MEAN ERGODICITY OF MULTIPLICATION OPERATORS IN WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS

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Dedicated to Heinz König on the occassion of his 80-th birthday

ABSTRACT. Multiplication operators in weighted Banach (and locally convex) spaces of functions holomorphic in the unit disc are well known. In this note we investigate the connection between power boundedness, mean ergodicity and uniform mean ergodicity of such operators.

1. INTRODUCTION

Our aim is to examine the mean ergodicity of multiplication operators in weighted Banach (and other types of) spaces consisting of holomorphic functions on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. All *weight functions* $v : \mathbb{C} \to (0, \infty)$ throughout this paper will be continuous, radial (i.e. $v(z) = v(|z|),\ z \in \mathbb{D}$) and satisfy $\lim_{r\to 1^-} v(r) = 0$. Well known examples, [10, §1], include $v(z) := (1 - |z|)^\alpha$ for $\alpha > 0$ (see also Example 3.7(i) below), $v(z) := (1 - \log(1 - |z|))^{-\beta}$ for $\beta > 0$ (see Example 3.7(ii) for equivalent weights) and $v(z) := \exp(-(1 - |z|)^{-\gamma})$ for $\gamma > 0$. Of relevance are the weighted Banach spaces

$$H_v^\infty := \{f \in H(\mathbb{D}) : \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}, \tag{1.1}$$

$$H_v^0 := \{f \in H(\mathbb{D}) : \lim_{|z|\to 1^-} v(z)|f(z)| = 0\} \tag{1.2}$$

endowed with the norm $\|\cdot\|_v$. Functions in H_v^0 are said to "vanish on the boundary of $\mathbb{D}$". The space H_v^0 is a closed subspace of H_v^∞. Moreover, the polynomials are dense in H_v^0. Indeed, for $f \in H(\mathbb{D})$, let $C_j(f)$ denote the Cesàro mean of the Taylor polynomial of degree j of

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f (about 0). It is known that $C_j(f) \to f$ as $j \to \infty$ for the compact open topology in $H(\mathbb{D})$, with $\|C_j(f)\|_v \leq \|f\|_v$ whenever $f \in H_v^\infty$, and also that $C_j(f) \to f$ in H_v^0 as $j \to \infty$ whenever $f \in H_v^0$, [4]. The spaces H_v^∞ and H_v^0 have been intensively studied, as well as certain aspects of operator theory in such spaces; see for example [4], [5], [8], [9], [10], [12], [21], [22], [23] and the references therein.

Given a locally convex Hausdorff space (briefly, lcHs), the space of all continuous linear operator from X into itself is denoted by $\mathcal{L}(X)$. The weak topology of X is denoted by $\sigma(X, X')$, where X' is the topological dual space of X. When $\mathcal{L}(X)$ is equipped with its strong operator topology τ_s (resp. the topology τ_b of uniform convergence on the bounded sets of X) we write $\mathcal{L}_s(X)$ (resp. $\mathcal{L}_b(X)$). Of course, if X is a Banach space, then τ_b is the operator norm topology. Given $T \in \mathcal{L}(X)$, its *Cesàro means* are defined by

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \qquad n \in \mathbb{N}. \tag{1.3}$$

It is routine to verify that

$$\frac{1}{n} T^n = T_{[n]} - \frac{(n-1)}{n} T_{[n-1]}, \qquad n \in \mathbb{N}, \tag{1.4}$$

where $T_{[0]} := I$ is the identity operator on X. We say that T is *mean ergodic* (resp. *uniformly mean ergodic*) if $\{T_{[n]}\}_{n=1}^\infty$ is a convergent sequence in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$). If the powers $\{T^n\}_{n=0}^\infty$ form an equicontinuous subset of $\mathcal{L}(X)$, then T is called *power bounded*. The standard text for mean ergodic operators in Banach spaces is [17]. For Fréchet and more general lcHs' we refer to [1], [2], [26, Ch. VIII, §3].

