C_0-SEMIGROUPS AND MEAN ERGODIC OPERATORS IN A CLASS OF FRÉCHET SPACES

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Abstract. It is shown that the generator of every exponentially equicontinuous, uniformly continuous C_0–semigroup of operators in the class of quojection Fréchet spaces X (which includes properly all countable products of Banach spaces) is necessarily everywhere defined and continuous. If, in addition, X is a Grothendieck space with the Dunford–Pettis property, then uniform continuity can be relaxed to strong continuity. Two results, one of M. Lin and one of H.P. Lotz, both concerned with uniformly mean ergodic operators in Banach spaces, are also extended to the class of Fréchet spaces mentioned above. They fail to hold for arbitrary Fréchet spaces.

Dedicated to the memory of V. B. Moscatelli

1. Introduction.

Consider a C_0-semigroup of operators $(T(t))_{t\geq 0}$ acting in a Banach space X and which is operator norm continuous. It is a classical result that its infinitesimal generator is then an everywhere defined, bounded linear operator on X, [17, Chap. VIII, Corollary 1.9]. If X happens to be a Grothendieck space with the Dunford-Pettis property (briefly, a GDP-space), then the operator norm continuity of $(T(t))_{t\geq 0}$ is *automatic* whenever the semigroup is merely strongly continuous. This is an elegant result due to H. P. Lotz, [26, 27], which had well known forrunners for *particular* GDP-spaces and C_0–semigroups of operators. For instance, it was known that every strongly continuous semigroup of *positive* operators in L^∞ has a bounded generator, [20]. Or, by a result of L.A. Rubel (see [7], for example), given any strongly continuous *group of isometries* $(T(t))_{t\in\mathbb{R}}$ in $H^\infty(\mathbb{D})$ there exists $\alpha \in \mathbb{R}$ such that $T(t) = e^{i\alpha t}I$, for $t \in \mathbb{R}$. Hence, $T(\cdot)$ is surely uniformly continuous.

Let T be a bounded linear operator on a Banach space X and consider its Cesàro means

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}.$$

If the sequence $\{T_{[n]}\}_{n=1}^\infty$ converges to some operator strongly in X (resp. in operator norm), then T is called *mean ergodic* (resp. *uniformly mean ergodic*). As a standard reference on this topic we refer to [24], for example. A useful result of M. Lin states that if $\mathrm{Ker}(I - T) = \{0\}$ and $\lim_{n\to\infty} \frac{1}{n}\|T^n\| = 0$, then T is

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uniformly mean ergodic (with $\|T_{[n]}\| \to 0$ as $n \to \infty$) if and only if $I - T_{[n]}$ is surjective for some $n \in \mathbb{N}$ if and only if $I - T$ is surjective, [25]. If X is a GDP-space, then another interesting result of H.P. Lotz states that a bounded operator T is uniformly mean ergodic in X whenever it merely satisfies $\lim_{n\to\infty} \frac{1}{n}\|T^n\| = 0$, [26, 27].

Suppose now that X is a Fréchet space. Then the natural analogue of operator norm convergence for Banach spaces is the topology τ_b of uniform convergence on the bounded subsets of X.

The basic theory of C_0-semigroups in the class of sequentially complete locally convex spaces (which includes Fréchet spaces) was developed by Komura and Yosida, [21, 39]. The recent (and growing) interest in the hypercyclicity of continuous linear operators and, in particular, of C_0-semigroups in non–normable Fréchet spaces, [8, 9, 12, 13, 37], suggests the need to determine whether or not certain important results concerning C_0-semigroups on Banach spaces continue to hold in the setting of Fréchet spaces. In particular, Conejero raised the question of whether every C_0-semigroup on ω has a continuous everywhere defined infinitesimal generator, [12, p.467]. We also point out that in the recent articles [3], [11], the theory of GDP-Fréchet spaces has been significantly advanced. So, do those classical Banach results mentioned above also carry over to the Fréchet space setting? Examples exist which show that in this generality the answer is surely no! The aim of this note is to show, nevertheless, that there is an important class of Fréchet spaces, namely the quojections, in which *all* of the above results are valid. This class of spaces was introduced in [5] and contains all countable products of Banach spaces. More precisely, it is shown in Section 3 that every (exponentially) equicontinuous, τ_b-continuous C_0-semigroup of operators in a quojection X necessarily has an everywhere defined and continuous infinitesimal generator. Furthemore, if X is also a GDP-space, then the τ_b-continuity of any (exponentially) equicontinuous C_0–semigroup in X follows automatically from its strong continuity. Concerning mean ergodic operators, it is shown in Section 4 that the Banach space criterion of M. Lin also carries over to quojections provided that $\lim_{n\to\infty} \frac{1}{n}\|T^n\| = 0$ is replaced with τ_b-$\lim_{n\to\infty} \frac{1}{n}T^n = 0$. In addition, a continuous linear operator T in a GDP-quojection is uniformly mean ergodic whenever merely τ_b-$\lim_{n\to\infty} \frac{1}{n}T^n = 0$.

2. Preliminaries.

Let X be a locally convex Hausdorff space (briefly, lcHs) and Γ_X a system of continuous seminorms determining the topology of X. Then the strong operator topology τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators from X into itself (from X into another lcHs Y we write $\mathcal{L}(X, Y)$) is determined by the family of seminorms

$$q_x(S) := q(Sx), \quad S \in \mathcal{L}(X),$$

for each $x \in X$ and $q \in \Gamma_X$, in which case we write $\mathcal{L}_s(X)$. Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X. The topology τ_b of uniform convergence on bounded sets is defined in $\mathcal{L}(X)$ via the seminorms

$$q_B(S) := \sup_{x\in B} q(Sx), \quad S \in \mathcal{L}(X),$$

for each $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$; in this case we write $\mathcal{L}_b(X)$. For X a Banach space, τ_b is the operator norm topology in $\mathcal{L}(X)$. If Γ_X is countable and X is complete, then X is called a Fréchet space. The identity operator on a lcHs X is denoted by I.

By X_σ we denote X equipped with its weak topology $\sigma(X, X')$, where X' is the topological dual space of X. The strong topology in X (resp. X') is denoted by $\beta(X, X')$ (resp. $\beta(X', X)$) and we write X_β (resp. X'_β); see [22, §21.2] for the definition. The strong dual space $(X'_\beta)'_\beta$ of X'_β is denoted simply by X''. By X'_σ we denote X' equipped with its weak–star topology $\sigma(X', X)$. Given $T \in \mathcal{L}(X)$, its *dual operator* $T^t \colon X' \to X'$ is defined by $\langle x, T^t x'\rangle = \langle Tx, x'\rangle$ for all $x \in X$, $x' \in X'$. It is known that $T^t \in \mathcal{L}(X'_\sigma)$ and $T^t \in \mathcal{L}(X'_\beta)$, [23, p.134].

Definition 2.1. [39, p.234] *Let X be a lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a one parameter family of operators satisfying the following properties.*

(i) $T(s)T(t) = T(s+t)$ *for all* $s, t \geq 0$, *with* $T(0) = I$.

(ii) $\lim_{t\to t_0} T(t)x = T(t_0)x$ *for all* $t_0 \geq 0$, $x \in X$, *i.e.*, $T(t) \to T(t_0)$ *in* $\mathcal{L}_s(X)$ *as* $t \to t_0$.

(iii) *There exists* $a \geq 0$ *such that* $(e^{-at}T(t))_{t\geq 0}$ *is an equicontinuous subset of* $\mathcal{L}(X)$, *i.e.*,

$$\forall p \in \Gamma_X\ \exists q \in \Gamma_X, M_p > 0 \text{ such that } p(T(t)x) \leq M_p e^{at} q(x)\ \forall t \geq 0, x \in X. \quad (2.1)$$

Such a family $(T(t))_{t\geq 0}$ is called an exponentially equicontinuous, C_0–semigroup on X. If $a = 0$, then we simply say equicontinuous.

If $(T(t))_{t\geq 0}$ satisfies the conditions (i) *and* (iii) *and the stronger condition*

(ii)' *for all* $t_0 \geq 0$ *we have* $T(t) \to T(t_0)$ *in* $\mathcal{L}_b(X)$ *as* $t \to t_0$,

then it is called an exponentially equicontinuous, uniformly continuous semigroup on X.

Observe that, given any exponentially equicontinuous C_0–semigroup $(T(t))_{t\geq 0}$ (resp. any exponentially equicontinuous, uniformly continuous semigroup) on a lcHs X, the condition (ii) (resp. the condition (ii)') in Definition 2.1 is equivalent to $T(t) \to I$ in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$) as $t \to 0^+$. This is a consequence of (i), namely, that $T(t_0 + h) - T(t_0) = T(t_0)(T(h) - I)$ for each $t_0 > 0$ and all h such that $t_0 + h \geq 0$.

If X is a *sequentially complete* lcHs and $(T(t))_{t\geq 0}$ is an exponentially equicontinuous C_0–semigroup on X, then the linear operator A defined by

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t},$$

for $x \in D(A) := \{x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists in } X\}$, is closed with $\overline{D(A)} = X$, [39, Ch. IX, Sections 3 & 4]. The operator $(A, D(A))$ is called the *infinitesimal generator* of $(T(t))_{t\geq 0}$.

It is known that every C_0–semigroup of operators in a Banach space is necessarily exponentially equicontinuous, [17, p.619]. For Fréchet spaces this need not be the case. Indeed, in the sequence space $\omega = \mathbb{C}^{\mathbb{N}}$ (with the topology of coordinate convergence) it is routine to check that

$$T(t)x := (e^{nt}x_n)_{n=1}^\infty, \quad t \geq 0,\ x = (x_n)_{n=1}^\infty \in \omega,$$

defines a C_0–semigroup which is not exponentially equicontinuous. Actually, since ω is a Montel space, $(T(t))_{t\geq 0}$ is even uniformly continuous.

Remark 2.2. Let X be a lcHs and $(T(t))_{t\geq 0}$ be an *equicontinuous* C_0–semigroup on X. Given any $p \in \Gamma_X$, define $\tilde{p}$ on X via the formula

$$\tilde{p}(x) := \sup_{t\geq 0} p(T(t)x), \quad x \in X.$$

By Definition 2.1(i)–(iii) $\tilde{p}$ is well-defined, is a seminorm and satisfies

$$p(x) \leq \tilde{p}(x) \leq M_p q(x) \leq M_p \tilde{q}(x), \quad x \in X. \tag{2.2}$$

Hence, $\tilde{\Gamma}_X := \{\tilde{p} : p \in \Gamma_X\}$ is also a system of continuous seminorms generating the given lc–topology of X, with the additional property that

$$\tilde{p}(T(t)x) = \sup_{s\geq 0} p(T(t)T(s)x) = \sup_{s\geq 0} p(T(t+s)x) \leq \tilde{p}(x), \quad x \in X,\ t \geq 0, \tag{2.3}$$

for all $\tilde{p} \in \tilde{\Gamma}_X$.

