Köthe coechelon spaces as locally convex algebras

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Abstract

We study those Köthe co-echelon sequence spaces $k_p(V), 1 \leq p \leq \infty$ or $p = 0$, which are locally convex (Riesz) algebras for the pointwise multiplication. We characterize in terms of the matrix $V = (v_n)_n$ when an algebra $k_p(V)$ is unital, locally m-convex, a $\mathcal{Q}$-algebra, it has a continuous (quasi)-inverse, all entire functions act on it or some transcendental entire functions act on it. It is proved that all multiplicative functionals are continuous and a precise description of all regular and all degenerate maximal ideals is given even for arbitrary solid algebras of sequences with pointwise multiplication. In particular, it is shown that all regular maximal ideals are solid.

The aim of this paper is to present a thorough investigation of those Köthe co-echeon sequence spaces $k_p(V), 1 \leq p \leq \infty$ or $p = 0$, which are commutative algebras for the pointwise multiplication. Let us recall that

$$k_p(V) =: \operatorname*{ind}_n \ell^p(v_n),\ p \in [1, \infty], \quad \text{and} \quad k_0(V) := \operatorname*{ind}_n c_0(v_n),$$

where $\ell_p(v_n) \subseteq \mathbb{C}^I$ and $c_0(v_n) \subseteq \mathbb{C}^I$ are the usual (weighted) Banach spaces of all scalar sequences $x = (x_i)_i$ such that $||x||_{n,p} = \left(\sum_{i \in I} (v_n(i)|x_i|)^p\right)^{1/p} < \infty$ and the corresponding supremum norm in the cases $p = 0$ and $p = \infty$. Here $V = (v_n)_{n \in \mathbb{N}}$ is a decreasing sequence of functions $v_n \colon \mathbb{N} \to]0, \infty[$.

These spaces appear naturally as coefficient spaces for some classical Schauder orthogonal bases in some classical spaces. For instance, the space of distributions on a compact interval $[-1, 1]$ with Chebyshev polynomials as a basis or the space of tempered distributions on the real line with Hermite functions as a basis. In some cases $k_p(V)$ are sequential representations of algebras with convolution as a product of functions or distributions defined on a compact group, where the unit basis corresponds to the basis of characters (see, for instance, spaces of distributions or ultradistributions over the unit circle, [32, Lemma 8.1]). In that case the convolution corresponds to the pointwise multiplication in $k_p(V)$. Co-echelon sequence spaces are not always closed with respect to pointwise multiplication (but this is so for the above named examples); that is why our first task will be to characterize which spaces $k_p(V)$ are algebras — it turns out that in that case they are automatically topological algebras with jointly continuous multiplication, see Proposition 2.1.

Surprisingly we cannot find in the literature papers devoted exactly to $k_p(V)$ algebras although there are many results which can be deduced from an investigation on more general classes of algebras. However, the picture was far from being complete.

Algebras $k_p(V)$ are Riesz algebras [1, Def. 2.53] or even the so-called f-algebras extensively studied, for instance, in [20], [15], [16]. Co-echelon sequence algebras are also ℓ_∞-submodules of the algebra of all sequences, this type of algebras was considered in [28]. If $p \neq \infty$, $k_p(V)$ has even the so-called orthogonal unconditional basis (unit vectors form such a basis), such algebras were studied in [5], [6], [13], [17], [18], [19]. In case $p = \infty$ the space $k_p(V)$ is algebraically equal to some space of sequences equipped with a family of weighted sup-norms (but topologically they

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may differ) [10, Th. 15]. Analogously for $p = 1$, $k_p(V)$ is equal even topologically to a space of sequences equipped with a family of weighted ℓ_1-norms [10, Th. 15]. These types of algebras were studied in [26] and [5], respectively. Finally, since the classical work of Michael [25] and Arens [2], several authors have investigated countable inductive limits of Banach algebras (see, for instance, [3], [23], [12], [14], [21], [29], [33], [34] etc.). Unfortunately, even if $k_p(V)$ is a topological algebra it need not be an inductive limit of a sequence of Banach *algebras* although it is by definition an inductive limit of a sequence of Banach *spaces*.

Let us summarize our main results and comment what has been known so far in this respect. The results are formulated and proved over the complex field but all of them (except in Section 3) remain valid with the same proof for spaces and algebras over the real field.