Given $\varphi \in H^\infty(\mathbb{D})$, i.e. $\varphi \in H(\mathbb{D})$ and $\|\varphi\|_\infty := \sup_{z \in \mathbb{D}} |\varphi(z)| < \infty$, we consider the multiplication operators $T_\varphi \in \mathcal{L}(H_v^\infty)$ and $S_\varphi \in \mathcal{L}(H_v^0)$ given by $T_\varphi : f \mapsto \varphi f$ and $S_\varphi : f \mapsto \varphi f$. The purpose of this note is determine precisely when these operators are power bounded, mean ergodic and uniformly mean ergodic. In the final section we examine multiplication operators defined in certain weighted inductive limit spaces $\operatorname{ind}_n H_{v_n}^\infty$ of holomorphic functions. Depending on the nature of the sequence of weights $(v_n)_{n=1}^\infty$, such spaces may even be lc-algebras (e.g. the Korenblum algebra). Moreover, unlike in the Banach space setting, in such inductive limit spaces multiplication by certain unbounded functions φ may generate elements of $\mathcal{L}(X)$.

2. The Banach space setting.

Let $v : \mathbb{D} \to (0, \infty)$ be a weight function. For each $z \in \mathbb{D}$, define the evaluation functional $\delta_z : H_v^\infty \to \mathbb{C}$ by $\langle f, \delta_z \rangle := f(z)$ in which case

$|\langle f, \delta_z\rangle| \leq \|f\|_v/v(z)$ for $f \in H_v^\infty$, i.e., $\delta_z \in (H_v^\infty)'$. The following fact (without proof) occurs in [9, Proposition 2.1].

Lemma 2.1. *If $\varphi \in H^\infty(\mathbb{D})$, then $T_\varphi \in \mathcal{L}(H_v^\infty)$ and $S_\varphi \in \mathcal{L}(H_v^0)$ with*

$$\|T_\varphi\| = \|\varphi\|_\infty = \|S_\varphi\|.$$

Proof. It is routine to check that $\|T_\varphi\| \leq \|\varphi\|_\infty$.

Concerning S_φ, we may assume that $\varphi \not\equiv 0$. Fix $f \in H_v^0$. Given $\varepsilon > 0$ select $0 < r < 1$ such that $v(z)|f(z)| < \varepsilon/\|\varphi\|_\infty$ for all $|z| > r$, i.e. $v(z)|\varphi(z)f(z)| < \varepsilon$ for $|z| > r$. Accordingly, $S_\varphi f \in H_v^0$. Since S_φ is the restriction of T_φ to H_v^0 it follows that $\|S_\varphi\| \leq \|\varphi\|_\infty$.

For the reverse inequalities, let $T_\varphi^t \in \mathcal{L}((H_v^\infty)')$ be the dual operator to T_φ. Fix $z \in \mathbb{D}$. Then

$$\begin{aligned}\|T_\varphi\| &= \|T_\varphi^t\| \geq \|T_\varphi^t(\delta_z/\|\delta_z\|_{(H_v^\infty)'})\|_{(H_v^\infty)'} = \|\delta_z\|^{-1}_{(H_v^\infty)'} \sup_{\|f\|_v \leq 1} |\langle T_\varphi f, \delta_z\rangle| \\ &= \|\delta_z\|^{-1}_{(H_v^\infty)'} |\varphi(z)| \sup_{\|f\|_v \leq 1} |\langle f, \delta_z\rangle| = |\varphi(z)|,\end{aligned}$$

from which it follows that $\|T_\varphi\| \geq \|\varphi\|_\infty$.

A similar argument yields $\|S_\varphi\| \geq \|\varphi\|_\infty$. □

Remark 2.2. It follows from Lemma 2.1 and the identities $T_\varphi^n = T_{\varphi^n}$ and $S_\varphi^n = S_{\varphi^n}$, for each $n \in \mathbb{N}$, that

$$\|T_\varphi^n\| = \|\varphi^n\|_\infty = \|\varphi\|_\infty^n = \|S_\varphi^n\|, \qquad n \in \mathbb{N}. \tag{2.1}$$

Proposition 2.3. *For $\varphi \in H^\infty(\mathbb{D})$ the following assertions are equivalent.*

(i) $\|\varphi\|_\infty \leq 1$.
(ii) $T_\varphi \in \mathcal{L}(H_v^\infty)$ *is power bounded.*
(iii) $S_\varphi \in \mathcal{L}(H_v^0)$ *is power bounded.*

Proof. (i) ⇒ (ii) and (i) ⇒ (iii) are immediate from (2.1).