A Fréchet space X is always a projective limit of continuous linear operators $S_k : X_{k+1} \to X_k$, for $k \in \mathbb{N}$, with each X_k a Banach space. If it is possible to choose X_k and S_k such that each S_k is surjective and X is isomorphic to $\mathrm{proj}_j(X_j, S_j)$, then X is called a *quojection*, [5, Section 5]. Banach spaces and countable products of Banach spaces are quojections. Actually, every quojection is the quotient of a countable product of Banach spaces, [10]. In [31] Moscatelli gave the first examples of quojections which are not isomorphic to countable products of Banach spaces. Concrete examples of quojections are the sequence space $\omega = \mathbb{C}^{\mathbb{N}}$, the function spaces $L^p_{loc}(\Omega)$, with $1 \leq p \leq \infty$ and Ω an open subset of $\mathbb{R}^N$, and $C^{(m)}(\Omega)$, with $m \in \mathbb{N}_0$ and Ω an open subset of $\mathbb{R}^N$, when equipped with their canonical lc–topology. Indeed, the above function spaces are isomorphic to countable products of Banach spaces. Moreover, the spaces of continuous functions $C(X)$, with X a σ–compact completely regular topological space, endowed with the compact open topology are also examples of quojections. However, Domański constructed a completely regular topological space X such that the Fréchet space $C(X)$ is a quojection which is not isomorphic to a complemented subspace of a product of Banach spaces [16, Theorem]. It is known that a Fréchet space X admits a continuous norm if and only if X contains no isomorphic copy of ω, see [19, Theorem 7.2.7]. On the other hand, a quojection X admits a continuous norm if and only if it is a Banach space, see [5, Proposition 3]. Hence, a quojection is either a Banach space or contains an isomorphic copy of ω, necessarily complemented, see [19, Theorem 7.2.7]. For further information on quojections we refer to the survey paper [29] and the references therein; see also [5], [15].

The following technical result plays a crucial role in later sections.

Lemma 2.3. *Let X be a quojection and let $\{q_j\}_{j=1}^\infty$ be an increasing sequence of seminorms generating the lc–topology of X. Let $\{S_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ be any sequence of operators satisfying the following properties.*

(i) For each $j \in \mathbb{N}$ there exists $c_j > 0$ such that

$$q_j(S_n x) \leq c_j q_j(x), \quad x \in X,\ n \in \mathbb{N},$$

that is, $\{S_n\}_{n=1}^\infty$ is equicontinuous in $\mathcal{L}(X)$.

(ii) For each $m \in \mathbb{N}$*, we have* $\lim_{n\to\infty}(I - S_m)S_n = 0$ *in* $\mathcal{L}_b(X)$*.*

(iii) $\lim_{n\to\infty} S_n^2 = 0$ *in* $\mathcal{L}_b(X)$*.*

Then also $\lim_{n\to\infty} S_n = 0$ *in* $\mathcal{L}_b(X)$*.*

Proof. For each $j \in \mathbb{N}$, set $X_j := X/q_j^{-1}(\{0\})$, endowed with the quotient lc–topology and denote by $Q_j : X \to X_j$ the canonical (surjective) quotient map, so that $\mathrm{Ker}(Q_j) = q_j^{-1}(\{0\})$. Then X_j is a Fréchet space whose lc–topology is generated by the sequence of seminorms $\{(\hat{q}_j)_k\}_{k=1}^\infty$ given by

$$(\hat{q}_j)_k(Q_jx) := \inf\{q_k(y) : \ y \in X \text{ satisfies } Q_jy = Q_jx\}, \quad x \in X.$$

Observe that

$$(\hat{q}_j)_k(Q_jx) \le q_k(x), \quad x \in X,\ k \in \mathbb{N}. \tag{2.4}$$

Moreover, $(\hat{q}_j)_j(Q_jx) = q_j(x)$, for all $x \in X$, thereby implying that $(\hat{q}_j)_j$ is a norm on X_j and hence, that $(\hat{q}_j)_k$ is a norm on X_j for all $k \ge j$. So, X_j is actually a Banach space because X is a quojection, [5, Proposition 3]. Then there exists $k(j) \ge j$ such that the norm $(\hat{q}_j)_{k(j)}$ generates the lc–topology of X_j. Consequently, X is isomorphic to the projective limit of the sequence $(X_j, (\hat{q}_j)_{k(j)})_{j=1}^\infty$ of Banach spaces with respect to the surjective linking maps $Q_{j,j+1} : X_{j+1} \to X_j$ defined by $Q_{j,j+1}(Q_{j+1}x) = Q_jx$ for all $x \in X$, i.e, $X = \mathrm{proj}_j(X_j, Q_{j,j+1})$.

Next, fix $j \in \mathbb{N}$ and define a sequence $\{S_n^{(j)}\}_{n=1}^\infty$ of operators on the Banach space X_j via

$$S_n^{(j)}Q_jx := Q_jS_nx, \quad x \in X. \tag{2.5}$$

Each $S_n^{(j)}$, for $n \in \mathbb{N}$, is a well defined continuous linear operator on X_j. Indeed, suppose that $Q_jx = Q_jy$ for some $x, y \in X$, i.e., $x - y \in \mathrm{Ker}(Q_j)$ so that $q_j(x - y) = 0$. This, together with (i), yields $q_j(S_n(x - y)) = 0$, i.e., $S_n(x - y) \in q_j^{-1}(\{0\}) = \mathrm{Ker}(Q_j)$. Therefore, $Q_jS_nx = Q_jS_ny$ which implies that $S_n^{(j)}Q_jx = S_n^{(j)}Q_jy$ by (2.5). So, $S_n^{(j)}$ is well defined and clearly linear. Moreover, via (i), (2.4) and (2.5) we obtain that

$$\begin{aligned}(\hat{q}_j)_{k(j)}(S_n^{(j)}\hat{x}) &= (\hat{q}_j)_{k(j)}(S_n^{(j)}Q_jx) = (\hat{q}_j)_{k(j)}(Q_jS_nx)\\ &\le q_{k(j)}(S_nx) \le c_{k(j)}q_{k(j)}(x)\end{aligned}$$

for all $\hat{x} \in X_j$ and $x \in X$ with $Q_jx = \hat{x}$. Taking the infimum with respect to $x \in Q_j^{-1}(\{\hat{x}\})$, it follows from the definition of the quotient seminorm $(\hat{q}_j)_{k(j)}$ that

$$(\hat{q}_j)_{k(j)}(S_n^{(j)}\hat{x}) \le c_{k(j)}(\hat{q}_j)_{k(j)}(\hat{x}), \quad \hat{x} \in X_j. \tag{2.6}$$

Since the norm $(\hat{q}_j)_{k(j)}$ induces the lc–topology of X_j, (2.6) ensures the continuity of $S_n^{(j)}$.

It follows from (2.5) that, for fixed $j \in \mathbb{N}$, we have

$$(S_n^{(j)})^2Q_jx = S_n^{(j)}Q_jS_nx = Q_jS_n^2x, \quad x \in X,\ n \in \mathbb{N}.$$

To see that $(S_n^{(j)})^2 \to 0$ in $\mathcal{L}_b(X_j)$ as $n \to \infty$ (i.e., in operator norm), let $\hat{B}_j$ denote the closed unit ball of the Banach space $(X_j, (\hat{q}_j)_{k(j)})$. Since X is a quojection, there is $B_j \in \mathcal{B}(X)$ such that $\hat{B}_j \subseteq Q_j(B_j)$, [15, Proposition 1]. Accordingly,

$$\begin{aligned}\sup_{\hat{x}\in\hat{B}_j} (\hat{q}_j)_{k(j)} \left((S_n^{(j)})^2\hat{x}\right) &\leq \sup_{\hat{x}\in Q_j(B_j)} (\hat{q}_j)_{k(j)} \left((S_n^{(j)})^2\hat{x}\right) \\ &= \sup_{x\in B_j} (\hat{q}_j)_{k(j)} \left((S_n^{(j)})^2 Q_j x\right) \\ &= \sup_{x\in B_j} (\hat{q}_j)_{k(j)} \left(Q_j(S_n^2 x)\right) \\ &\leq \sup_{x\in B_j} q_{k(j)} \left(S_n^2 x\right),\end{aligned}$$

where the last inequality follows from (2.4). But, $\sup_{x\in B_j} q_{k(j)}\left(S_n^2 x\right) \to 0$ as $n\to\infty$ by (iii). It follows that $(S_n^{(j)})^2 \to 0$ in $\mathcal{L}_b(X_j)$ as $n\to\infty$.

Now, fix $m \in \mathbb{N}$. Then, for each $j \in \mathbb{N}$, it follows from (2.5) that

$$\begin{aligned}(I - S_m^{(j)})S_n^{(j)}Q_j x &= Q_jS_nx - S_m^{(j)}Q_jS_nx = Q_jS_nx - Q_jS_mS_nx \\ &= Q_j(I-S_m)S_nx,\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Moreover, (ii) states that $(I - S_m)S_n \to 0$ in $\mathcal{L}_b(X)$ as $n\to\infty$ and so we can proceed as for $(S_n^{(j)})^2$ to conclude that

$$\lim_{n\to\infty} (I - S_m^j)S_n^j = 0 \;\text{ in }\; \mathcal{L}_b(X_j), \quad m \in \mathbb{N}. \tag{2.7}$$

Now, as X_j is a Banach space and $(S_n^{(j)})^2 \to 0$ in $\mathcal{L}_b(X_j)$ as $n\to\infty$, for all $j \in \mathbb{N}$ there exists $n(j) > n(j-1)$ (setting $n(0) := 1$) such that $(I - (S_{n(j)}^{(j)})^2)$ is invertible in $\mathcal{L}(X_j)$, [17, p.566]. Then the identity

$$(I - S_{n(j)}^{(j)})(I + S_{n(j)}^{(j)})(I - (S_{n(j)}^{(j)})^2)^{-1} = I$$

in X_j implies that $I - S_{n(j)}^{(j)}$ is invertible in $\mathcal{L}(X_j)$. But, for each $j \in \mathbb{N}$, it follows from (2.7) that $\lim_{n\to\infty}(I - S_{n(j)}^{(j)})S_n^{(j)} = 0$ in $\mathcal{L}_b(X_j)$ and so

$$\tau_b - \lim_{n\to\infty} S_n^{(j)} = \tau_b - \lim_{n\to\infty} (I - S_{n(j)}^{(j)})^{-1}(I - S_{n(j)}^{(j)})S_n^{(j)} = 0. \tag{2.8}$$

Finally, in order to conclude that $S_n \to 0$ in $\mathcal{L}_b(X)$, we fix $j \in \mathbb{N}$ and $B \in \mathcal{B}(X)$. Then, for each $n \in \mathbb{N}$, we have via (2.5) that

$$\begin{aligned}\sup_{x\in B} q_j(S_nx) &= \sup_{x\in B} (\hat{q}_j)_j(Q_jS_nx) = \sup_{x\in B} (\hat{q}_j)_j(S_n^{(j)}Q_jx) \\ &= \sup_{\hat{x}\in Q_j(B)} (\hat{q}_j)_j(S_n^{(j)}\hat{x}) \leq \sup_{\hat{x}\in Q_j(B)} (\hat{q}_j)_{k(j)}(S_n^{(j)}\hat{x}),\end{aligned}$$

with $Q_j(B) \in \mathcal{B}(X_j)$, after recalling that $q_j(x) = (\hat{q}_j)_j(Q_jx)$, for $x \in X$ and $j \in \mathbb{N}$ (because $\mathrm{Ker}(Q_j) = q_j^{-1}(\{0\})$). □

Recall that a lcHs X is a *Grothendieck space* if every sequence in X' which is convergent for $\sigma(X', X)$ is also convergent for $\sigma(X', X'')$. Clearly every reflexive lcHs is a Grothendieck space. A lcHs X is said to have the *Dunford–Pettis property* (briefly, DP) if every element of $\mathcal{L}(X, Y)$, for Y any quasicomplete lcHs, which transforms elements of $\mathcal{B}(X)$ into relatively $\sigma(Y, Y')$–compact subsets of Y, also transforms $\sigma(X, X')$–compact subsets of X into relatively compact subsets of Y, [18, pp.633-634]. Actually, it suffices if Y runs through the class of all Banach spaces, [11, p.79]. A reflexive lcHs satisfies the DP property if and only if it is Montel, [18, p.634]. A Grothendieck lcHs X with the DP property is called, briefly, a GDP space.

