0}v(r)e^r=:M$ for each $\zeta\in\overline{\mathbb{D}}$. We claim that H is holomorphic on $\mathbb{D}$. Since H is locally bounded (even bounded), by a result due to Grosse-Erdmann [23, Theorem 1], it is enough to find a $\sigma((Hv_0)',Hv_0)$-dense subset G of $(Hv_0)'$ such that $u\circ H:\mathbb{D}\to\mathbb{C}$ is holomorphic for each $u\in G$. Denote by G the set of all elements in $u\in(Hv_0)'$ which are continuous for the compact open topology. Since Hv_0 contains the polynomials, it is dense in the space $H(\mathbb{C})$ for the compact open topology. Therefore $\langle(Hv_0)',H(\mathbb{C})\rangle$ is a dual pair; consequently $H(\mathbb{C})'$, hence G, is $\sigma((Hv_0)',Hv_0)$-dense in $(Hv_0)'$. Now, the map $\mathbb{C}\to H(\mathbb{C}),\zeta\to e^{\zeta z}$, is holomorphic, hence its restriction to $\mathbb{D}$ is also holomorphic. This implies that $u\circ H:\mathbb{D}\to\mathbb{C}$ is holomorphic for each $u\in G$, and $H:\mathbb{D}\to Hv_0$ is holomorphic as we claimed.

We show that $H:\overline{\mathbb{D}}\to Hv_0$ is continuous. To do this, it is enough to prove the continuity at each ζ_0 in the boundary of $\overline{\mathbb{D}}$. Fix a sequence $(\zeta_j)_j$ in $\overline{\mathbb{D}}$ converging to ζ_0. We have

$$||H(\zeta_j)-H(\zeta_0)||_v=\sup_{z\in\mathbb{C}}v(z)\Big|e^{\zeta_j z}-e^{\zeta_0 z}\Big|.$$

Fix $\varepsilon>0$. Apply condition (3) to find $r_0>0$ such that $v(r)e^r<\varepsilon/4$ if $r\geq r_0$. Hence, if $|z|\geq r_0$, we have $v(z)|e^{\zeta_j z}-e^{\zeta_0 z}|<\varepsilon/2$. Since the map $\mathbb{C}\to H(\mathbb{C}),\zeta\to e^{\zeta z}$, is continuous, we find $\delta>0$ such that $|\zeta-\zeta_0|<\delta$ implies

$$\sup_{|z|\leq r_0}|e^{\zeta z}-e^{\zeta_0 z}|<\frac{\varepsilon}{2\max_{0\leq r\leq r_0}v(r)}.$$

Find $j_0\in\mathbb{N}$ with $|\zeta_j-\zeta_0|<\delta$ for $j\geq j_0$. Therefore, for $|z|\leq r_0$ and $j\geq j_0$, we get $v(z)|e^{\zeta_j z}-e^{\zeta_0 z}|<\varepsilon/2$. This implies $||H(\zeta_j)-H(\zeta_0)||_v<\varepsilon$, and H is continuous.

Assume that $u \in (Hv_0)'$ vanishes on P. We show $u = 0$ and Hahn Banach theorem permits us to conclude that P is dense in Hv_0. According to what we have proved above, the function $u \circ H : \overline{\mathbb{D}} \to \mathbb{C}$ belongs to the disc algebra and, since $u \circ H$ vanishes on all the roots of the unity and is continuous on $\overline{\mathbb{D}}$, it vanishes on the boundary of $\mathbb{D}$, consequently on the whole unit disc $\mathbb{D}$. In particular $(u \circ H)^{(n)}(0) = u(H^{(n)}(0)) = 0$ for each $n \in \mathbb{N} \cup \{0\}$. But $(H^{(n)}(0))(z) = z^n$ for each $n \in \mathbb{N} \cup \{0\}$, since $(H^{(n)}(\zeta))(z) = z^n e^{\zeta z}$. Accordingly u vanishes on all the polynomials. As the polynomials are dense in Hv_0, we conclude $u = 0$. □

The first simple criterion to ensure that an operator T on a separable Banach space E is hypercyclic was presented by Kitai in her Thesis. It was discovered independently by Gethner and Shapiro and was improved by several authors. The following form is due to Bès and Peris [6]; see also [4, 17].