(ii) ⇒ (i). For fixed $z \in \mathbb{D}$ and $n \in \mathbb{N}$ we have via (2.1) that

$$|\varphi(z)|^n = |\varphi(z)^n| \leq \|\varphi^n\|_\infty \leq C,$$

where $C := \sup_{m \in \mathbb{N}} \|T_\varphi^m\| < \infty$. Hence, $|\varphi(z)| \leq C^{1/n}$ for each $n \in \mathbb{N}$ from which it follows that $|\varphi(z)| \leq 1$. So, $\|\varphi\|_\infty \leq 1$.

(iii) ⇒ (i). Mimick the proof of (ii) ⇒ (i). □

Let $\varphi \in H^\infty(\mathbb{D})$. For each $f \in H_v^\infty$ and $n \in \mathbb{N}$ we have

$$\left((T_\varphi)_{[n]} f\right)(z) = \frac{f(z)}{n} \cdot \sum_{m=1}^{n} (\varphi(z))^m, \quad z \in \mathbb{D}, \tag{2.2}$$

and also

$$\left((T_\varphi)_{[n]}f\right)(z) = \frac{\varphi(z)f(z)}{n} \cdot \frac{(1-(\varphi(z))^n)}{(1-\varphi(z))}, \qquad z \in \mathbb{D} \setminus (\mathbf{1}-\varphi)^{-1}(\{0\}), \tag{2.3}$$

where $\mathbf{1}$ denotes the function constantly 1. For $f \in H_v^0$, the formulae (2.2) and (2.3) also hold with S_φ in place of T_φ.

In general, mean ergodic operators need not be power bounded, [15, §6]. However, in the present setting this is the case.

Proposition 2.4. *Let $\varphi \in H^\infty(\mathbb{D})$. If $T_\varphi \in \mathcal{L}(H_v^\infty)$ (resp. $S_\varphi \in \mathcal{L}(H_v^0)$) is mean ergodic, then T_φ (resp. S_φ) is power bounded.*

Proof. The mean ergodicity of T_φ and (1.4) imply that $\lim_{n\to\infty} \frac{1}{n}T_\varphi^n = 0$ in $\mathcal{L}_s(H_v^\infty)$. Evaluating at $\mathbf{1}$ yields $\lim_{n\to\infty} \frac{1}{n}\varphi^n = 0$ in H_v^∞. For fixed $z \in \mathbb{D}$, the inequalities $|(\varphi(z))^n/n| \leq v(z)^{-1}\|\varphi^n/n\|_v$ for $n \in \mathbb{N}$ show that $\lim_{n\to\infty}(\varphi(z))^n/n = 0$ in $\mathbb{C}$. It follows that $|\varphi(z)| \leq 1$, i.e. $\|\varphi\|_\infty \leq 1$. Via Proposition 2.3 the operator T_φ is power bounded.

Since also $\mathbf{1} \in H_v^0$, a similar proof applies to S_φ. □

For the operators S_φ, Proposition 2.4 admits a converse.

Proposition 2.5. *Let $\varphi \in H^\infty(\mathbb{D})$. Then $S_\varphi \in \mathcal{L}(H_v^0)$ is mean ergodic iff S_φ is power bounded iff $\|\varphi\|_\infty \leq 1$.*

Proof. In view of Propositions 2.3 and 2.4 we only need to establish that S_φ is mean ergodic whenever $\|\varphi\|_\infty \leq 1$. So, assume that $\|\varphi\|_\infty \leq 1$. We consider two cases.

First suppose that $|\varphi(z_0)| = 1$ for some $z_0 \in \mathbb{D}$. By the Maximum Principle there is $\xi \in \mathbb{C}$ with $|\xi| = 1$ such that $\varphi(z) = \xi$ for all $z \in \mathbb{D}$. Then $S_\varphi = \xi I$. If $\xi = 1$, then $(S_\varphi)_{[n]} = I$ and so $(S_\varphi)_{[n]} \to I$ in $\mathcal{L}_b(H_v^0)$ as $n \to \infty$. If $\xi \neq 1$, then it follows from (2.3), with S_φ in place of T_φ, that $(S_\varphi)_{[n]} = \frac{\xi(1-\xi^n)}{n(1-\xi)} \cdot I$, for $n \in \mathbb{N}$, and so $(S_\varphi)_{[n]} \to 0$ in $\mathcal{L}_b(H_v^0)$ as $n \to \infty$. In each case S_φ is mean ergodic (even uniformly).