Lemma 2.4. *Let X be a barrelled lcHs which is a GDP space. Let $\{S_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ be a sequence of operators satisfying the following properties.*

(i) *For each $m, n \in \mathbb{N}$ we have $S_m S_n = S_n S_m$.*

(ii) *For each $m \in \mathbb{N}$ we have $\lim_{n\to\infty}(S_n - I)S_m = 0$ in $\mathcal{L}_b(X)$.*

(iii) *$\lim_{n\to\infty} S_n = I$ in $\mathcal{L}_s(X)$.*

Then $\lim_{n\to\infty}(S_n - I)^2 = 0$ in $\mathcal{L}_b(X)$.

If, in addition, X is a quojection Fréchet space and there exists a fundamental sequence $\{q_j\}_{j=1}^\infty$ of seminorms generating the lc–topology of X which satisfy

(iv) *for each $j \in \mathbb{N}$ there exists $c_j > 0$ such that*

$$q_j(S_n x) \le c_j q_j(x), \quad x \in X,\ n \in \mathbb{N},$$

then also $(S_n - I) \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$, i.e., $\lim_{n\to\infty} S_n = I$ in $\mathcal{L}_b(X)$.

Proof. For $n \in \mathbb{N}$, set $T_n := I - S_n$. Observe, by the hypotheses, that for each $m \in \mathbb{N}$, we have via (i) and (ii) that

$$(I - T_m)T_n = T_n(I - T_m) = (I - S_n)S_m \to 0 \text{ in } \mathcal{L}_b(X) \text{ as } n \to \infty. \tag{2.9}$$

Moreover, (iii) gives

$$\lim_{n\to\infty} T_n = 0 \text{ in } \mathcal{L}_s(X). \tag{2.10}$$

Next, suppose that $T_n^2 \not\to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. Then there exist $B \in \mathcal{B}(X)$, $q \in \Gamma_X$ and $\varepsilon > 0$ such that $\sup_{x\in B} q(T_n^2 x) \ge \varepsilon$ for infinitely many n. Select increasing integers $n(s) \uparrow \infty$ and a sequence $\{x_s\}_{s=1}^\infty \subseteq B$ with $q(T_{n(s)}^2 x_s) \ge \varepsilon$ for all $s \in \mathbb{N}$. Arguing as in the proof of [11, Proposition 4.2], for each $s \in \mathbb{N}$ there is $x_s' \in X'$ with $|\langle \cdot, x_s'\rangle| \le q(\cdot)$ pointwise on X and such that $|\langle T_{n(s)}^2 x_s, x_s'\rangle| \ge \varepsilon$, i.e.,

$$|\langle T_{n(s)} x_s, T_{n(s)}^t x_s'\rangle| \ge \varepsilon, \quad s \in \mathbb{N}.$$

Because of (2.9) we can apply [3, Lemma 3.5(ii)] to the sequence $\{T_{n(s)}\}_{s=1}^\infty$ acting in the GDP space X to conclude that

$$\lim_{s\to\infty} T_{n(s)} x_s = 0, \quad \text{ for } \sigma(X, X'),$$

after noting that $\{x_s\}_{s=1}^\infty$ is bounded in X, and apply [3, Lemma 3.5(i)] to conclude that

$$\lim_{s\to\infty} T_{n(s)}^t x_s' = 0, \quad \text{ for } \sigma(X', X''),$$

after noting that $\{x_s'\}_{s=1}^\infty \subseteq X'$ is bounded for $\sigma(X', X)$. Then the DP property of X ensures that

$$\lim_{s\to\infty} |\langle T_{n(s)} x_s, T_{n(s)}^t x_s'\rangle| = 0,$$

[3, Proposition 3.3(i)], which is a contradiction. Thus, we must have that $T_n^2 \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$.

Suppose now, in addition, that X is a quojection Fréchet space and (iv) is satisfied. Then we can apply Lemma 2.3 to $\{T_n\}_{n=1}^\infty$ to conclude that $T_n \to 0$ in $\mathcal{L}_b(X)$, i.e., $\lim_{n\to\infty} S_n = I$ in $\mathcal{L}_b(X)$, after noting that condition (iv) implies

$$q_j(T_n x) \le (1 + c_j) q_j(x), \quad x \in X,\ j, n \in \mathbb{N}.$$

□

3. C_0-semigroups in Quojections

As already noted in the Introduction, every operator norm continuous semigroup in a Banach space X has its infinitesimal generator belonging to $\mathcal{L}(X)$. We begin with an example to show that this fails to hold in general Fréchet spaces.

Example 3.1. Let $B = (a_n(i))_{i,n\in\mathbb{N}}$ be a Köthe matrix, i.e., $1 \leq a_n(i) \leq a_{n+1}(i)$ for all i, $n \in \mathbb{N}$. Then we define the spaces

$$\lambda_1(B) := \{x = (x_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \ p_n(x) := \sum_{i\in\mathbb{N}} a_n(i)|x_i| < \infty, \forall n \in \mathbb{N}\},$$

$$\lambda_\infty(B) := \{x = (x_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \ q_n(x) := \sup_{i\in\mathbb{N}} a_n(i)|x_i| < \infty, \forall n \in \mathbb{N}\}.$$

These spaces are Fréchet spaces relative to the sequence of seminorms $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$, respectively. They are called Köthe sequence (or echelon) spaces. Moreover, they are nuclear if and only if for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with $m > n$ such that $(\frac{a_n(i)}{a_m(i)})_{i\in\mathbb{N}} \in \ell^1$, in which case $\lambda_1(B) = \lambda_\infty(B)$, [28, Proposition 28.16]. In particular, the space $s := \{x \in \mathbb{C}^N : \ p_n(x) = \sum_{i\in\mathbb{N}} i^n|x_i| < \infty \ \forall n \in \mathbb{N}\}$ of all rapidly decreasing sequences is a nuclear Köthe sequence space with Köthe matrix $B = (i^n)_{i,n\in\mathbb{N}}$.

Now, suppose that the Köthe sequence space $\lambda_1(B)$ is nuclear and that $\mu = (\mu_i)_{i\in\mathbb{N}}$ is a sequence of real numbers such that each $\mu_i > 0$ and $\lim_{i\to\infty} \mu_i = \infty$. For each $t \geq 0$, define a linear operator $T(t)$ on $\lambda_1(B)$ by

$$T(t)x := (e^{-\mu_i t}x_i)_{i\in\mathbb{N}}, \quad x \in \lambda_1(B).$$

We claim that $(T(t))_{t\geq 0}$ is an equicontinuous, uniformly continuous semigroup in $\lambda_1(B)$. Indeed, observe that $T(0) = I$. Moreover, we have

$$p_n(T(t)x) = \sum_{i\in\mathbb{N}} a_n(i)e^{-\mu_i t}|x_i| \leq \sum_{i\in\mathbb{N}} a_n(i)|x_i| = p_n(x), \quad x \in \lambda_1(B),$$

for each $t \geq 0$ and $n \in \mathbb{N}$. Accordingly, $(T(t))_{t\geq 0} \subseteq \mathcal{L}(\lambda_1(B))$ and is equicontinuous.

Fix any $x \in \lambda_1(B)$ and $\varepsilon > 0$. Then, for given $n \in \mathbb{N}$, there is $i_0 \in I$ such that $\sum_{i>i_0} a_n(i)|x_i| < \varepsilon/4$. On the other hand, there is $t_0 > 0$ such that $\sum_{i\leq i_0} |e^{-\mu_i t} - 1|a_n(i)|x_i| < \varepsilon/2$ for all $0 < t < t_0$. It follows, for every $0 < t < t_0$, that

$$\begin{aligned} p_n(T(t)x - x) &= \sum_{i\leq i_0} |e^{-\mu_i t} - 1|a_n(i)|x_i| + \sum_{i>i_0} |e^{-\mu_i t} - 1|a_n(i)|x_i| \\ &< \frac{\varepsilon}{2} + 2\sum_{i>i_0} a_n(i)|x_i| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

i.e., $\lim_{t\to 0^+} p_n(T(t)x - x) = 0$ for each $n \in \mathbb{N}$. Since x is arbitrary, we conclude that $\lim_{t\to 0^+} T(t) = I$ in $\mathcal{L}_s(\lambda_1(B))$. So, $(T(t))_{t\geq 0}$ is an equicontinuous C_0-semigroup in $\lambda_1(B)$ and hence, it is also uniformly continuous as $\lambda_1(B)$ is Montel, since it is nuclear.

A routine calculation shows that the infinitesimal generator $(A, D(A))$ of the C_0–semigroup $(T(t))_{t\geq 0}$ is given by

$$Ax = (-\mu_i x_i)_{i\in\mathbb{N}} \ \text{ for } \ x \in D(A) = \{x \in \lambda_1(B) : \ \mu\cdot x := (\mu_i x_i)_{i\in\mathbb{N}} \in \lambda_1(B)\}.$$

If the sequence $\mu = (\mu_i)_{i\in\mathbb{N}}$ grows fast enough, then $D(A) \neq \lambda_1(B)$ and hence, the operator A is neither everywhere defined nor continuous! The sequence μ can be selected as follows. Without loss of generality, we may suppose that $(\frac{a_n(i)}{a_{n+1}(i)})_{i\in\mathbb{N}} \in \ell^1$ for all $n \in \mathbb{N}$ (pass to a subsequence, if necessary). Then, for each $n \in \mathbb{N}$, we set $\mu_i := \sum_{n=1}^{i} a_n(i)$, for $i \in \mathbb{N}$. So, $\mu_i \to \infty$ (recall that $a_n(i) \geq 1$) and $\frac{1}{\mu} := (\frac{1}{\mu_i})_{i\in\mathbb{N}} \in \lambda_1(B)$ because

$$\begin{aligned} p_m(\frac{1}{\mu}) = \sum_{i\in\mathbb{N}} a_m(i)\frac{1}{\mu_i} &= \sum_{i=1}^{m} a_m(i)\frac{1}{\mu_i} + \sum_{i=m+1}^{\infty} \frac{a_m(i)}{\sum_{n=1}^{i} a_n(i)} \\ &\leq \sum_{i=1}^{m} a_m(i)\frac{1}{\mu_i} + \sum_{i=m+1}^{\infty} \frac{a_m(i)}{a_{m+1}(i)} < \infty, \end{aligned}$$

for all $m \in \mathbb{N}$. But $\mu\cdot\frac{1}{\mu} = (1)_{i\in\mathbb{N}} \notin \lambda_1(B)$, i.e., $\frac{1}{\mu} \notin D(A)$.

