NON COMPLETE MACKEY TOPOLOGIES ON BANACH SPACES

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ABSTRACT. Answering in the negative a question of W. Arendt and M. Kunze, we construct Banach spaces X and a norm closed weak*-dense subspace Y of the dual space X' of X such that the space X endowed with the Mackey $\mu(X,Y)$ of the dual pair $\langle X,Y\rangle$ is not complete.

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The following problem aroused in a natural way in connection with the study of Pettis integrability with respect to norming subspaces developed in his Ph.D. thesis by Markus Kunze [5] and was asked to the authors by Kunze himself and his thesis advisor W. Arendt.

Problem. *Suppose that $(X, \|\cdot\|)$ is a Banach space and Y is a subspace of its topological dual X' which is norm closed and weak*-dense. Is there a complete topology of the dual pair $\langle X,Y\rangle$ in X?*

We use freely the notation for locally convex spaces (shortly, lcs) as in [4, 6, 7]. In particular, we denote, respectively, by $\sigma(X,Y)$ and $\mu(X,Y)$ the weak and the Mackey topology in X associated to the dual pair $\langle X,Y\rangle$. For a Banach space X with topological dual X', the weak*-topology is $\sigma(X',X)$. By the Bourbaki Robertson lemma [4, §18.4.4], there is a complete topology in X of the dual pair $\langle X,Y\rangle$ if and only if the space $(X,\mu(X,Y))$ is complete. Therefore, the original question is equivalent to the following

Problem A: *Let $(X, \|\cdot\|)$ be a Banach space. Is $(X,\mu(X,Y))$ complete for every norm closed weak*-dense subspace Y of the dual space X'?*

Let $(X, \|\cdot\|)$ be a normed space. A subspace Y of X' is said to be *norming* if the function p of X given by $p(x) = \sup\{|x'(x)| : x' \in Y \cap B_{X'}\}$ is a norm equivalent to $\|\cdot\|$. Notice that Problem A is not affected by changing the given norm of X by any equivalent one. Thus if we want to study Problem A for some norming $Y \subset X'$ we can and will always assume that Y is indeed 1-norming, *i.e.*, $\|x\| = \sup\{|x'(x)| : x' \in Y \cap B_{X'}\}$.

We start by noting that, in the in the conditions of Problem A, if $(X,\mu(X,Y))$ is quasi-complete (in particular complete) then Krein-Smulyan's theorem, see [4, §24.5.(4)], implies that for every $\sigma(X,Y)$-compact subset H of X its $\sigma(X,Y)$-closed absolutely convex hull $M := \overline{\operatorname{aco} H}^{\sigma(X,Y)}$ is $\sigma(X,Y)$-compact. There are several papers dealing with the validity of Krein-Smulyan theorem for topologies weaker than the weak topology; see for instance [1, 2] where it is proved that for every Banach space X not containing $\ell^1\big([0,1]\big)$ and every 1-norming subspace $Y \subset X'$, if H is a norm bounded $\sigma(X,Y)$-compact subset of X then $\overline{\text{aco}H}^{\sigma(X,Y)}$ is $\sigma(X,Y)$-compact. It was proved in [3] that the hypothesis $\ell^1\big([0,1]\big) \not\subset X$ is needed in the latter.

We start with the following very useful observation:

Proposition 1. *Let $(X, \|\cdot\|)$ be a Banach space and let Y be a 1-norming subspace of X'. If $(X, \mu(X,Y))$ is quasi-complete, then every $\sigma(X,Y)$ compact of X is norm bounded.*