The second possibility is that $|\varphi(z)| < 1$ for each $z \in \mathbb{D}$. Since S_φ is power bounded (c.f. Proposition 2.3), it follows from the classical mean ergodic theorem (see (ii) ⇔ (iii) of Theorem 1.1 in [17, p.72]) that $\lim_{n\to\infty}(S_\varphi)_{[n]} = 0$ in $\mathcal{L}_s(H_v^0)$ (hence, S_φ is mean ergodic) *provided* we can show that $\lim_{n\to\infty}(S_\varphi)_{[n]}f = 0$ relative to $\sigma(H_v^0, (H_v^0)')$ for each $f \in H_v^0$. So, fix $f \in H_v^0$. Given $z \in \mathbb{D}$ it follows from (2.2), with S_φ in place of T_φ, and $\|\varphi\|_\infty \leq 1$ that $\left|\left((S_\varphi)_{[n]}f\right)(z)\right| \leq |f(z)|$. Accordingly, $\|(S_\varphi)_{[n]}f\|_v \leq \|f\|_v$, for $n \in \mathbb{N}$, and so $\{(S_\varphi)_{[n]}f\}_{n=1}^\infty$ is bounded in H_v^0. On the other hand, since $|\varphi| < \mathbf{1}$ pointwise on $\mathbb{D}$, it follows from (2.3), with S_φ in place of T_φ, that $(S_\varphi)_{[n]}f \to 0$ pointwise on $\mathbb{D}$ as $n \to \infty$. Actually, $(S_\varphi)_{[n]}f \to 0$ for the compact open topology. Indeed, given any $0 < r_0 < 1$ we have $m = m(\varphi, r_0) := \max_{|z|\le r_0} |\varphi(z)| < 1$ and so, for all $|z| \le r_0$ we have

$$\left|\left((S_\varphi)_{[n]} f\right)(z)\right| \le \frac{|\varphi(z) f(z)|}{n} \cdot \frac{|1-(\varphi(z))^n|}{|1-\varphi(z)|} \le \frac{2}{n(1-m)} \cdot \sup_{|z|\le r_0} |f(z)|$$

which converges to 0 as $n \to \infty$. It follows from [6, Lemma 13], that $(S_\varphi)_{[n]} f \to 0$ relative to $\sigma(H_v^0, (H_v^0)')$; this lemma is applicable as the assumptions on v ensure every $h \in H_v^0$ is the limit in the compact open topology of $\{C_j(h)\}_{j=1}^\infty$ and $\|C_j(h)\|_v \le \|h\|_v$; see Section 1. □

Proposition 2.6. *Let $\varphi \in H^\infty(\mathbb{D})$. Then $S_\varphi \in \mathcal{L}(H_v^0)$ is uniformly mean ergodic iff $\|\varphi\|_\infty \le 1$ and either*

(1) *there is $\xi \in \mathbb{C}$ with $|\xi| = 1$ such that $\varphi(z) = \xi$, for $z \in \mathbb{D}$, or*
(2) *$(\mathbf{1} - \varphi)^{-1} \in H^\infty(\mathbb{D})$.*

Proof. Let $\|\varphi\|_\infty \le 1$. If φ satisfies (1), then it was shown in the proof of Proposition 2.5 that S_φ is uniformly mean ergodic.

Suppose that (2) holds, i.e. there exists $\varepsilon > 0$ such that $|\mathbf{1} - \varphi| \ge \varepsilon \mathbf{1}$ pointwise on $\mathbb{D}$. Then $(\mathbf{1} - \varphi)^{-1}(\{0\}) = \emptyset$, and so it follows from (2.3), with S_φ in place of T_φ, that for each $f \in H_v^0$, $z \in \mathbb{D}$ and $n \in \mathbb{N}$ we have $\left|\left((S_\varphi)_{[n]} f\right)(z)\right| \le 2(n\varepsilon)^{-1}\|\varphi\|_\infty |f(z)|$. Accordingly, $\|(S_\varphi)_{[n]}\| \le 2\|\varphi\|_\infty / n\varepsilon$ for $n \in \mathbb{N}$ and hence, $\lim_{n\to\infty} \|(S_\varphi)_{[n]}\| = 0$, i.e., S_φ is uniformly mean ergodic.