Proposition 3.2. *Suppose that X is a Fréchet space which contains a complemented copy of some nuclear Köthe sequence space $\lambda_1(B)$. Then there exists an equicontinuous, uniformly continuous semigroup in X whose infinitesimal generator is not everywhere defined.*

Proof. Let $P\colon X \to X$ be any projection satisfying $\mathrm{Im(P)} = \lambda_1(\mathrm{B})$ and define $Y := \mathrm{Ker}(P)$. Next, let $(T_1(t))_{t\geq 0}$ be any equicontinuous, uniformly continuous semigroup on $\lambda_1(B)$ (see Example 3.1). Let $A \in \mathcal{L}(Y)$ be any operator for which $\{A^n\}_{n=0}^{\infty} \subseteq \mathcal{L}(Y)$ is equicontinuous. We may then assume that for each seminorm $p \in \Gamma_Y$ there exists $q \in \Gamma_Y$ such that

$$p(A^n y) \leq q(y), \quad y \in Y,\, n \in \mathbb{N}_0;$$

see for example [2, Remark 2.6(i)]. It follows that

$$p(e^{tA}y) \leq \sum_{n=0}^{\infty} p\left(\frac{t^n A^n y}{n!}\right) \leq e^t q(y), \quad y \in Y,\, t \geq 0.$$

Accordingly, we can define an equicontinuous, uniformly continuous semigroup $(T_2(t))_{t\geq 0}$ in Y by

$$T_2(t) := e^{-t}e^{tA}, \quad t \geq 0.$$

Then the one parameter family $(T(t))_{t\geq 0}$ of continuous linear operators on X defined via

$$T(t)x := T_1(t)Px + T_2(t)(I-P)x, \quad t \geq 0,\ x \in X,$$

is an equicontinuous, uniformly continuous semigroup in X whose infinitesimal generator is not everywhere defined. □

For the class of *quojection* Fréchet spaces, the phenomenon exhibited by Example 3.1 cannot occur.

Theorem 3.3. *Let X be a quojection Fréchet space and $(T(t))_{t\geq 0}$ be an exponentially equicontinuous, uniformly continuous semigroup on X. Then the infinitesimal generator A of $(T(t))_{t\geq 0}$ is everywhere defined, i.e., $D(A) = X$ and hence, $A \in \mathcal{L}(X)$.*

Proof. Let $a \geq 0$ be as in Definition 2.1. Then the rescaled semigroup $(e^{-at}T(t))_{t\geq 0}$ is equicontinuous and has infinitesimal generator $A-aI$ with domain $D(A-aI) = D(A)$. So, without loss of generality, we may suppose that $(T(t))_{t\geq 0}$ is equicontinuous.

According to Remark 2.2 there is a fundamental increasing sequence $\{q_j\}_{j=1}^{\infty}$ of continuous seminorms on X such that

$$q_j(T(t)x) \leq q_j(x), \quad x \in X,\ t \geq 0,\ j \in \mathbb{N}. \tag{3.1}$$

For each $j \in \mathbb{N}$, set $X_j := X/q_j^{-1}(\{0\})$, endowed with the quotient lc–topology and denote by $Q_j \colon X \to X_j$ the canonical (surjective) quotient map, so that $\mathrm{Ker}(Q_j) = q_j^{-1}(\{0\})$. As in the proof of Lemma 2.3, define the sequence of seminorms $\{(\hat{q}_j)_k\}_{k=1}^{\infty}$ in the Fréchet space X_j by

$$(\hat{q}_j)_k(Q_jx) := \inf\{q_k(y) :\ y \in X \ \text{ satisfies } \ Q_jy = Q_jx\}, \quad x \in X,$$

in which case $(\hat{q}_j)_k$ is a norm for each $k \geq j$, and select $k(j) \geq j$ such that the norm $(\hat{q}_j)_{k(j)}$ generates the lc–topology of X_j. That is, X is isomorphic to the projective limit of the sequence $(X_j, (\hat{q}_j)_{k(j)})_{j=1}^{\infty}$ of Banach spaces with respect to the surjective linking maps $Q_{j,j+1} \colon X_{j+1} \to X_j$ defined by $Q_{j,j+1}Q_{j+1}x = Q_jx$ for all $x \in X$.

Fix $j \in \mathbb{N}$. Define a family $(T_j(t))_{t\geq 0}$ of operators on the Banach space X_j via

$$T_j(t)Q_jx := Q_jT(t)x, \quad x \in X,\ t \geq 0. \tag{3.2}$$

Each $T_j(t)$, for $t \geq 0$, is a well defined linear continuous operator on X_j. Indeed, fix t and suppose that $Q_jx = Q_jy$ for some x, $y \in X$, i.e., $x-y \in \mathrm{Ker}(Q_j)$ so that $q_j(x-y) = 0$. This, together with (3.1), yields $q_j(T(t)(x-y)) = 0$ and hence, by (3.2), that $T(t)(x-y) \in q_j^{-1}(\{0\}) = \mathrm{Ker}(Q_j)$. Therefore, $Q_jT(t)x = Q_jT(t)y$ which implies that $T_j(t)Q_jx = T_j(t)Q_jy$; see (3.2). So, $T_j(t)$ is well defined, and clearly linear, for $t \geq 0$ and $j \in \mathbb{N}$, with $T_j(0) = I$. Moreover, via (2.4), (3.1) and (3.2) we obtain that

$$(\hat{q}_j)_{k(j)}(T_j(t)\hat{x}) = (\hat{q}_j)_{k(j)}(T_j(t)Q_jx) = (\hat{q}_j)_{k(j)}(Q_jT(t)x) \leq q_{k(j)}(T(t)x) \leq q_{k(j)}(x)$$

for all $\hat{x} \in X_j$ and $x \in X$ with $Q_jx = \hat{x}$. Taking the infimum with respect to $x \in Q_j^{-1}(\{\hat{x}\})$, it follows that

$$(\hat{q}_j)_{k(j)}(T_j(t)\hat{x}) \leq (\hat{q}_j)_{k(j)}(\hat{x}), \quad \hat{x} \in X_j. \tag{3.3}$$

Since $(\hat{q}_j)_{k(j)}$ is the norm of X_j, (3.3) ensures the continuity of $T_j(t)$, for all $t \geq 0$, and that $(T_j(t))_{t\geq 0} \subseteq \mathcal{L}(X_j)$ is uniformly bounded (i.e., equicontinuous).

Next observe that $(T_j(t))_{t\geq 0}$ satisfies the semigroup law. Indeed, by (3.2) and the surjectivity of Q_j we have

$$\begin{aligned} T_j(s)T_j(t)Q_jx &= T_j(s)Q_jT(t)x = Q_jT(s)T(t)x \\ &= Q_jT(s+t)x = T_j(s+t)Q_jx, \quad x \in X,\ s, t \geq 0. \end{aligned}$$

Finally, we claim that $(T_j(t))_{t\geq 0}$ is operator norm continuous on the Banach space X_j. Denote by $\hat{B}_j := \{\hat{x} \in X_j :\ (\hat{q}_j)_{k(j)}(\hat{x}) \leq 1\}$ the unit ball of X_j. As in the proof of Lemma 2.3, there is $B_j \in \mathcal{B}(X)$ such that $\hat{B}_j \subseteq Q_j(B_j)$. It follows from this containment and (3.2) that

$$\begin{aligned}\sup_{\hat{x}\in\hat{B}_j} (\hat{q}_j)_{k(j)}(T_j(t)\hat{x}-\hat{x}) &\leq \sup_{\hat{x}\in Q_j(B_j)} (\hat{q}_j)_{k(j)}(T_j\hat{x}-\hat{x}) \\ &= \sup_{x\in B_j} (\hat{q}_j)_{k(j)}(T_j(t)Q_jx-Q_jx) \\ &= \sup_{x\in B_j} (\hat{q}_j)_{k(j)}(Q_j(T(t)x-x)) \\ &\leq \sup_{x\in B_j} q_{k(j)}(T(t)x-x),\end{aligned}$$

where the last inequality follows from (2.4). But, $\sup_{x\in B_j} q_{k(j)}(T(t)x-x) \to 0$ as $t \to 0^+$, because $(T(t))_{t\geq 0}$ is uniformly continuous by assumption. Hence, $(T_j(t))_{t\geq 0}$ is operator norm continuous in X_j.

As noted in the Introduction, the operator norm continuity of $(T_j(t))_{t\geq 0}$ on X_j implies that its infinitesimal generator $A_j \in \mathcal{L}(X_j)$. In particular, $T_j(t) = e^{tA_j}$ for $t \geq 0$, and this holds for all $j \in \mathbb{N}$, which implies that the infinitesimal generator A of $(T(t))_{t\geq 0}$ is also everywhere defined. To show this we proceed as follows. Again, fix $j \in \mathbb{N}$. Observe, by (3.2) and the identity $Q_{j,j+1}Q_{j+1} = Q_j$, that

$$\begin{aligned}T_j(t)Q_jx &= Q_jT(t)x = Q_{j,j+1}Q_{j+1}T(t)x \\ &= Q_{j,j+1}T_{j+1}(t)Q_{j+1}x, \quad x \in X,\ t \geq 0.\end{aligned}$$

Thus, for each $x \in X$, we have

$$\begin{aligned}\frac{T_j(t)Q_jx - Q_jx}{t} &= \frac{Q_{j,j+1}T_{j+1}(t)Q_{j+1}x - Q_{j,j+1}Q_{j+1}x}{t} \\ &= Q_{j,j+1}\left(\frac{T_{j+1}(t)Q_{j+1}x - Q_{j+1}x}{t}\right), \quad t > 0. \qquad (3.4)\end{aligned}$$

Taking the limit in (3.4) for $t \to 0^+$, we obtain

$$A_jQ_jx = Q_{j,j+1}A_{j+1}Q_{j+1}x, \qquad (3.5)$$

which holds for all $x \in X$ and $j \in \mathbb{N}$. So, by (3.5) the operator $\overline{A}\colon X \to X$ defined by

$$Q_j\overline{A}x := A_jQ_jx, \quad x \in X,\ j \in \mathbb{N}, \qquad (3.6)$$

is well defined, linear (recall that $X = \operatorname{proj}_j(X_j, Q_{j,j+1})$) and satisfies

$$\begin{aligned}q_j(\overline{A}x) &= (\hat{q}_j)_j(Q_j\overline{A}x) = (\hat{q}_j)_j(A_jQ_jx) \leq (\hat{q}_j)_{k(j)}(A_jQ_jx) \\ &\leq c_j(\hat{q}_j)_{k(j)}(Q_jx) \leq q_{k(j)}(x), \quad x \in X, \qquad (3.7)\end{aligned}$$

for each $j \in \mathbb{N}$, i.e., $\overline{A} \in \mathcal{L}(X)$, where c_j denotes the operator norm of $A_j \in \mathcal{L}(X_j)$. Moreover, A coincides with $\overline{A}$ on $D(A)$. Indeed, if $x \in D(A)$, the we have the existence of

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t} = Ax \ \text{ in } X,$$

thereby implying, via (3.2), the existence of

$$\begin{aligned} Q_j A x &= \lim_{t\to 0^+} Q_j \left(\frac{T(t)x - x}{t}\right) \\ &= \lim_{t\to 0^+} \frac{T_j(t)Q_j x - Q_j x}{t} = A_j Q_j x, \end{aligned}$$

in X_j, for each $j \in \mathbb{N}$. Hence, (3.6) yields

$$Q_j A x = Q_j \overline{A} x, \quad j \in \mathbb{N},$$

which implies that $Ax = \overline{A}x$ as $X = \mathrm{proj}_j(X_j, Q_{j.j+1})$. On the other hand, if $j \in \mathbb{N}$ and $x \in X$, then we have from the previous calculation and (3.2) that

$$\lim_{t\to 0^+} Q_j \left(\frac{T(t)x - x}{t}\right) = \lim_{t\to 0^+} \frac{T_j(t)Q_j x - Q_j x}{t} = A_j Q_j x,$$

with the limit existing in X_j. As $X = \mathrm{proj}_j(X_j, Q_{j.j+1})$, this means exactly that