Proof. Let $H \subset X$ be $\sigma(X,Y)$-compact. As noted before, Krein-Smulyan's theorem, [4, §24.5.(4)], implies that the $\sigma(X,Y)$-closed absolutely convex hull $M := \overline{\text{aco}H}^{\sigma(X,Y)}$ is $\sigma(X,Y)$-compact. Therefore, M is an absolutely convex, bounded and complete subset of the locally convex space $(X, \sigma(X,Y))$. Now we can apply [4, §20.11.(4)] to obtain that M is a Banach disc, *i.e.*, $X_M := \bigcup_{n\in\mathbb{N}} nM$ is a Banach space with the norm

$$\|x\|_M := \inf\{\lambda \geq 0 : x \in \lambda M\}, x \in X_M.$$

Since M is bounded in $(X, \sigma(X,Y))$, the inclusion $J : X_M \to (X, \sigma(X,Y))$ is continuous, hence $J : X_M \to (X, \|\cdot\|)$ has closed graph, hence it is continuous by the closed graph theorem. In particular, the image of the closed unit ball M in X_M is bounded in $(X, \|\cdot\|)$, and the proof is complete. □

As an immediate consequence of the above we have the following:

Example A. *Let $X = C\big([0,1]\big)$ be with its sup norm and take*

$$Y := \text{span}\,\{\delta_x : x \in [0,1]\} \subset X'.$$

Then $(X, \mu(X,Y))$ is not quasi-complete.

Proof. Notice that $\sigma(X,Y)$ coincides with the topology τ_p of pointwise convergence on $C\big([0,1]\big)$. Since there are sequences τ_p-convergent to zero which are not norm bounded, $(X, \mu(X,Y))$ cannot be quasi-complete when bearing in mind Proposition 1. □

The subspace Y of X' in Example A is weak*-dense in X' but not closed. It is in fact easy to give even simpler examples: Take $X = c_0$, $Y = \varphi$, the space of sequences with finitely many non-zero coordinates, which is weak*-dense in $X' = \ell_1$. In this case $\mu(X,Y) = \sigma(X,Y)$, since every absolutely convex $\sigma(Y,X)$-compact subset of Y is finite dimensional by Baire category theorem. In this case $(X, \sigma(X,Y))$ is even not sequentially complete.

The following example, taken from Lemma 11 in [3], provides the negative solution to Problem A.

Example B. *Take $X = \big(\ell^1\big([0,1]\big), \|\cdot\|_1\big)$ and consider the space $Y = C\big([0,1]\big)$ of continuous functions on $[0,1]$ as a norming subspace of the dual $X' = \ell^\infty\big([0,1]\big)$. Then $(X, \mu(X,Y))$ is not quasi-complete.*

Proof. Let $H := \{e_x : x \in [0,1]\}$ be the canonical basis of $\ell^1\big([0,1]\big)$. The set H is clearly $\sigma(X,Y)$-compact but we will prove that $\overline{\text{aco}H}^{\sigma(X,Y)}$ is not $\sigma(X,Y)$-compact, and therefore $(X,\mu(X,Y))$ cannot be quasi-complete. Indeed, we proceed by contradiction and assume that $W := \overline{\text{aco}H}^{\sigma(X,Y)}$ is $\sigma(X,Y)$-compact. We write $M\big([0,1]\big) = \big(C\big([0,1]\big), \|\cdot\|_\infty\big)'$ to denote the space of Radon measures in $[0,1]$ endowed with its variation norm. The map

$$\phi : X \to M\big([0,1]\big)$$

given by $\phi\big((\xi_x)_{x\in[0,1]}\big) = \sum_{x\in[0,1]} \xi_x \delta_x$ is $\sigma(X,Y)$-w*-continuous. We notice that:

(1) $\phi(W) \subset \phi(\ell^1\big([0,1]\big))$;
(2) $\phi(W)$ is an absolutely convex w*-compact subset of $M\big([0,1]\big)$;
(3) $\{\delta_x : x \in [0,1]\} \subset \phi(W)$.

