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Spaces of Moscatelli type. A survey

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Abstract. We survey different developments on Fréchet, (LB)-spaces and (LF)-spaces of Moscatelli type, their properties, their applications to find counterexamples to different open problems, relevant examples of smooth function spaces in this frame and related results.

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Introduction

The investigation of the so-called Fréchet and (LB)-spaces of Moscatelli type was started by Susanne Dierolf and the first author of this survey in the late 1980's in [13, 14]. It was continued later by several authors, as will be seen below. The motivation to call these spaces "of Moscatelli type" came from important works of Bruno Moscatelli mainly related to the approximation theory and Schauder bases on Banach spaces, Fréchet spaces and more general locally convex spaces (l.c. spaces for brevity).

A sequence $(x_n)_n$ in a l.c. space E is called a *basis* if every $x \in E$ determines a unique sequence $(a_n)_n$ in the scalar field such that the series $\sum a_n x_n$ converges to x in the topology of E. Any l.c. space with a basis is separable. The basis $(x_n)_n$ is called a *Schauder basis* of E if its coefficient functionals $u_n(x) := a_n$, $n \in \mathbb{N}$, are continuous. Every basis in a Fréchet space is a Schauder basis. From this point on we will always write 'basis' and mean 'Schauder basis'.

The problem whether every separable Banach space has a basis appeared in 1931 for the first time in the Polish edition of Banach's book [9, Chapter 7, section 3]. It was clear to Banach, Mazur and Schauder that this question was related to an approximation problem mentioned by Mazur in the "Scottish Book" in 1936. This approximation problem was equivalent to the question

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whether every l.c. space has the approximation property, a question which was analyzed deeply by Grothendieck in his "thèse" [36]. A locally convex space E has the *approximation property (a.p.)* if the identity of E is the limit of a net of finite rank operators for the topology of uniform convergence on the absolutely convex compact subsets of E. If the net is equicontinuous, it is said that E has the *bounded a.p.*

Banach's problem was solved in the negative by Enflo in [31]: Each space ℓ_p ($1 \leq p \leq \infty$, $p \neq 2$), as well as c_0, has a closed subspace without the a.p. The case ℓ_p, $1 \leq p < 2$, is due to Szankowski in 1978. For a detailed account on approximation properties in Banach spaces see Casazza [23].

Grothendieck [36] proved that every nuclear space has the approximation property. In 1960, Dynin and Mitjagin proved that every equicontinuous basis in a nuclear space is absolute. For a long time it was an open problem whether there exists a nuclear Fréchet space without a basis. The first example of such a space was given by Mitjagin and Zobin; we refer the reader to [37]. It was an open problem of Grothendieck since 1955 if every nuclear Fréchet space had the bounded approximation property. This was solved in the negative by Dubinsky in 1981; the example was simplified considerably by Vogt in [55]. All the known examples are based in a celebrated observation of Pelczyński that a Fréchet space with a continuous norm and the bounded approximation property must be countably normable, i.e. the intersection of a decreasing sequence of Banach spaces. Moscatelli presented a new construction in [48] in 1991. Very recently, Vogt [58] gave an easy and transparent example of a nuclear Fréchet space failing the bounded approximation property and consisting of C^∞ functions on a subset of $\mathbb{R}^3$. A nuclear Fréchet space of C^∞ functions which has no basis had been constructed also by Vogt in [57].

A different way to construct examples of nuclear Fréchet spaces without basis had been presented by Moscatelli [47] in 1980. In fact, in this paper he answered negatively the following question posed by Dubinski: Is every Fréchet space without continuous norm isomorphic to a countable product of Fréchet spaces with a continuous norm? His approach is based on the following result due to Floret and Moscatelli [33] (see also [50, 8.4.38]): *Every Fréchet space with an unconditional basis is topologically isomorphic to a countable product of Fréchet spaces with a continuous norm and unconditional basis.* Moscatelli's idea is to use a "shifting" device which is implicit in the example of Grothendieck and Köthe of a non-distinguished echelon space. Moscatelli also utilized his method to construct a Fréchet space which is the projective limit of a sequence of Banach spaces with surjective linking maps and which is not isomorphic to a complemented subspace of a countable product of Banach spaces. A Fréchet space E is called a *quojection* if it is the projective limit of a sequence of

Banach spaces with surjective linking maps or, equivalently, if every quotient with a continuous norm is a Banach space for the quotient topology. Several authors proved that every quojection is a quotient of a countable product of Banach spaces. We refer the reader to the excellent survey by Metafune and Moscatelli [46] on quojections.