In Section 2 we characterize when $k_p(V)$ is an algebra (Prop. 2.1), when it is unital (Prop. 2.5, 2.6, Cor. 2.7), locally m-convex, $\mathcal{Q}$-algebra or an inductive limit of a sequence of Banach algebras (Theorem 2.8) — the last three conditions turned out to be equivalent. The implication that existence of unit implies nuclearity for $1 \le p < \infty$ was known for algebras with absolute orthogonal basis (i.e., for instance for $k_1(V)$) see [5, Cor. 2.6]. The characterization of $\mathcal{Q}$-algebras was known only for $p = \infty$ for unital algebras (for the non-unital case a necessary conditions was known) see [26, Prop. 3], a characterization of local m-convexity was known only for $p = \infty$ [26, Prop. 2]. The fact that the inductive limit of a sequence of Banach algebras is locally m-convex and a $\mathcal{Q}$-algebra is known [12] and [33] and references therein (for the unital case [3, Prop. 12, Thm. 2]). Let us emphasize that after adding unit to a non-unital algebra $k_p(V)$ it is no more a $k_p(V)$-algebra — that is why it makes sense to consider general non-necessarily unital case.

In Section 3 we characterize when all entire functions vanishing at zero act on the algebra $k_p(V)$ (Theorem 3.1) and when there is a transcendental function acting on $k_p(V)$ (Theorem 3.2).

In Section 4 we describe all maximal regular ideals in an arbitrary ℓ_∞-module (the class of algebras including $k_p(V)$ algebras for $1 \le p \le \infty$ or $p = 0$), explaining which of them have codimension 1 and which are of infinite codimension (these are the only possibilities), see Theorem 4.2, Proposition 4.3. As a consequence we prove that they are all solid. Moreover, we characterize which ideals are contained in some regular maximal proper ideal, Corollary 4.8. So far the description of multiplicative functionals were given in [28, Cor. 14.] or [26, Th. 1] for $p = \infty$, the description of maximal proper ideals in the *unital* case was given for spaces with orthogonal unconditional basis [18, Th. 1] (so for $p \neq \infty$) but the same proof works for all ℓ_∞-modules (so also for $p = \infty$). Nevertheless our proof seems to be easier also in the unital case. Some hint for the form of such maximal proper ideals in the unital case is given in [15, Th. 2.5]. For the general non-unital case a description of maximal closed ideals was given in [19, Th. 2.1] again for $p \neq \infty$. On the other hand the description of continuous multiplicative functionals was given for $p = \infty$ in [26, Th. 3].

In Section 5 we show that in $k_p(V)$ algebras all multiplicative functionals are continuous (Theorem 5.1), we clarify which maximal proper ideals are closed (Theorem 5.4) and show that all ideals in $k_p(V)$ are solid if and only if $k_p(V)$ is unital (Prop. 5.17). Finally we describe the ideal $(k_p(V))^2$ essential for the description of all maximal proper degenerate ideals (Prop. 5.6) and characterize when $k_p(V) = (k_p(V))^2$ (Prop. 5.10). As a consequence we find out that in some dual power series algebras there exist proper ideals not contained in any maximal proper ideal. In this respect only a description of closed ideals for $p \neq \infty$ was known [19, Th. 2.2.].

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1 Notation and Preliminaries

In this article we denote always by I a fixed countable index set and $V = (v_n)_{n\in\mathbb{N}}$ a decreasing sequence of functions $v_n : I \to]0, \infty[$. So, $k_p(V)$ is the increasing union of Banach spaces $\cup_{n=1}^{\infty} \ell_p(v_n)$ (resp. $\cup_{n=1}^{\infty} c_0(v_n)$) endowed with the strongest locally convex topology under which the natural injection of each of the Banach spaces $\ell^p(v_n)$ (resp. $c_0(v_n)$), for $n \in \mathbb{N}$, is continuous. The norm of $\ell_p(v_n)$ is denoted by $||\cdot||_{n,p}$ (resp. $||\cdot||_{n,0}$). The spaces $k_p(V)$ are called *co–echelon spaces* of order p. The natural map $k_0(V)\to k_\infty(V)$ is clearly continuous but, it is even a topological isomorphism into $k_\infty(V)$; see [9].