HYPERCYCLICITY CRITERION. Suppose that the continuous operator T on a separable Banach space E satisfies that there exist an incresing sequence $(n_k)_k$ of positive integers, two dense subsets V and W of E and a sequence $(S_{n_k})_k$ of maps, not necessarily linear nor continuous, $S_{n_k} : W \to E$, such that:

(i) $(T^{n_k} v)_k$ converges to 0 for each $v \in V$.

(ii) $(S_{n_k} w)_k$ converges to 0 for each $w \in W$.

(iii) $(T^{n_k} S_{n_k} w)_k$ converges to w for each $w \in W$.

Then T is hypercyclic.

Bès and Peris proved that an operator T satisfies (the assumptions of) the hypercyclicity criterion if and only if $T \oplus T$ is hypercyclic on $E \oplus E$. Only very recently De La Rosa and Read [15] were able to exhibit hypercyclic operators which do not satisfy the hypercyclicity criterion, thus solving a long standing problem. Their example was improved later by Bayart and Matheron [3], who presented examples defined on classical Banach sequence spaces. For the differentiation operator on Hv_0 the two conditions coincide as we show below.

The first part of our next lemma is well-known. The transpose of an operator T is denoted by T^t.

Lemma 2.2 *Let T be an operator on a separable Banach space E.*

(1) If T is hypercyclic, then the sequence $((T^t)^n(v))_n$ is unbounded in E' for each $v \in E', v \neq 0$.

(2) If T is topologically mixing, then $\lim_{n\to\infty} ||(T^t)^n(v)|| = \infty$ for each $v \in E', v \neq 0$.

Proof. We prove (2). Assume there are $v \in E', v \neq 0$, an increasing sequence $(n_k)_k \subset \mathbb{N}$ and $M > 0$ such that $||(T^t)^{n_k}(v)|| \leq M$ for each $k \in \mathbb{N}$. This implies $|v(T^{n_k}(x)| \leq M$

for each $k \in \mathbb{N}$ and each $x \in E$ with $||x|| < 1$. Consider the non empty open sets $V := \{x \in X \mid |\langle v, x\rangle| > M\}$ and the open unit ball U in E. Since T is topologically mixing, there is $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for each $n > N$. Therefore, for $n_k > N$, there is $x \in U$ with $T^{n_k}(x) \in V$, hence $|v(T^{n_k}(x)| > M$, a contradiction. $\square$

Theorem 2.3 *Assume that the differentiation operator $D : Hv_0 \to Hv_0$ is continuous. The following conditions are equivalent:*

(1) D satisfies the hypercyclicity criterion.

(2) D is hypercyclic on Hv_0.

(3) $\lim\inf_{n\to\infty} \frac{||z^n||_v}{n!} = 0$.

Proof. Condition (1) implies that D is hypercyclic by the hypercyclicity criterion. Assume now that D is hypercyclic on Hv_0. By Lemma 2.2(1), for $\delta_0 : Hv_0 \to \mathbb{C}, \delta_0(f) = f(0)$, the sequence $((D^t)^n(\delta_0))_n$ is unbounded in $(Hv_0)'$, hence, there is $f \in Hv_0$, such that $(f^{(n)}(0))_n$ is unbounded in $\mathbb{C}$. Fix $n \in \mathbb{N}$ and apply the Cauchy inequalities to obtain, for each $r > 0$,

$$v(r)\frac{|f^{(n)}(0)|}{n!}r^n \leq v(r)\max_{|z|=r}|f(z)| \leq ||f||_v.$$

This implies

$$\frac{|f^{(n)}(0)|}{n!}\sup_{z\in\mathbb{C}} v(z)|z^n| \leq ||f||_v,$$

which yields $|f^{(n)}(0)|\frac{||z^n||_v}{n!} \leq ||f||_v$ for each $n \in \mathbb{N}$. Since $(f^{(n)}(0))_n$ is unbounded, we conclude $\lim\inf_{n\to\infty}\frac{||z^n||_v}{n!} = 0$, which is condition (3).