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t} = \overline{A}x$$

with the limit existing in X. This completes the proof. □

In the notation of the proof of Theorem 3.3, we point out that the power series expansion of $e^{tA_j} = T_j(t)$ in the Banach space X_j yields

$$T_j(t)Q_j x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A_j^n (Q_j x), \quad x \in X,\ t \geq 0,\ j \in \mathbb{N}.$$

So, as $X = \mathrm{proj}_j(X_j, Q_{j.j+1})$, we obtain from (3.2) the expansion

$$T(t)x = \sum_{n=0} \frac{t^n}{n!} A^n x, \quad x \in X,\ t \geq 0.$$

A *prequojection* is a Fréchet space X such that X''_β is a quojection. Every quojection is a prequojection. A prequojection is called *non–trivial* if it is not itself a quojection. It is known that X is a prequojection if and only if X'_β is a strict (LB)–space. An alternative characterization is that X is a prequojecton if and only if X has no nuclear quotient which admits a continuous norm, see [5, 14, 33, 38]. The problem of the existence of non–trivial prequojections arose in a natural way in [5]; it has been solved, in the positive sense, in various papers, [6], [14], [32]. All of these papers employ the same method, which consists in the construction of the dual of a prequojection, rather than the prequojection itself, which is often difficult to describe (see the survey paper [29] for further information). However, in [30] an alternative method for constructing prequojections is presented which has the advantage of being direct. For an example of a concrete space (i.e., a space of continuous functions on a suitable topological space), which is a non–trivial prequojection, see [1]. The relevance of such spaces for this paper is the following extension of Theorem 3.3.

Proposition 3.4. *Let X be a prequojection Fréchet space. Then every exponentially equicontinuous, uniformly continuous semigroup in X has infinitesimal generator belonging to $\mathcal{L}(X)$.*

Proof. Suppose that X is a prequojection and that $(T(t))_{t\geq 0}$ is an exponentially equicontinuous, uniformly continuous semigroup in X. It is routine to check that the bidual operators $(T(t)^{tt})_{t\geq 0}$ form an exponentially equicontinuous, C_0-semigroup in X''_β. Via the definition of the bounded sets and 0–neighbourhoods in X''_β, it is straightforward to check that the uniform continuity of $(T(t)^{tt})_{t\geq 0}$ in $\mathcal{L}_b(X''_\beta)$ follows from that of $(T(t))_{t\geq 0}$ in $\mathcal{L}_b(X)$; see also [3, Lemma 2.1]. Hence, X''_β being a quojection, we can apply Theorem 3.3 to $(T(t)^{tt})_{t\geq 0}$ to conclude that its infinitesimal generator A^{tt} (which is the bi-dual of the infinitesimal generator A of $(T(t))_{t\geq 0}$) is everywhere defined, i.e., $D(A^{tt}) = X''$ and that $A^{tt} \in \mathcal{L}(X''_\beta)$. Since $A^{tt}|_{D(A)} = A$ and $D(A)$ is dense in X, it follows that A is also everywhere defined and hence, that $A \in \mathcal{L}(X)$. □

Recall that a Fréchet space is not a prequojection if and only if it admits a separated quotient isomorphic to an infinite dimensional nuclear Köthe echelon space, see [5, 14, 33, 38]. This fact, together with Propositions 3.2 and 3.4, suggest the following

Question 1. Is a Fréchet space X a prequojection if and only if every exponentially equicontinuous, uniformly continuous semigroup in X has its infinitesimal generator belonging to $\mathcal{L}(X)$?

For Banach spaces the following result is due to H.P. Lotz, [26], [27].

Theorem 3.5. *Let X be a quojection GDP–Fréchet space and $(T(t))_{t\geq 0}$ be an exponentially equicontinuous, C_0–semigroup on X. Then $(T(t))_{t\geq 0}$ is uniformly continuous and its infinitesimal generator belongs to $\mathcal{L}(X)$.*

Proof. Without loss of generality, we can suppose that $(T(t))_{t\geq 0}$ is actually equicontinuous (see the proof of Theorem 3.3). As in the proof of Theorem 3.3 we can select a fundamental increasing sequence $\{q_j\}_{j=1}^\infty$ of continuous seminorms generating the lc–topology in X such that (3.1) is satisfied. Denote by $\{R(\lambda) := (\lambda I - A)^{-1}\}_{\lambda\in\mathbb{C},\mathrm{Re}(\lambda)>0} \subseteq \mathcal{L}(X)$ the family of resolvent operators of the infinitesimal generator A, in which case (3.1) and Theorem 2 of [39, p.241] (see also its proof) yield

$$q_j((\lambda R(\lambda))^k x) \leq q_j(x), \quad x \in X,\ \lambda \in (0, +\infty),\ j, k \in \mathbb{N}.$$

Choose any increasing sequence $(\lambda_n)_{n=1}^\infty \subseteq (0, +\infty)$ with $\lambda_n \to \infty$ and set $S_n := \lambda_n R(\lambda_n)$. For $k = 1$ the previous inequalities yield

$$q_j(S_n x) \leq q_j(x), \quad x \in X,\ j, n \in \mathbb{N}. \tag{3.8}$$

On the other hand, the resolvent equation $R(\lambda)-R(\mu) = (\mu-\lambda)R(\lambda)R(\mu)$ implies that $(S_n - I)S_m = (\lambda_n - \lambda_m)^{-1}\lambda_m(S_m - S_n)$ for $\lambda_n \neq \lambda_m$. Hence, via (3.8) we obtain $\lim_{n\to\infty}(S_n - I)S_m = 0$ in $\mathcal{L}_b(X)$, for each $m \in \mathbb{N}$. Moreover, $S_n = \lambda_n R(\lambda_n) \to I$ in $\mathcal{L}_s(X)$ as $n \to \infty$, see [39, Corollary 2, p.241]. In view of (3.8) we can then apply Lemma 2.4 to the sequence $\{S_n\}_{n=1}^\infty$ of pairwise commuting operators in X to conclude that $\lambda_n R(\lambda_n) = S_n \to I$ in $\mathcal{L}_b(X)$ as $n \to \infty$.

As in the proof of Theorem 3.3, for each $j \in \mathbb{N}$, we set $X_j := X/q_j^{-1}(\{0\})$, denote by $Q_j : X \to X_j$ the canonical (surjective) quotient map, so that $\mathrm{Ker}(Q_j) = q_j^{-1}(\{0\})$, and define the sequence of seminorms $\{(\hat{q}_j)_k\}_{k=1}^\infty$ in X_j so that $(\hat{q}_j)_k$ is a norm for $k \geq j$. Again select $k(j) \geq j$ such that the norm $(\hat{q}_j)_{k(j)}$ generates the lc–topology of X_j. Consequently, X is isomorphic to the projective limit of the sequence $(X_j, (\hat{q}_j)_{k(j)})_{j=1}^{\infty}$ of Banach spaces with surjective linking maps $Q_{j,j+1}\colon X_{j+1} \to X_j$ defined by $Q_{j,j+1}Q_{j+1}x = Q_jx$ for all $x \in X$.

Fix $j \in \mathbb{N}$. Define a one parameter family $(T_j(t))_{t\geq 0}$ of operators on the Banach space X_j (see (3.2)) by setting

$$T_j(t)Q_jx := Q_jT(t)x, \quad x \in X, \ t \geq 0. \tag{3.9}$$

As in the proof of Theorem 3.3, one shows that $(T_j(t))_{t\geq 0}$ is an equicontinuous C_0–semigroup on X_j. It follows from (3.9) that the family $\{(R(\lambda))_j\}_{\lambda\in\mathbb{C},\mathrm{Re}(\lambda)>0} \subseteq \mathcal{L}(X_j)$ of resolvent operators of the infinitesimal generator of $(T_j(t))_{t\geq 0}$ satisfies

$$(R(\lambda))_jQ_j = Q_jR(\lambda), \quad \lambda \in \mathbb{C}, \ \mathrm{Re}(\lambda) > 0, \tag{3.10}$$

in $\mathcal{L}(X, X_j)$. Denote by $(A, D(A))$ and $(A_j, D(A_j))$ the infinitesimal generator of $(T(t))_{t\geq 0}$ and $(T_j(t))_{t\geq 0}$, respectively. Then $Q_j(D(A)) \subseteq D(A_j)$ and

$$A_jQ_jx = Q_jAx, \quad x \in D(A). \tag{3.11}$$

Indeed, it follows from the identity $t^{-1}(T_j(t) - I)Q_jx = Q_j(t^{-1}(T(t) - I)x)$, valid for $t > 0$, and the continuity of $Q_j\colon X \to X_j$ that if $x \in D(A)$, i.e., $\lim_{t\to 0^+} t^{-1}(T(t)-I)x = Ax$ exists in X, then $t^{-1}(T_j(t)-I)Q_jx = Q_jAx$ exists in X_j. That is, $Q_jx \in D(A_j)$ and (3.11) holds. Next, observe that $\lambda_n(R(\lambda_n))_j \to I$ in $\mathcal{L}_b(X_j)$ as $n \to \infty$. Indeed, denote by $\hat{B}_j := \{\hat{x} \in X_j : \ (\hat{q}_j)_{k(j)}(\hat{x}) \leq 1\}$ the closed unit ball of the Banach space X_j, in which case there is $B_j \in \mathcal{B}(X)$ such that $\hat{B}_j \subseteq Q_j(B_j)$, see [15, Proposition 1]. So, via (3.10) it follows that

$$\begin{aligned}
\sup_{\hat{x}\in\hat{B}_j} (\hat{q}_j)_{k(j)}(\lambda_n(R(\lambda_n))_j\hat{x} - \hat{x}) &\leq \sup_{\hat{x}\in Q_j(B_j)} (\hat{q}_j)_{k(j)}(\lambda_n(R(\lambda_n))_j\hat{x} - \hat{x}) \\
&= \sup_{x\in B_j} (\hat{q}_j)_{k(j)}(\lambda_n(R(\lambda_n))_jQ_jx - Q_jx) \\
&= \sup_{x\in B_j} (\hat{q}_j)_{k(j)}(Q_j(\lambda_nR(\lambda_n)x - x)) \\
&\leq \sup_{x\in B_j} q_{k(j)}(\lambda_nR(\lambda_n)x - x),
\end{aligned}$$

with $\sup_{x\in B_j} q_{k(j)}(\lambda_nR(\lambda_n)x - x) \to 0$ as $n \to \infty$. Since X_j is a Banach space, for each $j \in \mathbb{N}$, the operator $S_n^{(j)} := \lambda_n(R(\lambda_n))_j \in \mathcal{L}(X_j)$ is invertible for some $n(j) > n(j-1)$ (with $n(0) := 1$), i.e., the series $\sum_{k=0}^{\infty}(I - S_{n(j)}^{(j)})^k$ converges to $(S_{n(j)}^{(j)})^{-1}$ in operator norm, i.e., in $\mathcal{L}_b(X_j)$. This implies, in particular, that the range $\mathrm{Im}((R(\lambda_{n(j)}))_j) = \mathrm{Im}(S_{n(j)}^{(j)}) = X_j$ so that $D(A_j) = X_j$ and hence, $A_j = \lambda_{n(j)}(I - (S_{n(j)}^{(j)})^{-1}) \in \mathcal{L}(X_j)$. Accordingly, $(T_j(t))_{t\geq 0}$ is a uniformly continuous semigroup on X_j, see [17, Ch. VIII Corollary 1.9].