The space $k_p(V)$ is naturally included in a minimal sequence space with a family of ℓ_p-norms called the *projective hull of* $k_p(V)$ and denoted by $K_p(\overline{V})$ see [8], [10], [7]. It is known that for $p\neq 0,\infty$ the projective hull is topologically equal to $k_p(V)$, for $p=\infty$ it is algebraically equal to $k_\infty(V)$ (but the topologies may differ) while for $p=0$ the projective hull contains (in general properly) $k_0(V)$ as a topological subspace. For more details see [10] or [7].

The unit vectors of the sequence space will be denoted by $e_i:=(\delta_{i,j})_{j\in I}$. We denote by e the constant sequence whose elements are all equal to 1. If J is a subset of I, we denote by e_J the characteristic function on J. If $J\subset I$, the *sectional subspace* $k_p(V,J)$ is the subspace of $k_p(V)$ consisting of all the elements of $x\in k_p(V)$ whose coordinates x_i vanish if $i\notin J$.

For an increasing sequence $\alpha=(\alpha_i)_i$ of positive numbers tending to infinity, the *dual power series spaces* $\Lambda_0^p(\alpha)'$ of finite (resp. $\Lambda_\infty^p(\alpha))'$ of infinite) type are precisely the Köthe co-echelon spaces $k_p(V)$ of order p for the matrix $V=(v_n)_n$, $v_n(i):=\exp(\frac{1}{n}\alpha_i)$ (resp. $v_n(i):=\exp(-n\alpha_i)$).

For a systematic treatment of co–echelon spaces (and echelon spaces) see [7, 9, 10], [24, Ch. 27]. We refer the reader to the books by Mallios [23] and Żelazko [35], [36] for unexplained terminology about topological algebras, and for functional analysis and locally convex spaces to Meise, Vogt [24] and Bonet, Pérez Carreras [30].

2 Algebras $k_p(V)$

Proposition 2.1 *The space $k_p(V), 1\le p\le\infty$ or $p=0$, is an algebra with (separately continuous) pointwise multiplication if and only if for all n there is k such that $v_k/v_n^2\in\ell_\infty$. Moreover, if it is an algebra then its multiplication is automatically (jointly) continuous.*

Proof. Observe that the pointwise multiplication is well defined in $k_p(V)$ if and only if it is separately continuous by the closed graph theorem for LB-spaces. Since every LB-space is barrelled, separately continuous multiplication is then also a hypocontinuous bilinear form. We now can proceed as in the proof of [30, Prop 11.3.7] to conclude that the hypocontinuous multiplication is continuous, since every LB-space is also a (DF)-space. Summarizing: pointwise multiplication is well defined in $k_p(V)$ iff it is hypocontinuous and this, in turn, is equivalent to

$$\forall\ n\ \ \exists\ k, C>0 \qquad \|xy\|_{k,p}^p\le C\|x\|_{n,p}^p\|y\|_{n,p}^p. \tag{2.1}$$

Indeed, hypocontinuity follows from (2.1) by the fact that LB-spaces are ultrabornological and the unit balls of $(\|\cdot\|_{k,p})_k$ form a fundamental system of Banach discs. The converse follows from [31, Lemma2.1].

Sufficiency: Given $n\ge m$, we select k as in the assumption to conclude, for $y\in\ell_p(v_n)$,

$$\|xy\|_{k,p}^p=\sum_i|x_iy_iv_k(i)|^p\le C\sum_i(|x_i|v_n(i)|y_i|v_n(i))^p\le\|x\|_{n,p}^p\|y\|_{n,p}^p \tag{2.2}$$

.

Necessity. Applying (2.1) to $x=y=e_i, i\in I$, we get

$$\forall n\ \exists k, C\ \forall i\in I\ :\ v_k(i)\le Cv_n(i)^2.$$

□

Remark 2.2 (1) We have observed that in every topological LB-algebra multiplication is automatically jointly continuous.

(2) Analogously as in (2.2) one observes that

$$\|xy\|_{k,p}^p\le C\|x\|_{n,\infty}^p\|y\|_{n,p}^p,$$

which implies that if $k_p(V)$ is an algebra it is also a $k_\infty(V)$-module.