Constructions of Moscatelli type with a shifting device have been used since the mid 1980's several times to construct various counterexamples. For example, Dierolf and Moscatelli [27] used this construction to find Fréchet spaces with a continuous norm such that its bidual does not have a continuous norm. Examples of this type in the context of Köthe echelon spaces were obtained by Terzioglu and Vogt [54]. Moscatelli type constructions also played a role in Taskinen's counterexamples [52] to the problem of topologies of Grothendieck [36]. Bonet and Dierolf investigated this type of constructions in [13, 14]. Albanese and Moscatelli studied in a series of articles starting in 1996 the topological structure and the isomorphic classification of Fréchet spaces of smooth functions which appear as an intersection of two spaces and turned out to be of Moscatelli type in many cases. See Section 2 below.

It is well-known that every non-normable Fréchet space admits a quotient isomophic to ω, and that it has a subspace topologically isomorphic to ω if and only if it does not admit a continuous norm; e.g. see [50]. In 1961, Bessaga, Pełczyński and Rolewicz showed that *a Fréchet space contains a subspace which is topologically isomorphic to an infinite dimensional nuclear Fréchet space with basis and a continuous norm if and only if it is not isomorphic to the product of a Banach space and* ω. As a consequence of the results mentioned above, *every non-normable Fréchet space always contains a subspace which is isomorphic to a nuclear Köthe echelon space.* The situation for quotients is more complicated. Bellenot and Dubinsky in the separable case in 1982, and Önal and Terzioğlu in general in 1990, proved the following result: *A Fréchet space E does not have a quotient which is nuclear with a basis and a continuous norm if and only if the bidual E'' of E is a quojection.* Fréchet spaces satisfying this condition were introduced with another definition. Vogt showed that the original definition was equivalent to the one mentioned above. Dierolf, Moscatelli, Behrends and Harmand constructed Fréchet spaces E such that E'' is a quojection, but E is not a quojection. Fréchet spaces E such that E'' is a quojection are called *prequojections.* More information about prequojections and the results just mentioned can be seen in [46].

In Section 1 we recall the definition of Fréchet and (LB)-spaces of Moscatelli type, and collect the main results obtained by Susanne Dierolf and the present authors. Section 2 presents some results by Albanese and Moscatelli on Fréchet spaces of Moscatelly type consisting of smooth functions. (LF)-spaces are discussed in Section 3. A sample of further related results is contained in the final Section 4. Undefined notation about functional analysis and locally convex spaces can be seen in the books [38], [37], [41], [50] and the survey paper [11].

1 Fréchet spaces of Moscatelli type

Definition 1. A *normal Banach sequence space* is a Banach space $(L, \| \|)$ satisfying
(α) $\varphi \subset L \subset \omega$ with continuous inclusion,
(β) For all $a = (a_k)_k \in L$ and all $b = (b_k)_k \in \omega$ with $|b_k| \leq |a_k|$, $k \in \mathbb{N}$, we have $b \in L$ and $\| b \| \leq \| a \|$.

Let P_n denote the projection onto the first n-coordinates. We assume that for all $a \in L$
(γ) $\| a \| = \lim_n \| P_n(a) \|$.

We also consider the following properties on $(L, \| \|)$
(δ) If $a \in \omega$ and $\sup_n \| P_n(a) \| < \infty$, then $a \in L$ and $\| a \| = \lim_n \| P_n(a) \|$.
(ϵ) $\lim_n \| a - P_n(a) \| = 0$ for each $a \in L$.

If the sequence space L satisfies (ε), its dual can be identified with its α-dual (cf. [38]), hence it is also a normal Banach sequence space.

Typical examples of normal Banach sequence spaces are $(\ell_p, \| \|_p)$, $1 \leq p \leq \infty$ or $(c_0, \| \|)$.

Definition 2. Given a sequence of Banach spaces $(X_k, r_k)_{k\in\mathbb{N}}$ and a normal Banach sequence space $(L, \| \|)$, we put

$$L((X_k, r_k)_{k\in\mathbb{N}}) = \{ (x_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} X_k \mid (r_k(x_k))_{k\in\mathbb{N}} \in L \}$$

endowed with the norm $x \to \| (r_k(x_k))_{k\in\mathbb{N}} \|$. It is easy to check that $L((X_k, r_k)_{k\in\mathbb{N}})$ is a Banach space.

Definition 3. Given a normal Banach sequence space $(L, \| \|)$ and two sequences of Banach spaces $(X_k, r_k)_{k\in\mathbb{N}}$ and $(Y_k, s_k)_{k\in\mathbb{N}}$ with unit balls A_k and B_k respectively, and for every $k \in \mathbb{N}$ let $f_k : Y_k \to X_k$ be a linear map such that $f_k(B_k) \subset A_k$ and having dense range. For every $n \in \mathbb{N}$ we denote by $F_n = L((Y_k)_{k<n}, (X_k)_{k\geq n})$ and by $g_n : F_{n+1} \to F_n$ the norm decreasing linear map given by $g_n((z_k)_k) = ((z_k)_{k<n}, f_n(z_n), (z_k)_{k>n})$.