Thus, also $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup on X. Indeed, fix $B \in \mathcal{B}(X)$ and $j \in \mathbb{N}$. By (3.9) we then have

$$\begin{aligned} \sup_{x\in B} q_j(T(t)x - x) &= \sup_{x\in B} (\hat{q}_j)_j(Q_jT(t)x - Q_jx) \\ &= \sup_{x\in B} (\hat{q}_j)_j(T_j(t)Q_jx - Q_jx) \\ &= \sup_{\hat{x}\in Q_j(B)} (\hat{q}_j)_j(T_j(t)\hat{x} - \hat{x}) \\ &\leq \sup_{\hat{x}\in Q_j(B)} (\hat{q}_j)_{k(j)}(T_j(t)\hat{x} - \hat{x}). \end{aligned} \tag{3.12}$$

Since $(T_j(t))_{t\geq 0}$ is uniformly continuous in X_j with $Q_j(B) \in \mathcal{B}(X_j)$, we see that the right–side of (3.12) converges to 0 as $t \to 0^+$. That is, $\lim_{t\to 0^+} T(t) = I$ in $\mathcal{L}_b(X)$. Theorem 3.3 now implies that $A \in \mathcal{L}(X)$. □

Every Montel Fréchet space is a GDP–space, see [11, Remark 2.2]. Moreover, by virtue of the Montel property, every exponentially equicontinuous, C_0–semigroup in such a space is necessarily uniformly continuous. However, Example 3.1 shows that without the space being a quojection in Theorem 3.5 it is not possible to conclude that the infinitesimal generator is everywhere defined.

According to Theorems 3.3 and 3.5, the natural analogue in Fréchet spaces of C_0–semigroups of operators in Banach spaces are, perhaps, the exponentially equicontinuous ones. Nevertheles, other types of semigroups are always present.

Proposition 3.6. *Every Fréchet space X without a continuous norm admits a uniformly continuous semigroup of operators which fails to be exponentially equicontinuous.*

Proof. The hypotheses on X imply that it contains a complemented subspace Z which is isomorphic to ω (see the discussion before Lemma 2.3), that is, $X = Y \oplus Z$ with $Z \simeq \omega$.

Let $(T(t))_{t\geq 0} \subseteq \mathcal{L}(\omega)$ be the uniformly continuous semigroup given in the example just prior to Remark 2.2 (and which is not exponentially equicontinuous). Denote the operators $T(t)$ when transferred from ω to Z by $U(t)$, for $t \geq 0$. Then $S(t)\colon Y \oplus Z \to Y \oplus Z$ defined by $(y, z) \to (y, U(t)z)$, that is, $S(t) = I \oplus U(t)$ for $t \geq 0$, is a semigroup in X of the required type. □

We end this section with a result about semigroups on the space ω.

Proposition 3.7. *Let $(T(t))_{t\geq 0}$ be a C_0–semigroup on ω with infinitesimal generator $(A, D(A))$. Then $(T(t))_{t\geq 0}$ is uniformly continuous. Moreover, if there exists $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ is invertible, then the infinitesimal generator A belongs to $\mathcal{L}(\omega)$.*

Proof. Since ω is a Fréchet space, the semigroup $(T(t))_{t\geq 0}$ is locally equicontinuous and its infinitesimal generator $(A, D(A))$ is a closed densely defined operator in ω, see [21, Propositions 1.3 and 1.4, Corollary p.261], [34, Proposition 1.1].

Set $R := \lambda I - A$. By the definition of being invertible, R is injective and continuously maps ω onto $D(A)$, when $D(A)$ is endowed with the lc–topology induced on it by ω. Since ω is *minimal*, i.e., every injective, continuous linear operator from ω into any lcHs is open, see [35, p.66], the operator R is also open and hence, a topological isomorphism. It follows that $D(A)$, being isomorphic to

ω, must be a closed subspace of ω. But $D(A)$ is dense in ω and hence, $D(A) = \omega$. So, A is everywhere defined and accordingly, belongs to $\mathcal{L}(\omega)$. □

In [12, p.467] Conejero raised the question of whether every C_0–semigroup on ω is of the form $\{e^{tA}\}_{t\geq 0}$ for some $A \in \mathcal{L}(\omega)$. Proposition 3.7 provides a positive answer for an extensive class of C_0-semigroups on ω. We point out that [39, Ch.IX, §4, Corollary 1] implies that every exponentially equicontinuous C_0–semigroup in ω necessarily satisfies the hypothesis of Proposition 3.7 (since the resolvent set of A contains an interval of the form (a, ∞) for some $a \geq 0$). However, the example just prior to Remark 2.2 shows that Proposition 3.7 also applies to some C_0–semigroups which are not exponentially equicontinuous.

4. Mean Ergodic Operators

A continuous linear operator T in a lcHs X is called *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X, \tag{4.1}$$

exist in X. An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$. Of course, for a Banach space X, this means that $\sup_{m\geq 0} \|T^m\| < \infty$. A power bounded operator T is mean ergodic precisely when

$$X = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Im}(I - T)}, \tag{4.2}$$

where I is the identity operator, Im(I − T) denotes the range of $I - T$ and the bar denotes the "closure in X".

Given $T \in \mathcal{L}(X)$, let

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}, \tag{4.3}$$

denote the Cesàro means of T (see also (4.1)). For X a barrelled lcHs, T is mean ergodic precisely when $\{T_{[n]}\}_{n=1}^{\infty}$ is a convergent sequence in $\mathcal{L}_s(X)$. If $\{T_{[n]}\}_{n=1}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$, then T is called *uniformly mean ergodic*. The space X itself is called mean ergodic (resp. uniformly mean ergodic) if every power bounded operator on X is mean ergodic (resp. uniformly mean ergodic).

Given a lcHs X and $T \in \mathcal{L}(X)$ we have

$$(I - T)T_{[n]} = T_{[n]}(I - T) = \frac{1}{n}(T - T^{n+1}), \quad n \in \mathbb{N}, \tag{4.4}$$

and also, with $T_{[0]} := I$, that

$$\frac{1}{n} T^n = T_{[n]} - \frac{n-1}{n} T_{[n-1]}, \quad n \in \mathbb{N}. \tag{4.5}$$

If $T \in \mathcal{L}(X)$ is power bounded, then

$$\overline{\mathrm{Im}(I - T)} = \{x \in X : \lim_{n\to\infty} T_{[n]}x = 0\} \tag{4.6}$$

and hence, in particular,

$$\overline{\mathrm{Im}(I - T)} \cap \mathrm{Ker}(I - T) = \{0\}, \tag{4.7}$$

[39, Ch. VIII, §3]. Moreover, such a T clearly satisfies

$$\lim_{n\to\infty} \frac{1}{n} T^n = 0, \text{ in } \mathcal{L}_s(X). \tag{4.8}$$

In the Banach space setting the following result is due to M. Lin [25]. For general Fréchet spaces the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) occur in [2, Proposition 2.16] and for more general lcHs' in [4, Proposition 2.5]

Theorem 4.1. *Let X be a lcHs with the property that every continuous linear surjection from X onto itself is an open map. Let $T \in \mathcal{L}(X)$ satisfy* $\operatorname{Ker}(I-T) = \{0\}$ *and $\frac{1}{n}T^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. Consider the following statements.*

(i) *$I - T_{[n]}$ is surjective for some $n \in \mathbb{N}$.*
(ii) *$I - T$ is surjective.*
(iii) *$T_{[n]} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

If, in addition, X is a quojection Fréchet space, then also (iii) $\Rightarrow$ (i).

Proof. As indicated prior to the statement of Theorem 4.1, it only remains to establish (iii)$\Rightarrow$(ii), under the assumption that X is a quojection Fréchet space.

So, let $\{r_j\}_{j=1}^\infty$ be a fundamental increasing sequence of seminorms generating the lc–topology of X. As $\frac{1}{n}T^n \to 0$ in $\mathcal{L}_b(X)$ and X is a Fréchet space, the sequence $\{\frac{1}{n}T^n\}_{n=1}^\infty$ is equicontinuous. So, for each $j \in \mathbb{N}$, there exists $c_j \geq 1$ such that

$$r_j\left(\frac{1}{n}T^n x\right) \leq c_j r_{j+1}(x), \quad x \in X,\ n \in \mathbb{N}. \tag{4.9}$$

Define q_j on X by

$$q_j(x) := \max\left\{r_j(x), \sup_n r_j\left(\frac{1}{n}T^n x\right)\right\}, \quad x \in X.$$

According to (4.9), each q_j is well defined. It is routine to check that q_j is a seminorm and satisfies

$$r_j(x) \leq q_j(x) \leq c_j r_{j+1}(x) \leq c_j q_{j+1}(x), \quad x \in X.$$

Hence, $\{q_j\}_{j=1}^\infty$ is also a fundamental increasing sequence of seminorms generating the Fréchet-topology of X. Moreover, for $j \in \mathbb{N}$, we have

$$\begin{aligned} q_j(Tx) &= \max\left\{r_j(Tx), \sup_n r_j\left(\frac{1}{n}T^{n+1}x\right)\right\} \\ &= \max\left\{r_j(Tx), \sup_n r_j\left(\frac{n+1}{n}\frac{1}{n+1}T^{n+1}x\right)\right\} \\ &\leq 2q_j(x), \quad x \in X. \end{aligned} \tag{4.10}$$

For each $j \in \mathbb{N}$, set $X_j := X/q_j^{-1}(\{0\})$, endowed with the quotient lc–topology and denote by $Q_j\colon X \to X_j$ the canonical (surjective) quotient map. As in the proof of Lemma 2.3, define the sequence of seminorms $\{(\hat{q}_j)_k\}_{k=1}^\infty$ in X_j by

$$(\hat{q}_j)_k(Q_j x) := \inf\{q_k(y) : \ y \in X \ \text{ satisfies } \ Q_j y = Q_j x\}, \quad x \in X,$$

and select $k(j) \geq j$ such that the norm $(\hat{q}_j)_{k(j)}$ generates the lc–topology of X_j. That is, X is isomorphic to the projective limit of the sequence $(X_j, (\hat{q}_j)_{k(j)})_{j=1}^\infty$

of Banach spaces with respect to the surjective linking maps $Q_{j,j+1}\colon X_{j+1} \to X_j$ defined by $Q_{j,j+1}(Q_{j+1}x) = Q_j x$ for all $x \in X$.

Fix $j \in \mathbb{N}$. Define an operator T_j on the Banach space X_j via

$$T_j Q_j x := Q_j T x, \quad x \in X. \tag{4.11}$$

Each T_j is a well defined linear operator on X_j; this follows via the same argument used for $T_j(t)$ in the proof of Theorem 3.3. Moreover, by (2.4), (4.10) and (4.11) we obtain that

$$(\hat{q}_j)_{k(j)}(T_j \hat{x}) = (\hat{q}_j)_{k(j)}(T_j Q_j x) = (\hat{q}_j)_{k(j)}(Q_j T x) \le q_{k(j)}(Tx) \le 2q_{k(j)}(x)$$

for all $\hat{x} \in X_j$ and $x \in X$ with $Q_j x = \hat{x}$. Taking the infimum with respect to $x \in Q_j^{-1}(\{\hat{x}\})$, it follows that

$$(\hat{q}_j)_{k(j)}(T_j \hat{x}) \le 2(\hat{q}_j)_{k(j)}(\hat{x}), \quad \hat{x} \in X_j. \tag{4.12}$$

Since $(X_j, (\hat{q}_j)_{k(j)})$ is a Banach space, (4.12) ensures the continuity of T_j.