(3) If $v_n(i) \geq 1$, for all $i \in I$ and $n \in \mathbb{N}$, then the condition in Proposition 2.1 always holds taking $k = n$ and the constant $C = 1$. This implies in particular that all the Banach step spaces are in fact Banach algebras. In this case, the space $k_p(V)$ is a locally m-convex algebra [12, 14]. See below for a characterization of locally m-convex algebras of type $k_p(V)$.

(4) If we take $I = \mathbb{N} \times \mathbb{N}$ and define $v_n(i,j) := 1$ if $i \geq n$ and $v_n(i,j) := (1/j)^k$, if $i < k$, then k_∞ and k_0 are algebras by Proposition 2.1, but $k_0(V)$ is not equal to its projective hull $K_0(\overline{V})$ algebraically and the topology of k_∞ is strictly finer than the topology of the projective hull $K_\infty(\overline{V})$. In fact, the sequence V does not satisfy condition (D); cf. [10]. Another example of an algebra $k_0(V)$ strictly smaller than $K_0(\overline{V})$ can be seen in [24, 27.21].

Proposition 2.3 *(Comp. [19, Th.2.1]) The set of non zero continuous multiplicative functionals on an algebra $k_p(V)$, $1 \leq p < \infty$ or $p = 0$, coincides with the set of point evaluations on I.*

Proof. Continuous functionals on $k_p(V)$ are elements of $\lambda_q(V^{-1})$ for $\frac{1}{p} + \frac{1}{q} = 1$ for $p \neq \infty$. If a functional has two non-zero coordinates, it does not vanish on some pair of two vectors x, y such that $xy = 0$. □

Remark 2.4 If an algebra $k_\infty(V)$ is not Montel, then $k_\infty(V)$ contains complemented algebra (with a projection being an algebra homomorphism) isomorphic to l_∞ as an algebra. Therefore, there exists multiplicative functionals which are not point evaluations at points in I, since $\ell_\infty = C(\beta\mathbb{N})$ and the point evaluations on $\beta\mathbb{N}$ are multiplicative functionals. If $k_\infty(V) = K_\infty(\overline{V})$ holds topologically, the full description was given by Oubbi [28, Th. 3]. If $k_\infty(V)$ is Montel, either it is even a (DFS)-space and it coincides with $k_0(V)$, which happens precisely when V satisfies condition (S) (cf. [9]) or it is Montel but V does not satisfies condition (S). In this case, $k_\infty(V) = K_\infty(\overline{V}) = K_0(\overline{V})$ is the completion of $k_0(V)$; cf. [9]. In both cases, the set of non zero continuous multiplicative functionals on an algebra $k_\infty(V)$ coincides with the set of point evaluations on I.

Proposition 2.5 *Suppose that $k_p(V), 1 \leq p < \infty$, is an algebra. The following assertions are equivalent:*

(a) $k_p(V)$ is unital.

(b) There is m such that $v_m \in \ell_p$.

(c) There is m such that $v_m \in \ell_1$.

(d) $k_p(V)$ is nuclear and there is m such that $v_m \in \ell_\infty$.

In particular, if any of the above conditions hold then $k_p(V)$ is isomorphic to a quotient of s' the dual of the space of rapidly decreasing sequences.

Proof. The equivalence of (a) and (b) and implication (c)⇒(b) are trivial.

(b)⇒(c): Take $2^k > p$, then $v_m \in \ell_{2^k}$. By Prop. 2.1 applied k-times, there is n such that $v_n \leq C v_m^{2^k}$ thus $v_n \in \ell_1$.

To see that (c) implies nuclearity of $k_p(V)$, given $n \geq m$ we apply Proposition 2.1 to find k such that $v_k/v_n^2 \in \ell_\infty$. Then

$$\sum_i \frac{v_k(i)}{v_n(i)} \leq C \sum_i v_n(i) < \infty,$$

since v_m, hence v_n, belongs to ℓ_1 The conclusion follows by Grothendieck Pietsch criterion, cf. [24, 28.16].

Now suppose that condition (d) holds. We apply Grothendieck Pietsch criterion to find $k > m$ such that $v_k/v_m \in \ell_1$. Since $v_m \in \ell_\infty$, we get

$$\sum_i v_k(i) \le \sum_i v_m(i)\left(\frac{v_k(i)}{v_m(i)}\right) \le C\sum_i \left(\frac{v_k(i)}{v_m(i)}\right) < \infty,$$

thus $v_k \in \ell_1$ and (c) holds.