The *Fréchet space of Moscatelli type with respect to* $(L, \| \|)$, $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$ and $f_k : Y_k \to X_k$ is the projective limit of the projective sequence of Banach spaces $(F_n)_{n\in\mathbb{N}}$ with linking maps $(g_n)_{n\in\mathbb{N}}$.

Remark 1. The Fréchet space of Moscatelli type defined above coincides algebraically with $F = \{ (y_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} Y_k \mid (f_k(y_k))_{k\in\mathbb{N}} \in L((X_k, r_k)_{k\in\mathbb{N}}) \}$

and it has the initial topology with respect to the inclusion $j : F \to \prod_{k\in\mathbb{N}} Y_k$ and the map $f : F \to L((X_k, r_k)_{k\in\mathbb{N}})$, defined as $f((y_k)_{k\in\mathbb{N}}) = (f_k(x_k))_{k\in\mathbb{N}}$.

In case $X_k = X$ and $Y_k = Y$ for all $k \in \mathbb{N}$ the assumption on the unit balls simply means that there is a continuous linear map $f : Y \to X$. If in addition Y is continuously included in X, the Fréchet space of Moscatelli type is $Y^{\mathbb{N}} \bigcap L(X)$ endowed with the intersection topology.

Fréchet spaces of Moscatelli type are Montel only when they reduce to a finite dimensional space or to the space of all scalar sequences.

Proposition 1. *The Fréchet space of Moscatelli type with respect to* $(L, \|\,\|)$, $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$ *and* $f_k : Y_k \to X_k$ *is Montel if and only if each* Y_k *is finite dimensional and there exists* k_0 *such that* $X_k = 0$ *for all* $k \geq k_0$.

Albanese [1] proved even more.

Proposition 2. *Complemented Montel subspaces of a Fréchet space* F *of Moscatelli type are isomorphic to* ω *or finite dimensional.*

To study other relevant properties of Fréchet spaces in this frame we need to investigate the structure of duals. To this end we introduce the (LB)-spaces of Moscatelli type.

Definition 4. Given a normal Banach sequence space $(L, \|\,\|)$ and two sequences of Banach spaces $(X_k, r_k)_{k\in\mathbb{N}}$ and $(Y_k, s_k)_{k\in\mathbb{N}}$ such that, for every $k \in \mathbb{N}$, Y_k is a subspace of X_k and $s_k \geq r_k|_{Y_k}$. We set, for every $n \in \mathbb{N}$, $E_n = L((X_k)_{k<n}, (Y_k)_{k\geq n})$, obtaining this way an increasing sequence of Banach spaces. The *(LB)-space of Moscatelli type* with respect to $(L, \|\,\|)$, $(X_k, r_k)_{k\in\mathbb{N}}$ and $(Y_k, s_k)_{k\in\mathbb{N}}$ is the inductive limit $E = \mathrm{ind}_n E_n$.

In fact (LB)-spaces of Moscatelli type were studied before Fréchet spaces in connection with complete and regular (LB)-spaces. The last question of Grothendieck on functional analysis which still remains open is whether every regular (LF)-space is complete. It is open even for (LB)-spaces. A positive solution to this problem would imply that the completion of every (LB)-space is also an (LB)-space. Since Bonet and Dierolf [14] showed that each regular (LB)-space of Moscatelli type is complete and Dierolf and Kuß [30] proved that the completion of an (LB)-space of Moscatelli type is again an (LB)-space, no counterexample to Grothendieck's problem can be constructed with this shifting device.

Theorem 1. *Let* $(L, \|\,\|)$ *be a normal Banach sequence space such that* L *satisfies property* (ε) *and let* F *be the Fréchet space of Moscatelli type with respect to* $(L, \|\,\|)$, $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$ *and* $f_k : Y_k \to X_k$. *Then the inductive dual* F'_i *of* F *coincides algebraically and topologically with the (LB)-space of Moscatelli type with respect to the duals.*

Definition 5. A locally convex space F is *distinguished* if its strong dual F'_b is barrelled. As Grothendieck proved, a metrizable space is distinguished if and only if F'_b is bornological.

Distinguished Fréchet spaces have a good behavior concerning the duality between (reduced) projective limits of Banach spaces and inductive limits of Banach spaces. In fact, if F is the reduced projective limit of a sequence $(F_n)_{n\in\mathbb{N}}$ of Banach spaces, F is distinguished if and only if the strong dual F'_b coincides with the inductive dual $F'_i := \mathrm{ind}_n F'_n$.

Proposition 3. *Let F be the Fréchet space of Moscatelli type with respect to $(L \,\|\,\|)$, $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$, and $f_k : Y_k \to X_k$.*

(i) *If L and its dual L' have property (ε), then F is always distinguished.*

(ii) *If $X_k = X$, $Y_k = Y$ and $f_k = f$ for all k and*

(ii1) L has property (ε) but L' does not, or

(ii2) L does not satisfy property (ε),

then F is distinguished if and only if f is surjective.