It follows from (4.11) that, for fixed $j \in \mathbb{N}$, we have

$$T_j^n Q_j x = T_j^{n-1} Q_j T x = \ldots = T_j Q_j T^{n-1} x = Q_j T_j^n x, \quad x \in X,\, n \in \mathbb{N}.$$

To see that $\frac{1}{n}T_j^n \to 0$ in $\mathcal{L}_b(X_j)$ as $n \to \infty$, recall that $\hat{B}_j$ denotes the closed unit ball of the Banach space $(X_j, (\hat{q}_j)_{k(j)})$. Since X is a quojection, there is $B_j \in \mathcal{B}(X)$ such that $\hat{B}_j \subseteq Q_j(B_j)$. So, it follows that

$$\begin{aligned}\sup_{\hat{x}\in \hat{B}_j} (\hat{q}_j)_{k(j)}\left(\frac{1}{n}T_j^n \hat{x}\right) &\le \sup_{\hat{x}\in Q_j(B_j)} (\hat{q}_j)_{k(j)}\left(\frac{1}{n}T_j^n \hat{x}\right) = \sup_{x\in B_j} (\hat{q}_j)_{k(j)}\left(\frac{1}{n}T_j^n Q_j x\right)\\ &= \sup_{x\in B_j} (\hat{q}_j)_{k(j)}\left(Q_j\left(\frac{1}{n}T^n x\right)\right) \le \sup_{x\in B_j} q_{k(j)}\left(\frac{1}{n}T^n x\right).\end{aligned}$$

Since $\sup_{x\in B_j} q_{k(j)}\left(\frac{1}{n}T^n x\right) \to 0$ as $n \to \infty$ (by assumption), it follows that $\frac{1}{n}T_j^n \to 0$ in $\mathcal{L}_b(X_j)$ as $n \to \infty$. Because of

$$(T_j)_{[n]} Q_j x = \frac{1}{n}\sum_{i=1}^{n} T_j^i Q_j x = \frac{1}{n}\sum_{i=1}^{n} Q_j T^i x = Q_j T_{[n]} x, \quad x \in X,\ n \in \mathbb{N},$$

with $T_{[n]} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ (by assumption), we can proceed as in the previous argument (showing that $\frac{1}{n}T_j^n \to 0$ in $\mathcal{L}_b(X_j)$) to establish that $(T_j)_{[n]} \to 0$ in $\mathcal{L}_b(X_j)$ as $n \to \infty$. On the other hand, $\mathrm{Ker}(I - T_j) = \{0\}$; this follows by observing (see (4.6)) that $\overline{\mathrm{Im}(I - T_j)} = \{x \in X_j : \lim_{n\to\infty}(T_j)_{[n]}x = 0\}$, i.e., $\overline{\mathrm{Im}(I - T_j)} = X_j$, and hence, $\mathrm{Ker}(I - T_j) = \{0\}$ via (4.7). Since X_j is a Banach space, by Lin's result, [25], we can conclude that $I - T_j$ is invertible, i.e., $I - T_j$ is a topological isomorphism on X_j.

To conclude that $I - T$ is invertible in $\mathcal{L}(X)$, first observe, by (4.11) and the identity $Q_{j,j+1}Q_{j+1} = Q_j$, that

$$Q_j(I-T) = (I-T_j)Q_j \;\text{ and }\; Q_{j,j+1}(I-T_{j+1}) = (I-T_j)Q_{j,j+1}, \quad j \in \mathbb{N}, \tag{4.13}$$

with the first equalities holding in $\mathcal{L}(X, X_j)$ and the second equalities holding in $\mathcal{L}(X_{j+1}, X_j)$. Now, $I - T_j$ is bijective, for all $j \in \mathbb{N}$. Hence, given any $y \in X$, define $x_j := (I-T_j)^{-1}Q_jy$, for each $j\in\mathbb{N}$, and $x := (x_j)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} X_j$. Then, by (4.13),

$$(I-T_j)Q_{j,j+1}x_{j+1} = Q_{j,j+1}(I-T_{j+1})x_{j+1} = Q_{j,j+1}Q_{j+1}y = Q_jy, \quad j\in\mathbb{N},$$

and hence, by the injectivity of $I-T_j$, we have

$$Q_{j,j+1}x_{j+1} = (I-T_j)^{-1}Q_jy = x_j, \quad j\in\mathbb{N}.$$

Since $X = \mathrm{proj}_j(X_j, Q_{j,j+1}) = \{(u_j)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} X_j :\ Q_{j,j+1}u_{j+1} = u_j\ \forall j\in\mathbb{N}\}$ we can conclude that $x\in X$. Moreover, $(I-T)x = y$ because (4.13) implies that $Q_j(I-T)x = (I-T_j)Q_jx = Q_jy$ for all $j\in\mathbb{N}$. This shows that $I-T$ is surjective. Since $I-T$ is injective by hypothesis, it follows that $I-T$ is invertible in $\mathcal{L}(X)$. □

Remark 4.2. The proof of (iii)⇒(i) in Theorem 4.1, for the *particular* sequence $\{T_{[n]}\}_{n=1}^{\infty}$, relies on the facts that X is a quojection and that $\{T^n\}_{n\in\mathbb{N}}$ is a semigroup, in the sense that $T^nT^m = T^{n+m}$ for all n, $m\in\mathbb{N}$. The conclusion does not extend to a general sequence $\{R_n\}_{n\in\mathbb{N}}$ of operators on a quojection satisfying $R_n\to 0$ in $\mathcal{L}_b(X)$; see Remark 4.5(i) below.

We now present a result concerning any mean ergodic operator T satisfying $\frac{1}{n}T^n \to 0$ in $\mathcal{L}_b(X)$ and defined in a lcHs X which is a GDP–space. Namely, there exists a projection $P\in\mathcal{L}(X)$ such that the sequence of squares $(T_{[n]})^2\to P$ in $\mathcal{L}_b(X)$ as $n\to\infty$. If, in addition, X is a quojection Fréchet space, then actually $\lim_{n\to\infty} T_{[n]} = P$ in $\mathcal{L}_b(X)$, i.e., T is uniformly mean ergodic. For GDP–Banach spaces we recover a classical result of H.P. Lotz, [26, Theorem 8], [27, Theorem 5].

Theorem 4.3. *Let X be a barrelled lcHs which is a GDP–space and $T\in\mathcal{L}(X)$ be a mean ergodic operator satisfying $\lim_{n\to\infty}\frac{1}{n}T^n = 0$ in $\mathcal{L}_b(X)$. Then the following properties hold.*

- (i) *There exists a projection $P\in\mathcal{L}(X)$ such that $\lim_{n\to\infty}(T_{[n]})^2 = P$ in $\mathcal{L}_b(X)$.*
- (ii) *If, in addition, X is a quojection Fréchet space, then $\lim_{n\to\infty} T_{[n]} = P$ in $\mathcal{L}_b(X)$, i.e., T is uniformly mean ergodic.*

Proof. (i) Observe that $T_{[n]}T_{[m]} = T_{[m]}T_{[n]}$ for all n, $m\in\mathbb{N}$. Moreover,

$$\lim_{n\to\infty}(I-T_{[m]})T_{[n]} = 0 \ \text{ in } \mathcal{L}_b(X),\ \forall m\in\mathbb{N}. \tag{4.14}$$

Indeed, it is routine to verify that $g_m(T) := \frac{1}{m}\sum_{k=0}^{m-1}(\sum_{r=0}^{k} T^r)$ satisfies the identity $I-T_{[m]} = g_m(T)(I-T)$ and hence, via (4.3) and (4.4) that

$$(I-T_{[m]})T_{[n]} = g_m(T)(I-T)T_{[n]} = g_m(T)\frac{1}{n}(T-T^{n+1}), \quad n\in\mathbb{N}.$$

Since $g_m(T)$ is a continuous linear operator and m is fixed, these identities, together with τ_b-$\lim_{n\to\infty}\frac{1}{n}T^n = 0$, yield (4.14).

According to [2, Theorem 2.4], [35, Proposition 2.2], there is a projection $P\in\mathcal{L}(X)$, commuting with T, such that τ_s-$\lim_{n\to\infty} T_{[n]} = P$ and with the closed subspaces $\mathrm{Im(P)} = \mathrm{Ker(I-T)}$ and $\mathrm{Ker}(P) = \overline{\mathrm{Im}(I-T)}$ satisfying (4.2). If $x\in\mathrm{Im(P)}$, then $Tx = x$ and so $T_{[n]}x = x$ for all $n\in\mathbb{N}$. That is, the restriction

$$T_{[n]}|_{\mathrm{Ker}(I-T)} = I \ \text{ in } \ \mathcal{L}(\mathrm{Ker}(I-T)), \quad n\in\mathbb{N}. \tag{4.15}$$

On the other hand, since $\mathrm{Ker}(P) = \overline{\mathrm{Im}(I-T)}$ is complemented in X, it is both barrelled, see [35, Corollary 4.2.2], and a GDP–space, see [3, Proposition 3.1]. Since $PT_{[n]} = T_{[n]}P$, it is clear that $Z := \mathrm{Ker}(P)$ is $T_{[n]}$–invariant and hence, $T_{[n]}$ defines an element $S_{[n]} := T_{[n]}|_Z$ of $\mathcal{L}(Z)$, for all $n \in \mathbb{N}$. Of course, with $S := T|_Z$ we have $S_{[n]} = (T|_Z)_{[n]}$ and the operators $\{S_{[n]}\}_{n=1}^\infty$ are pairwise commuting. Moreover, $Z = \mathrm{Ker}(P) = \overline{\mathrm{Im}(I-T)}$ together with $\lim_{n\to\infty} T_{[n]}x = 0$, for all $x \in \mathrm{Ker}(P)$, imply that $\lim_{n\to\infty} S_{[n]} = 0$ in $\mathcal{L}_s(Z)$. So, defining $S_n \in \mathcal{L}(Z)$ via $S_n := I - S_{[n]}$, for $n \in \mathbb{N}$, we have that $S_nS_m = S_mS_n$ for all $m, n \in \mathbb{N}$ together with $S_n \to I$ in $\mathcal{L}_s(Z)$ as $n \to \infty$ and, that $(S_n - I)S_m = S_{[n]}(I - S_{[m]}) \to 0$ in $\mathcal{L}_b(Z)$ as $n \to \infty$ (for each fixed $m \in \mathbb{N}$). Hence, applying Lemma 2.4 to $\{S_n\}_{n=1}^\infty \subseteq \mathcal{L}(Z)$ we conclude that

$$\lim_{n\to\infty} (S_{[n]})^2 = 0 \ \text{ in } \mathcal{L}_b(Z). \tag{4.16}$$

Combining (4.15) and (4.16) it follows that $(T_{[n]})^2 \to P$ in $\mathcal{L}_b(X)$ as $n \to \infty$.

(ii) Suppose now that X is a quojection Fréchet space with the lc–topology generated by the increasing sequence of continuous seminorms $\{q_j\}_{j=1}^\infty$.