If $k_p(V)$ is nuclear then it is reflexive and its strong dual is a Köthe sequence space with a Köthe matrix $\left(\frac{1}{v_k(j)}\right)_{k,j}$ (comp. [24, Sec. 27]). Moreover, by (d), sup-norm is a continuous norm on this dual space. By Prop. 2.1,

$$\forall\ n\ \exists\ k \quad \frac{1}{v_n^2} \le C\frac{1}{v_k}\cdot 1,$$

which gives a (DN)-condition and, by [24, Th. 31.5], the dual of $k_p(V)$ is isomorphic to a subspace of s. By duality, the last claim of the proposition follows. □

Proposition 2.5 does not hold for $p = 0$. Indeed, it is enough to take $v(i) \to 0$ as i goes to ∞ such that $v_n(i) := v(i)^n$ is not summable for each $k \in \mathbb{N}$. For example $v_n(i) := \frac{1}{(\log i)^n}$. However, we have the following result.

Proposition 2.6 *Suppose that $k_0(V)$ is an algebra. The following assertions are equivalent:*

(a) $k_0(V)$ is unital.

(b) There is m such that $v_m \in c_0$.

(c) $k_0(V)$ is Schwartz and there is m such that $v_m \in \ell_\infty$.

Proof. The proof is similar to the one of Proposition 2.5, since $k_0(V)$ is Schwartz if and only if for each n there is k such that $v_k/v_n \in c_0$; cf. [9]. □

Since for some V we have $k_\infty(V) = \ell_\infty$ the above results do not hold for $p = \infty$. However, we have the following general result.

Corollary 2.7 *The algebra $k_p(V)$, $1 \le p \le \infty$ or $p = 0$, has a unit if and only if $k_p(V) = k_\infty(V)$ and there is m such that $v_m \in \ell_\infty$.*

Clearly the condition $e \in k_\infty(V)$ does not imply that the algebra $k_\infty(V)$ is Montel, since one can take $v_n(i) = 1$ for each n and obtain $k_\infty(V) = \ell_\infty$.

An element x of a (not necessarily unital) commutative algebra is called *quasi-invertible* if there is $y \in A$, necessarily unique, such that $x + y = xy$. The element y is called the *quasi-inverse* of A. An element x of an algebra A with unit e is quasi-invertible with quasi-inverse y if and only if $e - x$ is invertible and $e - y$ is its inverse. An algebra A is called a *$\mathcal{Q}$-algebra* if the set of quasi-invertible elements is open. For unital algebras this concept coincides with the fact that the set of invertible elements is open. Moreover, the inverse is continuous in an unital algebra if and only if the map sending quasi-invertible elements into their quasi-inverses is continuous (i.e., *quasi-inverse is continuous*). We refer to [23] for more information.

Now we present a theorem characterizing many basic properties of algebras $k_p(V)$.

Theorem 2.8 *Suppose that $k_p(V), 1 \le p \le \infty$ or $p = 0$, is an algebra. The following assertions are equivalent:*

(a) $k_p(V)$ is locally m-convex.

(b) $k_p(V)$ is a $\mathcal{Q}$-algebra.

(c) The map sending each quasi-invertible element of $k_p(V)$ to its quasi-inverse is continuous.

(d) Every sectional subspace $k_p(V,E)$ with unit coincides algebraically (and topologically) with $\ell_\infty(E)$.

(e) $\inf_{i\in I} v_n(i) > 0$ *for each* $n \in \mathbb{N}$.

(f) $k_p(V) \subset \ell_\infty$ *holds algebraically.*

(g) Infinitely many / all the steps are Banach algebras.