Parts (i) and (ii1) for $L = \ell_1$ follow from [13, 14] and use the representation of the duals given in Theorem 1. For $L = \ell_\infty$ the characterization was given in [18]. The picture was completed in [32].

The first example of a non-distinguished Fréchet space was given by Köthe and Grothendieck in 1954 [35]. It is the echelon space $\lambda_1(A)$ where $A = (\mathrm{a_n})$ where $a_n(i,j) = 1$ for $i \geq n$ and $a_n(i,j) = j$ when $i < n$. This space is of Moscatelli type. However these two classes, echelon spaces and those of Moscatelli type, are fairly different. The former contains nontrivial Schwartz (hence Montel and quasinormable) spaces but no non-normable quojections, whereas the latter contains only trivial Montel spaces and nontrivial quojections.

We refer the reader to [11] and [41] for the density condition, the property (Ω) and quasinormable Fréchet spaces.

Proposition 4. *Let F be the Fréchet space of Moscatelli type with respect to $(L, \|\,\|)$, $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$, and $f_k : Y_k \to X_k$. The following conditions are equivalent:*

(i) *F has the density condition.*

(ii) *F is quasinormable.*

(iii) *F has property (Ω).*

(iv) *F is a prequojection.*

(v) *F is a quojection.*

(vi) *There is m such that for all $k \geq m$ the map f_k is surjective.*

Corollary 1. *Let F be the Fréchet space of Moscatelli type with respect to $(L, \| \|)$, X, Y and $f : Y \to X$. If L has property (ε) but L' does not or L does not satisfy property (ε), F is distinguished if and only if it is a quojection.*

The following result is due to Moscatelli [47].

Theorem 2. *Let F be the Fréchet space of Moscatelli type with respect to $(L, \| \|)$, $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$, and $f_k : Y_k \to X_k$. Assume that each $f_k : Y_k \to X_k$ is surjective. The Fréchet space F is isomorphic to a countable product of Banach spaces if and only if the kernel of f_k is complemented in X_k for each $k \geq k_0$.*

Theorem 3. *Let X, Y be two reflexive Banach spaces such that Y is a proper subspace of X with continuous and dense inclusion. Let $F = Y^{\mathbb{N}} \bigcap c_0(X)$ be the corresponding Fréchet space of Moscatelli type, then F is distinguished but*

(i) *$\ell_1 \tilde{\bigotimes}_\pi F$ is not distinguished, and*

(ii) *F'' is not distinguished.*

Part (i) follows from the theorems stated above and [10]. Part (ii) uses that F'' is the Fréchet space of Moscatelli type with respect to ℓ_∞, X and Y. This result was presented in [18] and solved in the negative a problem of Grothendieck [35]. The first example of a distinguished Fréchet space such that $\ell_1 \tilde{\bigotimes}_\pi F$ is not distinguished was given in [26], thus solving in the negative a question by Grothendieck whether the complete projective tensor product of two distinguished Fréchet spaces is also distinguished.

Proposition 5.

(i) *A Fréchet space of Moscatelli type has a continuous norm if and only if there is m such that for all $k \geq m$ the map f_k is injective.*

(ii) *If L has (ε), F''_b has a continuous norm if and only if there is m such that for all $k \geq m$ the set $f^t_k(X'_K)$ is norm dense in Y'_k.*

Taking L with (ε), $Y_k = \ell_1$, $X_k = \ell_2$ and f_k the inclusion for all k in Proposition 5, the corresponding Fréchet space of Moscatelli type has a continuous norm but its bidual does not. Compare with [27].

Definition 6. (a) ([54]) A locally convex space Z is called *locally normable* if there is a continuous norm on Z such that on every bounded set in Z the norm topology and the space topology coincide.
(b) Let Z be a Fréchet space with fundamental sequence of seminorms $(\| \|_n)$. Z satisfies property (DN_φ) for some increasing continuous function $\varphi :]0, \infty[\to]0, \infty[$ with $\lim_{r\to\infty} \varphi(r) = \infty$ if
$\exists n_0 \ \forall m \ \exists n, C > 0 \ \forall z \in Z \ \forall r > 0$

$$\| z \|_m \leq C\varphi(r) \| z \|_{n_0} + \frac{1}{r} \| z \|_n .$$

If $\varphi(r) = r$, (DN_φ) is called (DN).