In the case that T is power bounded, there exist constants $c_j \geq 1$, for each $j \in \mathbb{N}$, such that

$$q_j(T^n x) \leq c_j q_j(x), \quad x \in X,\, n \in \mathbb{N}.$$

It follows that

$$q_j(T_{[n]})x \leq \frac{1}{n}\sum_{m=1}^{n} q_j(T^m x) \leq c_j q_j(x), \quad x \in X,\ j, n \in \mathbb{N}.$$

So, still in the notation of the proof of part (i), the result follows by recalling (4.15) and applying Lemma 2.4 to the sequence $\{S_n\}_{n=1}^\infty \subseteq \mathcal{L}(Z)$.

Otherwise, as the sequence $\{\frac{1}{n}T^n|_Z\}_{n=1}^\infty \subseteq \mathcal{L}(Z)$ is equicontinuous and the GDP–space Z, see [3, Proposition 3.1(i)], is a quojection (being a quotient space of the quojection X), we proceed as in the proof of (iii)⇒(ii) in Theorem 4.1. So, we first construct a sequence $\{Z_j\}_{j=1}^\infty$ of Banach spaces and a sequence $\{Q_j\}_{j=1}^\infty$, with $Q_j \colon Z \to Z_j$, of continuous linear surjective operators such that $Z = \mathrm{proj}_j(Z_j, Q_{j,j+1})$, where $Q_{j.j+1} \in \mathcal{L}(Z_{j+1}, Z_j)$ satisfies $Q_{j,j+1}Q_{j+1} = Q_j$. Then, for each $j \in \mathbb{N}$, we define $T_j \in \mathcal{L}(Z_j)$ satisfying the following properties:

$$T_jQ_j = Q_jT, \quad \text{(which implies } (T_j)_{[n]}Q_j = Q_jT_{[n]},\ n \in \mathbb{N}), \tag{4.17}$$

$$\lim_{n\to\infty} \frac{1}{n} T_j^n = 0 \ \text{ in } \mathcal{L}_b(Z_j), \tag{4.18}$$

$$\lim_{n\to\infty} (I - (T_j)_{[m]})(T_j)_{[n]} = 0 \ \text{ in } \mathcal{L}_b(Z_j),\ \forall m \in \mathbb{N}. \tag{4.19}$$

$$\lim_{n\to\infty} ((T_j)_{[n]})^2 = 0 \ \text{ in } \mathcal{L}_b(Z_j). \tag{4.20}$$

We point out that (4.20) follows from (4.16) and (4.17) because of $S = T|_Z$. Now, the facts that $((T_j)_{[n]})^2 \to 0$ in $\mathcal{L}_b(Z_j)$ with Z_j is Banach space imply that $I - ((T_j)_{[m]})^2$ is invertible in $\mathcal{L}(Z_j)$ for some $m \in \mathbb{N}$. Then the identity $(I - (T_j)_{[m]})(I + (T_j)_{[m]})(I - ((T_j)_{[m]})^2)^{-1} = I$ in $\mathcal{L}(Z_j)$ shows that $I - (T_j)_{[m]}$ is invertible in $\mathcal{L}(Z_j)$. But, (4.19) implies that

$$\lim_{n\to\infty} (T_j)_{[n]} = \lim_{n\to\infty} (I - (T_j)_{[m]})^{-1}(I - (T_j)_{[m]})(T_j)_{[n]} = 0$$

in $\mathcal{L}_b(Z_j)$. It follows that $S_{[n]} = T_{[n]}|_Z \to 0$ in $\mathcal{L}_b(Z)$ as $n \to \infty$. Indeed, fix any $j \in \mathbb{N}$ and $B \in \mathcal{B}(Z)$. Then, again in the notation of (iii)⇒(ii) in Theorem 4.1, we have

$$\begin{aligned}
\sup_{z\in B} q_j(S_{[n]}z) &= \sup_{z\in B} q_j(T_{[n]}z) = \sup_{z\in B} (\hat{q}_j)_j(Q_jT_{[n]}z) \\
&= \sup_{z\in B} (\hat{q}_j)_j((T_j)_{[n]}Q_jz) = \sup_{\hat{z}\in Q_j(B)} (\hat{q}_j)_j((T_j)_{[n]}\hat{z}) \\
&\le \sup_{\hat{z}\in Q_j(B)} (\hat{q}_j)_{k(j)}((T_j)_{[n]}\hat{z}), \quad n \in \mathbb{N},
\end{aligned}$$

with $\sup_{\hat{z}\in Q_j(B)} (\hat{q}_j)_{k(j)}((T_j)_{[n]}\hat{z}) \to 0$ as $n \to \infty$ because $(T_j)_{[n]} \to 0$ in $\mathcal{L}_b(Z_j)$. Finally, because of (4.15), we conclude that $T_{[n]} \to P$ in $\mathcal{L}_b(X)$ as $n \to \infty$. □

Corollary 4.4. *Let X be a quojection GDP–Fréchet space. If X is mean ergodic, then X is uniformly mean ergodic.*

Proof. Let $T \in \mathcal{L}(X)$ be power bounded. According to [2, Remark 2.6(i)], given any $q \in \Gamma_X$ there is $p \in \Gamma_X$ such that

$$q(T^n x) \le p(x), \quad x \in X,\ n \in \mathbb{N}.$$

So, for any $B \in \mathcal{B}(X)$, we have $q_B(T^n) \le \sup_{x\in B} p(x) < \infty$ for all $n \in \mathbb{N}$ and hence, $\lim_{n\to\infty} \frac{1}{n}T^n = 0$ in $\mathcal{L}_b(X)$. Since T is mean ergodic (as X is), Theorem 4.3(ii) implies that T is uniformly mean ergodic. □

Remark 4.5. (i) The argument used in the proof of Theorem 4.3(ii) to show that the sequence $\{(T_j)_{[n]}\}_{n=1}^\infty$ tends to 0 in the Banach space $\mathcal{L}_b(Z_j)$ is essentially the idea behind the proof given in [26, Theorem 8], [27, Theorem 5].

Unfortunately, if X is non-normable, then this strategy is not applicable. The problem lies in the fact that if a sequence $\{R_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ satisfies $\lim_{n\to\infty} R_n = 0$ in $\mathcal{L}_b(X)$, then $I - R_n$ may fail to be invertible for *every* $n \in \mathbb{N}$. Indeed, consider $X = \omega$ and the projection operators $R_n x := (0, \ldots, 0, x_n, x_{n+1}, \ldots)$, for $x \in \omega$ and $n \in \mathbb{N}$. Clearly $R_n \to 0$ in $\mathcal{L}_s(\omega)$ and hence, also $R_n \to 0$ in $\mathcal{L}_b(\omega)$ because ω is a Montel space. But, for every $n \in \mathbb{N}$, the operator $I - R_n$ is finite rank and hence, is surely not invertible.

(ii) There is another class of Fréchet spaces for which the conclusion of Theorem 4.3(ii) holds. As already noted, any Montel Fréchet space X is necessarily a GDP–space. An examination of the proof of Proposition 2.8 in [2] then shows that every mean ergodic operator $T \in \mathcal{L}(X)$ is necessarily uniformly mean ergodic in X (even without the hypothesis that $\lim_{n\to\infty} \frac{1}{n}T^n = 0$ in $\mathcal{L}_b(X)$).

A classical example of a quojection GDP–Fréchet space which is not Montel is $L^\infty_{loc}(\Omega)$, with Ω any open subset of $\mathbb{R}^N$. It was noted in Section 2 that $L^\infty_{loc}(\Omega)$ is a quojection. Since $L^\infty_{loc}(\Omega)$ is isomorphic to a countable product of Banach spaces, each one isomorphic to the GDP–Banach space $L^\infty([0,1])$, it follows that $L^\infty_{loc}(\Omega)$ is a GDP–space, see [3, Proposition 3.1(ii)]. But, since $L^\infty_{loc}(\Omega)$ contains a complemented copy of the Banach space $L^\infty([0,1])$, it cannot be a Montel space. By the same argument, any countably infinite product of infinite dimensional GDP–Banach spaces is a (non–normable) quojection GDP–Fréchet space which is not Montel. On the other hand, every infinite dimensional Montel Fréchet space which admits a continuous norm (see Example 3.1 for such spaces) cannot be a quojection (see the discussion prior to Lemma 2.3).

It is worth pointing out that the implication (iii)⇒(i) in Theorem 4.1, valid whenever X is a quojection, *fails* to hold in Montel Fréchet spaces; see [2, Example 2.17].

Question 2. Let X be a Fréchet GDP–space which is non–Montel and not a quojection (examples of such spaces of the kind $\lambda_\infty(A)$ can be found in [11]). Let $T \in \mathcal{L}(X)$ be mean ergodic with τ_b-$\lim_{n\to\infty} \frac{1}{n}T^n = 0$. Is T necessarily uniformly mean ergodic?

We end with an application of Theorem 4.3.

Corollary 4.6. *Let X be a prequojection Fréchet space such that X''_β is a GDP–space. Suppose that $T \in \mathcal{L}(X)$ satisfies $\lim_{n\to\infty} \frac{1}{n}T^n = 0$ in $\mathcal{L}_b(X)$ and that T^{tt} is mean ergodic in X''_β. Then T is uniformly mean ergodic.*

Proof. Observe that $(T_{[n]})^{tt} = (T^{tt})_{[n]}$ for all $n \in \mathbb{N}$. By assumption there is a projection $Q \in \mathcal{L}(X''_\beta)$ such that $(T^{tt})_{[n]} \to Q$ in $\mathcal{L}_s(X''_\beta)$ as $n \to \infty$. It follows that $Q(X) \subseteq X$ and $T_{[n]} \to P$ in $\mathcal{L}_s(X)$ as $n \to \infty$ with $P = Q|_X \in \mathcal{L}(X)$.

Since $\lim_{n\to\infty} \frac{1}{n}T^n = 0$ in $\mathcal{L}_b(X)$ and $(T^{tt})^n = (T^n)^{tt}$ for all $n \in \mathbb{N}$, we have that $\lim_{n\to\infty} \frac{1}{n}(T^{tt})^n = 0$ $\mathcal{L}_b(X''_\beta)$. So, we can apply Theorem 4.3 to T^{tt} to conclude that $(T^{tt})_{[n]} \to Q$ in $\mathcal{L}_b(X''_\beta)$ as $n \to \infty$. This implies that $T_{[n]} \to P$ in $\mathcal{L}_b(X)$ and hence, T is uniformly mean ergodic. □

We note that the Banach space c_0 itself is not a GDP–space, but its strong bidual $c_0'' = \ell^\infty$ is. For a non–normable example, consider the product space $X = \prod_{n=1}^\infty X(n)$, where $X(n) = c_0$ for each $n \in \mathbb{N}$. Then X is a quojection and hence, also a prequojection. Since X contains a complemented copy of c_0, it cannot be a GDP–space, see [3, Proposition 3.1(i)]. However, its strong bidual $X''_\beta = \prod_{n=1}^\infty X(n)'' = \prod_{n=1}^\infty \ell^\infty$ (see [22, p.287]) is a GDP–space [3, Proposition 3.1(ii)].

For an example of a non–trivial prequojection which itself is not a GDP–space, but with strong bidual a GDP–space, we refer to [1]. Indeed, the example of the prequojection constructed in [1] contains a complemented copy of c_0 and its strong bidual is isomorphic to $\prod_{n=1}^\infty \ell^\infty$.

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