(h) $k_p(V)$ *is an inductive limit of a sequence of Banach algebras.*

If $p \neq \infty$ *then the above conditions are also equivalent to*

(i) No infinite dimensional sectional subspace of $k_p(V)$ *has unit.*

Proof. (a) ⇒ (i) for finite p: Suppose that a sectional subspace $k_p(V,J)$, $J \subset I$, with J infinite, has a unit. By (a), $k_p(V)$ is locally m-convex, then $k_p(V,J)$ is a unital locally m-convex algebra of type $k_p(V)$. By Żelazko [35, Th 12.3], an element $x \in k_p(V,J)$ is invertible as soon as every non zero multiplicative continuous functional does not vanish on x. Non zero multiplicative continuous functionals on $k_p(V)$ are evaluations on the elements of the index set (see Prop. 2.3); thus $x \in k_p(V,J)$ is invertible if and only if all its coordinates are non-zero. However, if $x \in k_p(V,J)$ is invertible in the unital algebra $k_p(V,J)$, there are k and $C > 0$ such that, for all $j \in J$, $\frac{1}{|x_j|} < \frac{C}{v_k(j)}$. If we select $y_j := \min(1, j^{-1}v_j(j)), j \in J$, we have that $y = (y_j)_{j\in J} \in k_p(V,J)$, since $k_p(V,J)$ has a unit. On the other hand, for each $k \in \mathbb{N}$ and each $j > k$, $j \in J$.

$$v_k(j)\frac{1}{|y_j|} \geq v_k(j)\frac{j}{v_j(j)} \geq j,$$

which implies that y is not invertible, although all its coordinates are non zero. A contradiction.

(i) ⇒ (d) for finite p: Obvious.

(d) ⇒ (e). Assume that there exists n such that $\inf_i v_n(i) = 0$. Thus there exists an infinite subset $J \subset I$ such that

$$\sum_{j\in J} v_n(j) < \infty.$$

The sectional subspace $k_p(V,J)$ contains unit by Propositions 2.5, 2.6 and Corollary 2.7 and $k_p(V,J) \neq \ell_\infty$.

(e) ⇒ (f): Obvious.

(f) ⇒ (e): If (e) does not hold we find m and a sequence $(i_j)_j$ in I such that $v_m(i_j) < 2^{-2j}$ for each j. The element $x = (x_i)_i$ given by $x_{i_j} := 1/v_m(i_j)^{1/2}, j \in \mathbb{N}$ and $x_i = 0$ otherwise, is unbounded but it belongs to $\ell_p(v_m) \subset k_p(V)$.

(a) ⇒ (d) for $p = \infty$: Since $k_0(V)$ is a topological subalgebra of $k_\infty(V)$ [9], it is a locally m-convex algebra. Thus by (a) ⇒ (d) for finite p and (d) ⇒ (e) ⇔ (f) already proved we get (e) and (f). Clearly, every unital $k_\infty(V,E)$ contains $\ell_\infty(E)$, thus (f) implies that $k_\infty(V,E) = \ell_\infty(E)$.

(e) ⇒ (g) If $\inf_i v_n(i) > 0$, then

$$\frac{v_n(i)}{v_n(i)^2} = \frac{1}{v_n(i)}$$

is bounded and the step $\ell_p(v_n)$ or $c_0(v_n)$ is an algebra.

(g) ⇒ (h): Obvious.

(h) ⇒ (a) If infinitely many steps are commutative Banach algebras then the inductive limit is an m-convex algebra. See e.g. [12] and [14].

(a) ⇒ (b): See [33], where the proof is given in the unital case. For non-unital case obvious changes are required.

(b) ⇒ (e): Suppose that (e) does not hold. Find n and a sequence $(i(k))_k$ in I such that $v_n(i(k)) < 1/k$ for each k. This implies that $(e_{i(k)})_k$ tends to 0 in $\ell_p(v_n)$, hence in $k_p(V)$. Now, 0 is quasi-invertible but $e_{i(k)}$ is not, since $e_{i(k)} + y = e_{i(k)}y$ implies $1 + y_{i(k)} = y_{i(k)}$, a contradiction.

(a) $\Rightarrow$ (c): Every locally m-convex algebra has a continuous quasi-inverse operation by [23, II Lemma 3.1].

(c) $\Rightarrow$ (e): We proceed again by contradiction and assume that (e) is not satisfied. Select n and a sequence $(i(k))_k$ such that $\lim_{k\to\infty} v_n(i(k)) = 0$. Select a sequence $(\nu_k)_k \subset]0, 1/2[$ tending to 0 such that $(v_m(i(k))/\nu_k)_k$ is unbounded for each m. Each element $(1-\nu_k)e_{i(k)}$ is quasi-invertible with quasi-inverse $\frac{1-\nu_k}{\nu_k} e_{i(k)}$ as a direct computation shows. On the other hand, $((1-\nu_k)e_{i(k)})_k$ converges to 0 in $\ell_p(v_n)$, hence in $k_p(V)$, but $\left(\frac{1-\nu_k}{\nu_k} e_{i(k)}\right)_k$ is unbounded in every step of $k_p(V)$.