Proposition 6. *Let F be the Fréchet space of Moscatelli type with respect to $(L, \| \|)$, $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$, and $f_k : Y_k \to X_k$. The following conditions are equivalent:*

(i) *F is locally normable.*

(ii) *F has property* $(\mathrm{DN})_\varphi$ *for some φ.*

(iii) *F has property (DN).*

(iv) *F is a Banach space.*

(v) *There is m such that for all $k \geq m$ the map f_k is a topological isomorphism.*

2 Moscatelli spaces of smooth functions

In this section we see how several natural spaces of continuous or smooth functions can be represented as spaces of Moscatelli type. Several properties of these spaces are then easily obtained. These representations were obtained by Albanese, Metafune and Moscatelli in a series of papers. See for instance [4, 3, 5, 7]. In this section $H^{k,p}$ denotes the Sobolev space of measurable functions with distributional derivatives up to order k in L^p.

In [53] Taskinen proved that $\mathcal{C}(\mathbb{R}) \cap L^1(\mathbb{R})$ is not distinguished showing that this space has a complemented subspace isomorphic to a non-distinguished space of Moscatelli type. Because of the criterion in Proposition 3, it is easy to decide when a Fréchet space of Moscatelli type is distinguished. Up to that moment all examples of non-distinguished Fréchet spaces were artificial and constructed on purpose. A direct and easier proof of this result was given by Bonet and Taskinen in [21]. In that paper they showed that the spaces $\mathcal{C}^m(\Omega) \cap H^{k,p}(\Omega)$, $0 \leq k \leq m$, $1 < p < \infty$, are distinguished, by proving that they are isomorphic to complemented subspaces of distinguished Fréchet spaces of Moscatelli type. It was natural to conjecture that all spaces $C^m(\Omega) \cap H^{k,p}(\Omega)$ should themselves be of Moscatelli type.

Theorem 4. [5] For $1 \leq p \leq \infty$ and $m \in \mathbb{N}_0$, $\mathcal{C}^m(\mathbb{R}) \cap L^p(\mathbb{R})$ is isomorphic to $\mathcal{C}^m([0,1])^{\mathbb{N}} \cap \ell_p(L^p([0,1])$

Corollary 2.

(i) $\mathcal{C}^m(\mathbb{R}) \cap L^p(\mathbb{R})$ *is distinguished if and only if* $1 < p < \infty$.

(ii) $\mathcal{C}^m(\mathbb{R}) \cap L^p(\mathbb{R})$ *does not satisfy the density condition, does not have neither property (DN) nor property* (Ω) *and is not locally normable.*

The spaces $\mathcal{C}^m(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ for $1 \leq p \leq \infty$ and $0 \leq k \leq m$, endowed with their natural intersection topology, were represented as Fréchet spaces of Moscatelli type in [3]. The obtained representations complemented earlier ones by the same authors [4]. More precisely, if Q_N denotes the $N-$dimensional cube $[0,1]^N$, the space $\mathcal{C}^m(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ is isomorphic to the Fréchet space of Moscatelli type with respect to $L = \ell_p$, $Y = \mathcal{C}^m(Q_N)$ and $X = H^{k,p}(Q_N)$.

Theorem 5.

(i) $\mathcal{C}^m(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ *is isomorphic to* $\mathcal{C}^m(Q_N)^{\mathbb{N}} \cap \ell_p(H^{k,p}(Q_N))$.

(ii) *The spaces* $\mathcal{C}^m(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$, $1 \leq p < \infty$, *have basis.*

From statement (i) we conclude that these spaces are distinguished only for $1 < p < \infty$ and that they do not satisfy the density condition, they are not locally normable and do not have properties (DN) and (Ω) of Vogt. Moreover, the strong duals of $\mathcal{C}^m(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$, $1 \leq p < \infty$, are (LB)-spaces of Moscatelli type.

The Fréchet-Sobolev spaces $\mathcal{C}^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ are isomorphic to spaces belonging to a more general class of Fréchet spaces of Moscatelli type. See Section 4.

Theorem 6. [7] $\mathcal{C}^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$, $1 \leq p \leq \infty$ is isomorphic to $\mathcal{C}^\infty(Q_N)^{\mathbb{N}} \cap \ell_p(H^{k,p}(Q_N))$.

The topological properties of a more general class of Fréchet spaces of Moscatelli type were investigated in [28]. In particular, the first two statements in the next result, previously obtained by Bonet and Taskinen [21], also follow from the representation in the previous theorem and [28].

Proposition 7. [21], [7]

(i) The spaces $\mathcal{C}^\infty(\mathbb{R}^N) \cap H^{k,1}(\mathbb{R}^N)$ are not distinguished.

(ii) The spaces $\mathcal{C}^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$, $1 < p < \infty$, are distinguished but do not satisfy the density condition, hence they are not quasinormable.

(iii) $\mathcal{C}^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$, $1 \le p < \infty$, has a basis.