Conversely, suppose that S_φ is uniformly mean ergodic in which case $\|\varphi\|_\infty \le 1$ (c.f. Proposition 2.5). Suppose that (1) does not hold. We need to establish (2). Since (1) fails to hold, the Maximum Principle guarantees that $|\varphi| < \mathbf{1}$ pointwise on $\mathbb{D}$. Via the proof of Proposition 2.5 we can conclude that $\lim_{n\to\infty} (S_\varphi)_{[n]} = 0$ in $\mathcal{L}_s(H_v^0)$. It is routine to check that $\ker(I - S_\varphi) = \ker(S_{\mathbf{1}-\varphi}) = \{0\}$. Moreover, S_φ is power bounded (c.f. Proposition 2.3) and hence, $\frac{1}{n} S_\varphi^n \to 0$ in $\mathcal{L}_b(H_v^0)$ as $n \to \infty$. By a criterion of M. Lin, [18, Theorem], the operator S_φ is uniformly mean ergodic iff $I - S_\varphi = S_{\mathbf{1}-\varphi}$ is an isomorphism of H_v^0 onto itself (for Lin's criterion in this form see [1, Proposition 2.16]), which in turn is equivalent to $(\mathbf{1} - \varphi)^{-1} \in H^\infty(\mathbb{D})$, [9, Lemma 2.3]. □

Remark 2.7. It is known that H_v^0 is isomorphic to either c_0 or to the c_0-sum $(\oplus_{n=1}^\infty H_n)_{c_0}$, where H_n is the space of polynomials on $\mathbb{D}$ with degree at most n and equipped with the sup-norm, [23, Theorem 1.1]. Now, the spaces H_n, for $n \in \mathbb{N}$, have bases with uniformly bounded basis constants, [7], [11], and hence, $(\oplus_{n=1}^\infty H_n)_{c_0}$ has a basis. So, as pointed out by W. Lusky, H_v^0 has a basis. It follows from [13, Corollary 3] that H_v^0 admits power bounded operators which fail to be uniformly mean ergodic. Proposition 2.6 provides *particular* operators with these features, e.g. S_φ with $\varphi(z) := z$, for $z \in \mathbb{D}$. Since H_v^0 is not reflexive, Corollary 1 of [13] implies that H_v^0 also admits power bounded operators which fail to be mean ergodic. The operators S_φ do not exhibit these features (c.f. Proposition 2.5). □

The final result of this section illustrates that the operators T_φ behave somewhat differently to the operators S_φ.

Proposition 2.8. *Let $\varphi \in H^\infty(\mathbb{D})$ satisfy $\|\varphi\|_\infty \le 1$. The following assertions for $T_\varphi \in \mathcal{L}(H_v^\infty)$ are equivalent.*

(i) *T_φ is mean ergodic.*
(ii) *T_φ is uniformly mean ergodic.*
(iii) *Either one of* (1) *or* (2) *in Proposition* 2.6 *is satisfied.*

Proof. (ii) $\Rightarrow$ (i) is immediate from the definitions involved.

(iii) $\Rightarrow$ (ii) is proved as for S_φ in Proposition 2.6.

(i) $\Rightarrow$ (ii). Proposition 2.4 implies that T_φ is power bounded and hence, $\lim_{n\to\infty} \frac{1}{n}\|T_\varphi^n\| = 0$. By a result of W. Lusky, [23], H_v^∞ is isomorphic to either ℓ^∞ or $H^\infty(\mathbb{D})$, both of which are Grothendieck spaces with the Dunford-Pettis property, [19, p.121]. Fix $f \in H_v^\infty$. Since $\{(T_\varphi)_{[n]}\}_{n=1}^\infty$ is convergent in $\mathcal{L}_s(H_v^\infty)$, the sequence $\{(T_\varphi)_{[n]}f\}_{n=1}^\infty$ is relatively $\sigma(H_v^\infty, (H_v^\infty)')$-compact. It follows from [19, Theorem 8], [20, Theorem 5], that T_φ is uniformly mean ergodic.