Now we prove that (3) implies (1). First of all, since D is continuous, there is $C \geq 1$ such that $||f^{(j)}||_v \leq C^j||f||_v$ for each $f \in Hv_0$ and each $j \in \mathbb{N}$. Set $n_0 = 0$ and use (3) inductively to find $n_k \in \mathbb{N}$ with $n_{k+1} > n_k + k + 1$ and

$$\frac{||z^{n_k+k+1}||_v}{(n_k + k + 1)!} \leq \frac{1}{kC^k}.$$

This is the increasing sequence of natural numbers required in the hypercyclicity criterion. Take $V = W$ as the set of all polynomials and define $S_{n_k} := S^{n_k}$ on W, with S the integration map defined on the monomials by $S(z^n) := \frac{z^{n+1}}{n+1}$. Since $D \circ S(g) = g$ for each polynomial g, conditions (i) and (iii) in the hypercyclicity criterion hold trivially. It remains to show that $\lim_{k\to\infty} S^{n_k}w = 0$ in Hv_0 for each polynomial w. To see this, fix $s \in \mathbb{N} \cup \{0\}$ and take $k \geq s$. Observe that

$$S^{n_k}(z^s) = \frac{s!}{(n_k + s)!}z^{n_k+s}$$

and

$$D^{k+1-s}(z^{n_k+k+1}) = \frac{(n_k+k+1)!}{(n_k+s)!} z^{n_k+s}.$$

This implies

$$||S^{n_k}(z^s)||_v = \frac{s!}{(n_k+s)!}||z^{n_k+s}||_v =$$

$$\frac{s!}{(n_k+k+1)!}||D^{k+1-s}(z^{n_k+k+1})||_v \le s!C^{k+1-s}\frac{||z^{n_k+k+1}||_v}{(n_k+k+1)!} < s!\frac{1}{k},$$

and the proof is complete. □

Theorem 2.4 *Assume that the differentiation operator* $D : Hv_0 \to Hv_0$ *is continuous. The following conditions are equivalent:*

(1) D *is topologically mixing.*

(2) $\lim_{n\to\infty} \frac{||z^n||_v}{n!} = 0$.

Proof. Assume first that condition (1) holds. By Lemma 2.2(2), $\lim_{n\to\infty} ||\delta_0 \circ D^n|| = \infty$. We can apply the Cauchy inequalities to obtain, for each $r > 0$, n and $f \in Hv_0$ with $||f||_v \le 1$,

$$v(r)|\delta_0 \circ D^n(f)|\frac{r^n}{n!} = v(r)|f^{(n)}(0)|\frac{r^n}{n!} \le v(r)\max_{|z|=r}|f(z)| \le 1.$$

Therefore, for each n and $r > 0$,

$$v(r)\frac{r^n}{n!}||\delta_0 \circ D^n|| \le 1,$$

hence

$$\frac{||z^n||_v}{n!}||\delta_0 \circ D^n|| \le 1$$

for each n, from where it follows $\lim_{n\to\infty} \frac{||z^n||_v}{n!} = 0$, which is condition (2).

Conversely, assume that condition (2) is satisfied. By Costakis, Sambarino [13, Theorem 1.1], to conclude that the differentiation operator D is topologically mixing, it is enough to show that D satisfies the assumptions of the hypercyclicity criterion for the sequence (n_k) of all positive integers. As in the proof of Theorem 2.3, we take $V = W$ the set of all polynomials and denote by S the operator of integration in the set of polynomials. Clearly $(D^n)_n$ tends pointwise to 0 in the set V of polynomials, $D \circ S$ coincides with the identity on W and it remains only to prove that $(S^n(g))_n$ converges to 0 in Hv_0. Since $S^n(z^k) = k!z^{k+n}/(k+n)!$ for each k, it is enough to show that $(z^n/n!)_n$ converges to 0 in Hv_0, but this is condition (2). □

Corollary 2.5 *Let* v *be a weight such that the differentiation operator* $D : Hv_0 \to Hv_0$ *is continuous. Then*

(1) If there is $A > 0$ such that $1/v(r) \leq A r^{-1/2} e^r,\ r > 0$, then D is not hypercyclic on Hv_0. In particular D is not hypercyclic for $v(r) = \exp(-\log^2 r)$ or for $v(r) = \exp(-\alpha r), 0 < \alpha < 1$.