In [2] Albanese, Díaz and Metafune represented the spaces $H^k_{loc}(\mathbb{R}) \cap L^2(\mathbb{R})$ as Köthe spaces of Moscatelli type. More precisely, denote by $\lambda(n^k)$ the Köthe sequence space

$$\{ x = (x_{ij}) \mid p_k(x) =: \sum_{i=1}^{k} \sum_{j=1}^{\infty} j^{\,i} |x_{ij}|^2 + \sum_{i>k} \sum_{j=1}^{\infty} |x_{ij}|^2 < \infty, \text{ for all } m \in \mathbb{N} \},$$

which is the Fréchet space of Moscatelli type with respect to $L = \ell_2$, $Y_k = \ell_2(j^k)$ and $X_k = \ell_2$. This space is also the Köthe sequence space $\lambda_2(\mathbb{N}^2, A)$ where the Köthe matrix $A = (\tilde{a_k})$ is defined by

$$\tilde{a_k}(i,j) = \begin{cases} 1 & \text{if} \quad i > k,\ j \in \mathbb{N} \\ j^{\,i} & \text{if} \quad i \le k,\ j \in \mathbb{N} \end{cases}$$

Then,

Theorem 7. *For each* $k \in \mathbb{N}$

$$H^k_{loc}(\mathbb{R}) \cap L^2(\mathbb{R}) \simeq \lambda(n^k)$$

This representation permitted the authors of [2] to conclude that $H^k_{loc}(\mathbb{R}) \cap L^2(\mathbb{R})$ and $H^r_{loc}(\mathbb{R}) \cap L^2(\mathbb{R})$ are isomorphic only if $k = r$.

3 (LF)-spaces of Moscatelli type

We refer to Chapter 6 of the monograph of Wengenroth [60] and the survey of Vogt [56] for undefined terminology on (LF)-spaces.

In 1988 Bonet and Dierolf [12] used a variant of the Moscatelli shifting device to give a negative solution to the following open problem of Grothendieck [35]: Is the bidual of a strict (LF)-space again an (LF)-space?

They proceeded as follows: Let X be a Fréchet space and $Y \subset X$ a Fréchet subspace, and for each n denote by F_n the topological product $\prod_{k\leq n} X \times \prod_{k>n} Y$. Clearly F_n is a Fréchet space and the natural inclusion $F_n \hookrightarrow F_{n+1}$ is a topological isomorphism onto its range. We may form the inductive limit $F := \mathrm{ind}_n F_n$, which is a strict (LF)-space.

If $Y = \lambda_1(A)$ is the Köthe-Grothendieck non-distinguished echelon space and X is a countable product of Banach spaces containing Y as a topological subspace, they showed that the bidual of the strict (LF)-space constructed above is not an (LF)-space.

In the eighties and early nineties, the following questions about arbitrary (LF)-spaces were open:

(A) Does every sequentially retractive (LF)-space satisfy property (M) of Retakh?

(B) Is every regular (LF)-space complete?

The first question was known to have a positive answer for (LB)-spaces [49], whereas, as we have already mention, the second one is still open even in the case of (LB)-spaces. The positive answer to the first question was given by Wengenroth in 1996 [59]. Mainly motivated by these two questions a variant of the Moscatelli shifting device was used to construct (LF)-spaces as follows.

Definition 7. Let $(L, \|\,\|)$ be a normal Banach sequence space and let Y and X be two Fréchet spaces such that Y is continuously included in X. For every n we form the space $F_n := \bigoplus_{k<n} X \bigoplus \times L((Y)_{k\geq n})$, where

$$L(Y) = \{\, (y_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} Y \mid (p(y_k))_{k\in\mathbb{N}} \in L \ \text{ for each continuous seminorm } \ p \,\}.$$

We define the *(LF)-space of Moscatelli type* with respect to L, X and Y as $F := \mathrm{ind}\, F_n$.

These spaces were introduced and studied by Bonet, Dierolf and Fernández [20] for $L = \ell_\infty$ and the generalization for arbitrary L was investigated by Meléndez [42].

One has the continuous inclusions $L(Y) \subset F \subset L(X)$ and $\bigoplus_{k\in\mathbb{N}} X \subset F$. Moreover, the continuous linear surjection

$$\bigoplus_{k\in\mathbb{N}} X \times L(Y) \to F,\ ((x_k)_k, (y_k)_k) \to (x_k + y_k)_k,$$

is open. Therefore a basis of zero-neighborhoods is $\bigoplus_{k\in\mathbb{N}} U_k + V^{\mathbb{N}} \cap L(Y)$, where $(U_k)_{k\in\mathbb{N}}$ is a sequence of absolutely convex zero neighborhoods in X and V is a balanced and convex neighborhood in Y.

Proposition 8. *For the (LF)-space of Moscatelli type F with respect to L, X and Y, the following conditions are equivalent:*

(i) *F is sequentially retractive.*

(ii) *F is strict.*

(iii) *Y is a topological subspace of X.*

Proposition 9. *For the (LF)-space of Moscatelli type F with respect to L, X and Y, the following conditions are equivalent:*

(i) *F is regular.*

(ii) *F is complete.*

(iii) *Y has an X-closed 0 neighborhood.*

To prove this result a convenient projective hull, generalizing the one in [14], was constructed.