(ii) $\Rightarrow$ (iii) is verified by the same argument as in Proposition 2.6. □

Remark 2.9. Combining Propositions 2.3 and 2.8 we see that T_φ, for $\varphi(z) := z$, $z \in \mathbb{D}$, is an example of a power bounded operator in H_v^∞ which fails to be mean ergodic. Since H_v^∞ does not have a basis, the existence of such operators is not guaranteed by general theory, e.g., [13, Corollary 1].

3. Inductive limits

Let $v : \mathbb{D} \to (0, \infty)$ be a weight function and B_v denote the closed unit ball of H_v^∞. The *associated weight* $\widetilde{v}$ is defined by

$$\widetilde{v}(z) := \|\delta_z\|^{-1}_{(H_v^\infty)'} = 1/\sup\{|f(z)| : f \in H(\mathbb{D}),\ |f| \le |v|^{-1} \text{ on } \mathbb{D}\},$$

for $z \in \mathbb{D}$. We summarize a few facts concerning $\widetilde{v}$, which can be either found in [5], [8] or are deducible from facts in these papers. Firstly, $\widetilde{v}$ is continuous, radial, satisfies $\lim_{r\to 1^-} \widetilde{v}(r) = 0$ and also $0 < v(z) \le \widetilde{v}(z) \le v(0)$ for $z \in \mathbb{D}$. Moreover, the Banach spaces H_v^∞ and $H_{\widetilde{v}}^\infty$ coincide with identical norms.

Throughout this section $V = (v_n)_{n=1}^\infty$ is a decreasing sequence (i.e. $v_{n+1} \le v_n$ pointwise on $\mathbb{D}$, for $n \in \mathbb{N}$) of weight functions, in which case

$H^\infty_{v_n} \subseteq H^\infty_{v_{n+1}}$ with $\|f\|_{v_{n+1}} \le \|f\|_{v_n}$ for each $f \in H^\infty_{v_n}$. The weighted inductive limit $VH :=\ \mathrm{ind}_n H^\infty_{v_n}$ is equipped with the finest lc-topology such that all natural inclusions $H_n \hookrightarrow VH$ are continuous. Then VH is a complete (LB)-space (hence, also regular, i.e., each bounded set of VH is contained and bounded in some step $H^\infty_{v_n}$) with a topology finer than the compact open topology, [3]. The following fact is routine to verify and is essentially known, [5], [12, Proposition 3.1].

Lemma 3.1. *Let v, w be weight functions on $\mathbb{D}$ and $\varphi \in H(\mathbb{D})$. The following assertions are equivalent.*

(i) *$\varphi \cdot B_v \subseteq KB_w$ for some constant $K > 0$.*
(ii) *$\widetilde{w}|\varphi| \le M\widetilde{v}$ pointwise on $\mathbb{D}$ for some constant $K > 0$.*

Whenever $\varphi \in H(\mathbb{D})$ satisfies $\varphi \cdot VH \subseteq VH$ the linear map given by $T_\varphi : f \mapsto \varphi f$ is well defined on VH.

Lemma 3.2. *For $\varphi \in H(\mathbb{D})$ the following statements are equivalent.*

(i) *$\varphi \cdot VH \subseteq VH$.*
(ii) *T_φ is continuous, that is, $T_\varphi \in \mathcal{L}(VH)$.*
(iii) *For each $n \in \mathbb{N}$ there exist $m(n) \ge n$ and $C_n > 0$ with $T_\varphi(B_{v_n}) \subseteq C_n B_{v_{m(n)}}$.*
(iv) *For each $n \in \mathbb{N}$ there exist $m(n) \ge n$ and $C_n > 0$ with $\widetilde{v}_{m(n)}|\varphi| \le C_n \widetilde{v}_n$ pointwise on $\mathbb{D}$.*

Proof. The Closed Graph Theorem for (LB)-spaces, [24, Theorem 24.31], gives (i) $\Leftrightarrow$ (ii). Moreover, (ii) $\Leftrightarrow$ (iii) is a general fact about continuity of linear maps between (LB)-spaces, [24, Proposition 24.7 & Theorem 24.33]. Finally, (iii) $\Leftrightarrow$ (iv) is a consequence of Lemma 3.1. □