From the above we obtain that

$$B_{M\big([0,1]\big)} = \overline{\text{aco}\{\delta_x : x \in [0,1]\}}^{w^*} \subset \phi(W) \subset \phi(\ell^1\big([0,1]\big),$$

which is a contradiction because there are Radon measures on $[0,1]$ which are not of the form $\sum_{x\in[0,1]} \xi_x \delta_x$. The proof is complete. □

Proposition 2. *If X is a Banach space such that $\ell^1\big([0,1]\big) \subset X$, then there is a subspace $Y \subset X'$ norm closed and norming such that $(X,\mu(X,Y))$ is not quasi-complete.*

Proof. In the proof of [3, Proposition 3] it is constructed a norming subspace $E \subset X'$ and $H \subset X$ norm bounded $\sigma(X,E)$-compact such that $\overline{\text{aco}H}^{\sigma(X,E)}$ is not $\sigma(X,E)$-compact. If we take $Y = \overline{E} \subset X'$, norm closure, then $\sigma(X,E)$ and $\sigma(X,Y)$ coincide on norm bounded sets of X. Thus $H \subset X$ is $\sigma(X,Y)$-compact with $\overline{\text{aco}H}^{\sigma(X,E)}$ not $\sigma(X,E)$-compact and therefore $(X,\mu(X,Y))$ cannot be quasi-complete. □

We conclude this note with a few comments about the relation of the questions considered here with Mazur property. We say that a lcs $(E,\mathfrak{T})$ is Mazur if every sequentially $\mathfrak{T}$-continuous form defined on E is $\mathfrak{T}$-continuous. We quote the following result:

Theorem 3. [7, Theorem 9.9.14] *Let $\langle X,Y\rangle$ be a dual pair. If $(X,\sigma(X,Y))$ is Mazur and $(X,\mu(X,Y))$ is complete, then $(Y,\mu(Y,X))$ is complete.*

Proposition 4. *Let X be a Banach space, $Y \subset X'$ proper subspace and w*-dense. Assume that:*

(1) *the norm bounded $\sigma(X,Y)$-compact subsets of X are weakly compact.*
(2) *$(X,\sigma(X,Y)$ is Mazur.*

Then $(X,\mu(X,Y))$ is not complete.

Proof. Assume that $(X,\mu(X,Y))$ is complete. Then Proposition 1 implies that every $\sigma(X,Y)$-compact subset of X is norm bounded. Therefore the family of $\sigma(X,Y)$-compact subset coincide with the family of weakly compact sets. So the Mackey topology $\mu(Y,X)$ in Y associated to the pair $\langle X,Y\rangle$ is the topology induced in Y by the Mackey topology $\mu(X',X)$ in X' associated to the dual pair

$\langle X, X' \rangle$. If we use now Theorem 3 we obtain that Y is $\mu(Y, X)$ is complete, what implies that $Y \subset X'$ is $\mu(X', X)$ closed. Thus:

$$Y = \overline{Y}^{\mu(X',X)} = \overline{Y}^{w^*} = X',$$

that is a contradiction with the fact that Y is a proper subspace of X'. □

We observe that hypothesis (1) in the above Proposition is satisfied for Banach spaces without copies of $\ell^1([0,1])$ whenever Y contains a boundary for the norm, see [1, 2].

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REFERENCES

[1] B. Cascales, G. Manjabacas, G. vera, A Krein-Smulian type result in Banach spaces, *Quart. J. Math. Oxford Ser.* (2) 48(190) (1997), 161-167.
[2] B. Cascales, R. Shvydkoy, On the Krein-Smulian theorem for weaker topologies, *Illinois J. Math.* 47(4) (2003), 957-976.
[3] A.S. Granero, M. Sánchez, The class of universally Krein-Smulian spaces, *Bull. London Math. Soc.* 39(4) (2007), 529-540.
[4] G. Köthe, *Topological Vector Spaces I and II*, Springer Verlag, Berlin 1969 and 1979.
[5] M.C. Kunze, Semigropus on norming dual pairs and transition operators for Markov processes, Ph.D. Thesis, Universität Ulm, 2008.
[6] Meise, R., Vogt, D., *Introduction to Functional Analysis*, Clarendon, Oxford, 1997.
[7] A. Wilanski, *Modern Methods in Topological Vector Spaces*, McGraw-Hill International Book Co. New York, 1978.

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