In [20] another more complicated class of (LF)-spaces was constructed. Starting with three Fréchet spaces $Y \subset Z \subset X$ with continuous inclusions, we form $G := \bigoplus_{k\in\mathbb{N}} X + c_0(Z) + \ell_\infty(Y)$, which is the countable inductive limit of the Fréchet spaces $G_n = \bigoplus_{k<n} X \times (c_0((Z)_{k\geq n}) + \ell_\infty((Y)_{k\geq n}))$. For these (LF)-spaces the equivalence between regularity and completeness was shown only under additional assumptions. For instance, this was the case if Y was supposed to be Banach or a Fréchet-Schwartz space.

In a recent paper, Dierolf and Kuß [30] consider the class of (LF)-spaces obtained when ℓ_∞ and c_0 are replaced by any other normal Banach sequence space L and by $\overline{\varphi}^L$, respectively. They were able to show the following result

Theorem 8. *Let $Y \subset Z \subset X$ be Fréchet spaces with continuous inclusions, let L be a normal Banach sequence space and $M := \overline{\varphi}^L$. If the (LF)-space $G := \bigoplus_{k\in\mathbb{N}} X + M(Z) + L(Y)$ is regular, then it is complete.*

As a consequence they proved that *the completion of every (LB)-space of Moscatelli type is again an (LB)-space.*

4 Further Results

4. 1 Generalized projective and inductive limits of Moscatelli type

In 1990 in her thesis written under the advice of Bonet and Dierolf, Meléndez [44] applied Moscatelli's shifting device to general locally convex spaces X, Y

and thus obtained a larger class of inductive and projective limits of Moscatelli type. In the case of projective limits she considered locally convex spaces of the form

$$F = \{ (y_k)_{k\in\mathbb{N}} \in Y^{\mathbb{N}} \mid (f(y_k))_{k\in\mathbb{N}} \in L(X) \}$$

where X and Y are locally convex spaces, $f : Y \to X$ is a continuous linear map and L is a normal Banach sequence space. In 1994, S. Dierolf and Khin Aye Aye [28] characterized semireflexivity and quasinormability for projective limits of Moscatelli type when the entries X and Y are general locally convex spaces. They also characterized when F has the density condition and when it is a quojection under the assumption that X and Y are Fréchet spaces. A characterization of distinguishedness is given in case Y is Fréchet, X is Banach and $L = \ell_1$.

In 1998 Albanese and Moscatelli [6] studied the class of Fréchet spaces obtained when all entries in the former construction are allowed to be Fréchet spaces. More precisely, let L be a Fréchet sequence space which is an intersection of a decreasing sequence of normal Banach spaces, and let $(Y_k)_{k\in\mathbb{N}}$ and $(X_k)_{k\in\mathbb{N}}$ be two sequences of Fréchet spaces. For each k denote by $(p_{k,n})_{n\in\mathbb{N}}$ and $(q_{k,n})_{n\in\mathbb{N}}$ fixed increasing sequences of seminorms defining the topologies of Y_k and X_k respectively. Assume that for each k a continuous linear map $u_k : Y_k \to X_k$ is given such that

$$q_{k,n}(u_k(y_k)) \le p_{k,n}(y_k)$$

for all $y_k \in Y_k$ and all k, n. Proceeding as in the Section 1 they obtain the Fréchet space $F = F((Y_k)_{k\in\mathbb{N}}, (X_k)_{k\in\mathbb{N}}, (u_k)_{k\in\mathbb{N}}, L)$

$$F = \{ (y_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} Y_k \mid (q_{k,n}(u_k(y_k)))_{k\in\mathbb{N}} \in L, \text{ for all } n \}$$

with its natural topology.

When all entries are Banach spaces, we recover the spaces studied in [13]. If all entries are nuclear Fréchet spaces and the maps u_k are nuclear, the spaces are like the ones in [47]. This larger class contains non-trivial Montel, Schwartz or nuclear spaces.

Theorem 9. *The space $F((Y_k)_{k\in\mathbb{N}}, (X_k)_{k\in\mathbb{N}}, (u_k)_{k\in\mathbb{N}}, L)$ is*

(i) *nuclear (resp. Schwartz) provided that L, Y_k and X_k for all k are nuclear (resp. Schwartz.)*

(ii) *Montel if L is Schwartz and each Y_k is Montel.*

As an application of the construction we have just described, Mangino [40] proved that the class of Fréchet spaces satisfying property $(DN)_\varphi$ for some strictly increasing unbounded function φ is not closed under complete projective tensor product.