Lemma 3.3. *Let $X = \mathrm{ind}_n X_n$ be a regular (LB)-space and B_n denote the closed unit ball of X_n, $n \in \mathbb{N}$. For a subset $\mathcal{H} \subseteq \mathcal{L}(X)$, the following assertions are equivalent.*

(i) *$\mathcal{H}$ is equicontinuous.*
(ii) *There is an increasing sequence $m(n) \uparrow \infty$ in $\mathbb{N}$ and constants $C_n > 0$ such that*

$$\bigcup_{T \in \mathcal{H}} T(B_n) \subseteq C_n B_{m(n)}, \qquad n \in \mathbb{N}. \tag{3.1}$$

Proof. (i) $\Rightarrow$ (ii). Since $\mathcal{H}$ is equicontinuous, $D_n := \bigcup_{T\in\mathcal{H}} T(B_n)$ is bounded in X for each $n \in \mathbb{N}$. By regularity of $\mathrm{ind}_n X_n$ there is $m(1) \in \mathbb{N}$ and $C_1 > 0$ with $D_1 \subseteq C_1 B_{m(1)}$, i.e. (3.1) holds for $n = 1$. Proceeding inductively, suppose that (3.1) holds for $(n-1)$. Then we can select $m(n) > m(n-1)$ with $m(n) \ge n$ and $D_n \subseteq C_n B_{m(n)}$ for some

$C_n > 0$, again by regularity of $\text{ind}_n X_n$ and the fact that $X_k \subseteq X_{k+1}$ continuously, for each $k \in \mathbb{N}$.

(ii) $\Rightarrow$ (i). Since X is barrelled, [24, Proposition 24.16] and (ii) imply that $\mathcal{H}$ is bounded in $\mathcal{L}_s(X)$. The equicontinuity of $\mathcal{H}$ follows via [16, p.137]. $\square$

Our first result for the operators T_φ is the following one.

Proposition 3.4. *For $\varphi \in H(\mathbb{D})$ satisfying $\varphi \cdot VH \subseteq VH$ the following assertions are equivalent.*

(i) *$T_\varphi \in \mathcal{L}(VH)$ is power bounded.*
(ii) *$\varphi \in H^\infty(\mathbb{D})$ with $\|\varphi\|_\infty \leq 1$.*

Proof. (i) $\Rightarrow$ (ii). By Lemma 3.3 and the equicontinuity of $\{T_{\varphi^k}\}_{k=1}^\infty = \{T_\varphi^k\}_{k=1}^\infty \subseteq \mathcal{L}(VH)$ there exist $C_1 > 0$ and $m(1) \geq 1$ with $\varphi^k \cdot B_{v_1} = T_{\varphi^k}(B_{v_1}) \subseteq C_1 B_{v_{m(1)}}$, for $k \in \mathbb{N}$. Lemma 3.1 implies that $\widetilde{v}_{m(1)}|\varphi|^k \leq C_1\widetilde{v}_1$ pointwise on $\mathbb{D}$, i.e. $|\varphi(z)| \leq \left(C_1\widetilde{v}_1(z)/\widetilde{v}_{m(1)}(z)\right)^{1/k}$ for all $k \in \mathbb{N}$, $z \in \mathbb{D}$. Consequently, $\|\varphi\|_\infty \leq 1$.

(ii) $\Rightarrow$ (i). Since $|\varphi^k| \leq \mathbf{1}$ pointwise on $\mathbb{D}$, for $k \in \mathbb{N}$, it follows from Lemma 3.1 that $T_\varphi^k(B_{v_n}) = T_{\varphi^k}(B_{v_n}) \subseteq B_{v_n}$ for all $k, n \in \mathbb{N}$. Equicontinuity of $\{T_\varphi^k\}_{k=1}^\infty \subseteq \mathcal{L}(VH)$ follows from Lemma 3.3. $\square$

Proposition 3.5. *Let $\varphi \in H(\mathbb{D})$. If $T_\varphi \in \mathcal{L}(VH)$ is mean ergodic, then $\|\varphi\|_\infty \leq 1$.*