For $p \neq 0$ this implies that $\left(\frac{1-\nu_k}{\nu_k} e_{i(k)}\right)_k$ is not convergent by regularity of $k_p(V)$. For $p = 0$ we observe that $\left(\frac{1-\nu_k}{\nu_k} e_{i(k)}\right)_k$ is also unbounded in every step of $k_\infty(V)$ so it cannot be convergent to zero in $k_\infty(V)$ and also in $k_0(V)$ since the latter space is a topological subspace of $k_\infty(V)$ [9].

This shows that the quasi-inverse operation is not continuous. □

As a consequence of Propositions 2.5, 2.6, Corollary 2.7 and Theorem 2.8, we get the following corollaries.

Corollary 2.9 *If $1 \leq p < \infty$ or $p = 0$ and $k_p(V)$ is a unital algebra, then $k_p(V)$ is not a $\mathcal{Q}$-algebra and the inversion is not continuous on the group of invertible elements $\mathcal{G}(k_p(V))$.*

Corollary 2.10 *An unital algebra $k_\infty(V)$ is a $\mathcal{Q}$-algebra if and only if it coincides with the Banach algebra ℓ_∞.*

Proof. It is a consequence of Corollary 2.7 and Theorem 2.8, since a locally m-convex unital algebra $k_\infty(V)$ must coincide algebraically and topologically with ℓ_∞. □

From the above it follows immediately:

Corollary 2.11 *$\Lambda_0^p(\alpha)'$ is a non-unital locally m-convex $\mathcal{Q}$-algebra. $\Lambda_\infty^p(\alpha)'$ are locally convex (Riesz) algebras which are never locally m-convex $\mathcal{Q}$-algebras and they are never inductive limits of a sequence of Banach algebras. They have always unit for $p = \infty$ or $p = 0$ and only in nuclear case for $1 \leq p < \infty$.*

Let us observe that $\Lambda_0^p(\beta)$ and $\Lambda_\infty^p(\beta)$ for $\beta = (\beta_n)$, $\beta_n = n$, are isomorphic as locally convex spaces to spaces of germs of holomorphic functions over the closed unit disc or over a point, respectively. Thus they are also topological algebras equipped with the multiplication of germs — clearly, these are completely different (always unital) algebras than our $k_p(V)$-algebras.

3 Entire functions acting on algebras $k_p(V)$

We say that an entire function $f(z) = \sum_{n=0}^\infty \alpha_n z^n$ *acts on the algebra* A if for every $x \in A$ the series $\sum_{n=0}^\infty \alpha_n x^n$ converges to an element in A. We say that a space of entire functions *acts on an algebra* if all its elements act on the algebra. In our next results we identify the entire function $f \in H(\mathbb{C})$, $f(z) = \sum_{n=0}^\infty \alpha_n z^n$, with the sequence of coefficients $(\alpha_n)_{n=0}^\infty$.

If an algebra $k_p(V)$ is not unital, then constant functions do not act on $k_p(V)$. This is why we restrict our attention to entire functions vanishing at 0.

Theorem 3.1 *The space $H_0(\mathbb{C})$ of entire functions vanishing at 0 acts on an algebra $k_p(V), 1 \leq p \leq \infty$ or $p = 0$, if and only if $\inf_i v_n(i) > 0$ for all $n \in \mathbb{N}$.*

Proof. We give the details of the proof for $1 \leq p < \infty$, the other cases are easier.

Assume first that there exists k such that $\inf_j v_k(j) = 0$. By Theorem 2.8 (e)$\Leftrightarrow$(f), there is $t = (t_j)_j \in k_p(V) \setminus \ell_\infty$ with a subsequence $(t_{j_k})_k$ tending to ∞. Take $(w_{j_k})_k$ such that $\sum_k v_n(j_k)|w_{j_k}|^p = \infty$ for all $n \in \mathbb{N}$. It is well known that there is $f \in H_0(\mathbb{C})$, $f(t_{j_k}) = w_{j_k}$, $k \in \mathbb{N}$. Clearly, f does not act on $k_p(V)$. Otherwise $(f(t_j))_j \in k_p(V)$ which is not the case.