4. 2 The problem of topologies of Grothendieck

Grothendieck [36] asked whether every bounded set in the complete projective tensor product $E\tilde{\bigotimes}_\pi F$ of two Fréchet spaces can be lifted by bounded sets, i.e. whether it is contained in the closed absolutely convex hull of a set $A\bigotimes B = \{\, x\bigotimes y \mid x \in A, y \in B \,\}$ where A and B are bounded sets in E and F, respectively. This is the so-called Grothendieck's *problème des topologies.* In his thesis J. Taskinen [52] solved in the negative this long standing open problem. The counterexample is a very elaborate Fréchet space of Moscatelli type. In fact, starting with a Banach space (G, p) he constructs suitable equivalent norms $p_{n,k} \geq p$, takes $(Y, s) := \ell_1((G, p_{n,k})_{n,k})$, $(X, r) := \ell_1((G, p)_{n,k})$ and forms the Fréchet space of Moscatelli type with respect to $L = \ell_1$, (Y, s), (X, r) and the non-surjective injection $j : (Y, s) \to (X, r)$. This space is not a quojection. In [22] Bonet and Taskinen refined the original construction of Taskinen to obtain a quojection E such that $E\tilde{\bigotimes}_\pi \ell_2$ contains a bounded set which cannot be lifted by bounded sets. In [25] Díaz and Metafune gave new examples of quojections F and Banach spaces X such that the problem of topologies of Grothendieck has a negative answer for $F\tilde{\bigotimes}_\pi X$, and they characterized those quojections F of Moscatelli type such that every bounded subset of $F\tilde{\bigotimes}_\pi X$ is localized for every Banach space X. This is precisely the case when the bidual F'' of F is a countable product of Banach spaces. We refer the reader to Section 6 of the survey paper by Bierstedt and Bonet [11] and the original papers by Taskinen for more details on the problem of topologies of Grothendieck and related results.

4. 3 Primary Fréchet spaces

A Fréchet space E is said to be *primary* if $E = G \bigoplus H$ implies that G or H is isomorphic to E. While this property has been thoroughly studied for Banach spaces, very little is known for non-Banach Fréchet spaces. Metafune and Moscatelli [45] proved that $\ell_p^{\mathbb{N}}$ $(1 \leq p \leq \infty)$ is primary. The space ℓ_{p+} is also primary by [46, Theorem 3.9]. Köthe sequence spaces of Moscatelli type are not primary [24]. The complemented subspaces of the spaces of Moscatelli type $(l^p)^{\mathbb{N}} \cap l^q(l^q)$, with $1 \leq p < q < \infty$ or $q = 0$ are studied in [8]. They show that if $(l^p)^N \cap l^q(l^q) = F \bigoplus G$ then either F or G contains a complemented copy of the whole space. They conjecture that these sequence spaces are primary.

4. 4 The relevance of (LB)-spaces of Moscatelli type

(LB)-spaces of Moscatelli type of the form $\bigoplus_{k\in\mathbb{N}} E + c_0(F)$ have been used by Bonet and Dierolf in [16] to answer in the negative a question by Rump that asked whether there is a pullback in the category of bornological spaces. The relevance of this example to solve a longstanding conjecture of V.A. Raikov in category theory can be seen in [51] and [34]. The same construction was used by Bonet, Dierolf and Aye-Aye [17] to construct (DF)-spaces, in fact a countable direct sums of Banach spaces, containing a dense subspace which is not quasinormable.

In 1995, Mangino [39] constructed an (LB)-space $E = \operatorname{ind} E_n$ of Moscatelli type such that $E\tilde{\bigotimes}_\pi M(S) \neq \operatorname{ind}(E_n\tilde{\bigotimes}_\pi M(S))$. Here $M(S)$ is the Banach space of measures on the unit circle. In this way she solved in the negative the following question which was open for a long time: If $E = \operatorname{ind}_n E_n$ and $F = \operatorname{ind}_n F_n$ are complete inductive limits of Banach spaces E_n and F_n, is it true that $\operatorname{ind}_n(E_n\tilde{\bigotimes}_\pi F_n) = (\operatorname{ind}_n E_n)\tilde{\bigotimes}_\pi(\operatorname{ind}_n F_n)$ holds algebraically and hence topologically?

(LB)-spaces of Moscatelli type have also played a role in the theory of projective spectra of (LB)-spaces, derived functors and functional analytic applications of category theory [60]. In fact, examples of (LB)-spaces of Moscatelli type were used by S. Dierolf, Frerick, Mangino and Wengenroth [29] to clarify the relation between the vanishing of the first derived functor of a projective spectra of (LB)-spaces and the properties of the inductive spectrum of the duals.

4. 5 Weighted spaces of Moscatelli type

Weighted Fréchet and (LB)-spaces are studied in [43]. The idea is to combine the structure of Köthe sequence spaces and the structure of Fréchet and (LB)-spaces of Moscatelli type. In this setting one has non-trivial Montel or Schwartz spaces, too.

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