Proof. According to (1.4) we have $\frac{1}{n}T_\varphi^n = \frac{1}{n}T_{\varphi^n} \to 0$ in $\mathcal{L}_s(VH)$ as $n \to \infty$. In particular, $\lim_{n\to\infty} \frac{1}{n}T_{\varphi^n}\mathbf{1} = 0$ in VH and hence, since $\delta_z \in (VH)'$ for each $z \in \mathbb{D}$, [24, Proposition 24.7], also $\frac{1}{n}T_{\varphi^n}\mathbf{1} \to 0$ pointwise on $\mathbb{D}$ as $n \to \infty$. It follows that $\|\varphi\|_\infty \leq 1$. $\square$

The converse of Proposition 3.5 does not hold in general. Indeed, if $v_n := v$, for all $n \in \mathbb{N}$ and some fixed weight function $v : \mathbb{D} \to (0, \infty)$, then $\text{ind}_n H_{v_n}^\infty$ is isomorphic to the Banach space H_v^∞. Then Remark 2.9 applies. However, for a certain large class of weight sequences $V = (v_n)_{n=1}^\infty$ this phenomenon cannot occur. We say that V satisfies *property* (S) if, for each $n \in \mathbb{N}$ there is $m(n) > n$ satisfying $\lim_{r\to 1^-} v_{m(n)}(r)/v_n(r) = 0$, [3]. In this case each natural inclusion map $H_{v_n}^\infty \hookrightarrow H_{v_{m(n)}}^\infty$ is compact, for $n \in \mathbb{N}$, and the (LB)-space $\text{ind}_n H_{v_n}^\infty$ is a (DFS)-space, i.e. the strong dual of a Fréchet Schwartz space (in particular, a Montel space), [3], [24, Proposition 25.20].

Proposition 3.6. *Let $V = (v_n)_{n=1}^\infty$ be a decreasing sequence of weight functions with property (S) and $\varphi \in H(\mathbb{D})$ satisfy $T_\varphi \in \mathcal{L}(VH)$. The following statements are equivalent.*

(i) *T_φ is power bounded.*
(ii) *T_φ is mean ergodic.*
(iii) *T_φ is uniformly mean ergodic.*

(iv) $\varphi \in H^\infty(\mathbb{D})$ *with* $\|\varphi\|_\infty \leq 1$.

Proof. (i) $\Leftrightarrow$ (iv) is immediate from Proposition 3.4 and (iii) $\Rightarrow$ (ii) is clear from the definitions involved. Proposition 3.5 gives (ii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (iii). Recall that VH is a Montel (LB)-space and hence, is a reflexive (DF)-space. Since T_φ is power bounded, the reflexivity of VH implies that T_φ is mean ergodic, [26, Corollary, p.214]. Then the Montel property of the (DF)-space VH ensures that T_φ is uniformly mean ergodic, [1, Proposition 2.8]. □

We end with two classical examples to which Proposition 3.6 applies.

Example 3.7. (i) The Korenblum algebra. For each $n \in \mathbb{N}$, define $v_n(z) := (1 - |z|)^n$ for $z \in \mathbb{D}$, in which case $V = (v_n)_{n=1}^\infty$ is a decreasing sequence of weight functions which satisfies property (S) because $\lim_{r\to 1^-} v_{n+1}(r)/v_n(r) = 0$ for each $n \in \mathbb{N}$. The space $VH = \mathrm{ind}_n\, H_{v_n}^\infty$ is traditionally denoted by $A^{-\infty}$. Its study was initiated in 1975 by B. Korenblum; see [14] for the detailed theory of $A^{-\infty}$, which is actually a commutative lc-topological algebra for pointwise multiplication. In particular, Lemma 3.2 applied to $A^{-\infty}$ provides a supply of continuous multiplication operators T_φ with $\varphi \in H(\mathbb{D}) \setminus H^\infty(\mathbb{D})$.

(ii) Taskinen's weighted inductive limit space for continuity of the Bergman projection, [25]. Here $V = (v_n)_{n=1}^\infty$ is defined by

$$v_n(z) := \begin{cases} 1 \text{ if } |z| \leq (e-1)/e \\ |\log(1-|z|)|^{-n}, \qquad (e-1)/e \leq |z| < 1. \end{cases}$$

Then $\lim_{r\to 1^-} v_{n+1}(r)/v_n(r) = 0$ for each $n \in \mathbb{N}$ and so V satisfies property (S). Again $\mathrm{ind}_n\, H_{v_n}^\infty$ is a commutative lc-algebra.

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