(2) If there are $B > 0, \alpha \geq 1, r_0 > 0$ such that $v(r) \leq B exp(-\alpha r)$ for each $r \geq r_0$, then D is topologically mixing on Hv_0. In particular, D is topologically mixing for $v(r) = \exp(-\alpha r), \alpha \geq 1$.

Proof. (1) In order to apply Theorem 2.3 we estimate the norm of the monomial z^n.

$$||z^n||_v = \sup_{r\geq 0} r^n v(r) \geq A^{-1} \sup_{r\geq 0} r^{n+\frac{1}{2}} e^{-r} = A^{-1}\Big(n+\frac{1}{2}\Big)^{n+\frac{1}{2}} e^{-n-\frac{1}{2}}.$$

This implies by Stirling's formula that $\lim\inf_{n\to\infty} \frac{||z^n||_v}{n!} > 0$, and D is not hypercyclic by Theorem 2.3.

(2) Now we wish to apply Theorem 2.4 and, since $v(r)$ is non increasing, we estimate as follows

$$||z^n||_v = \sup_{r\geq 0} r^n v(r) \leq r_0^n v(0) + B \sup_{r\geq 0} r^n e^{-\alpha r} = r_0^n v(0) + B \frac{n^n e^{-n}}{\alpha^n}.$$

This implies

$$\lim_{n\to\infty} \frac{||z^n||_v}{n!} = 0,$$

by the Stirling formula. The conclusion follows from Theorem 2.4. □

Corollary 2.6 *Let $v_\alpha(r) = e^{-\alpha r}, \alpha > 0$. Then the differentiation operator D on $H(v_\alpha)_0$ satisfies:*

(1) If $0 < \alpha < 1$, then D is not hypercyclic and has no periodic point different from 0.

(2) If $\alpha = 1$, then D is topologically mixing but has no periodic point different from 0.

(3) If $\alpha > 1$, then D is topologically mixing and has a dense set of periodic points; in particular D is chaotic in the sense of Devaney.

The case $\alpha = 1$ in Corollary 2.6 already appeared in our proof of Theorem 11 in [10].

We conclude the paper with some remarks about weaker or stronger notions of hypercyclicity. An operator T on a separable Banach space E is called *supercyclic* if there exists a vector $x \in E$ such that the set of scalar multiples of the orbit of x is dense in E. This concept was introduced by Hilden and Willis. The study of supercyclic operators has experimented a great development during the last years; see e.g. [4, 30].

Proposition 2.7 *Every continuous differentiation operator $D : Hv_0 \to Hv_0$ is supercyclic.*

Proof. The generalized kernel $\bigcup_{n=0}^{\infty} \ker D^n$ of the operator D contains the polynomials, hence it is dense in the separable Banach space Hv_0 by [7, Proposition 1.2]. Moreover, the range of D is also dense in Hv_0, since it contains the polynomials. The conclusion now follows from Bermúdez, Bonilla, Peris [4, Corollary 3.3]. □

Bayart and Grivaux [1] introduced the following concept which has attracted much attention. The operator T on a Banach space E is *frequently hypercyclic* if there is $x \in E$ such that, for every non-empty open subset U of E, the lower density of the set $\{n \in \mathbb{N}\ ;\ T^n x \in U\}$ is strictly greater than 0. The proof of Blasco, Bonilla, Grosse-Erdmann [9, Theorem 3] yields the following result.

Proposition 2.8 *Let v be a weight such that the differentiation operator $D : Hv_0 \to Hv_0$ is continuous. If $\lim_{r\to\infty} v(r)e^r = 0$, then D is frequently hypercyclic on Hv_0.*

In fact, the proof of [9, Theorem 3] shows that, if $\lim_{r\to\infty} v(r)e^r = 0$, then the differentiation operator on Hv_0 satisfies the frequent hypercyclicity criterion in [11, Theorem 2.4]. By [11, Remark 2.2.(b)], every operator satisfying the frequent hypercyclicity criterion is chaotic in the sense of Devaney. Accordingly, the three conditions in Theorem 2.1 are also equivalent to the fact that D satisfies the frequent hypercyclicity criterion. Note that Bayart and Grivaux constructed in [2] frequently hypercyclic operators on c_0 which are not chaotic.

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