To show the converse, first observe that, by Theorem 2.8 (e)$\Leftrightarrow$(g), steps of $k_p(V)$ are Banach algebras with pointwise multiplication and, as it is well known, $H_0(\mathbb{C})$ acts on each Banach algebra. $\square$

Theorem 3.2 *Let $k_p(V), 1 \leq p \leq \infty$ or $p = 0$, be an algebra. The following conditions are equivalent:*

(a) Only the polynomials act on $k_p(V)$.

(b) There is k such that for all $l > k$ there is n such that $\sup_{i\in I} \frac{v_l(i)}{v_k(i)^n} = \infty$.

(c) There is k such that for all $l > k$ there is n such that for all $m \geq n$ we have $\sup_{i\in I} \frac{v_l(i)}{v_k(i)^m} = \infty$.

Proof. To prove that (a) implies (b), we assume that (b) does not hold, and select an increasing sequence $(l(k))_k \subset \mathbb{N}$ and $C_{k,n} \geq 1,\ k, n \in \mathbb{N}$ such that

$$\sup_{i\in I} \frac{v_{l(k+1)}(i)}{v_{l(k)}(i)} \leq C_{k,n}, \quad k, n \in \mathbb{N}.$$

Find a sequence $(\beta_n)_n$ of strictly positive numbers such that $M_{k,\lambda} := \sup_n \beta_n C_{k,n} \lambda^n < \infty$ for each $k \in \mathbb{N}$ and each $\lambda \in \mathbb{N}$. Put $\alpha_n := 2^{-n}\beta_n > 0$. Clearly $f(z) := \sum_n \alpha_n z^n, z \in \mathbb{C}$, defines an entire function which is not a polynomial. If $x \in k_p(V)$, there are $l(k)$ and $\lambda \in \mathbb{N}$ such that

$$||x||_{l(k),p} = \left(\sum_i (|x_i| v_{l(k)}(i))^p\right)^{1/p} \leq \lambda.$$

For n arbitrary, we have

$$||x^n||_{l(k+1),p} = \left(\sum_i (|x_i^n| v_{l(k+1)}(i))^p\right)^{1/p} \leq C_{k,n} ||x||^n_{l(k),p} \leq C_{k,n}\lambda^n.$$

Therefore

$$\sum_n |\alpha_n| ||x^n||_{l(k+1),p} \leq \sum_n 2^{-n}\beta_n C_{k,n}\lambda^n \leq M_{k,\lambda}.$$

Consequently, $\sum_n \alpha_n x^n$ converges in $\ell_p(v_{l(k+1)})$, and $f(z)$ acts on $k_p(V)$. The details for the cases $p = 0$ and $p = \infty$ are even easier.

We now show that (b) implies the stronger condition (c). To see this, select k as in (b) and, for $l > k$ use (b) to take n. Clearly $n \geq 2$. Observe that, if $v_k(i) \geq 1$, then $\frac{v_l(i)}{v_k(i)^n} \leq \frac{1}{v_k(i)^{n-1}} \leq 1$. Hence, $\sup_{\{i\in I | v_k(i)<1\}} \frac{v_l(i)}{v_k(i)^n} = \infty$, since $\sup_{i\in I} \frac{v_l(i)}{v_k(i)^n} = \infty$. On the other hand, if $m > n$ we get, for $i \in I$ with $v_k(i) < 1$, $\frac{v_l(i)}{v_k(i)^n} \leq \frac{v_l(i)}{v_k(i)^m}$. This yields $\sup_{i\in I} \frac{v_l(i)}{v_k(i)^m} = \infty$; and condition (c) is proved.

Finally, we assume that condition (a) does not hold but condition (c) is satisfied. There is an entire function $f(z) = \sum_n \alpha_n z^n$ not a polynomial acting on $k_p(V)$. Select an increasing sequence of natural numbers $1 < n_1 < n_2 < ... < n_j < ...$ such that $\alpha_{n_j} \neq 0$ for each j. Proceeding by induction we select a subsequence of $(n_j)_j$ as follows. We apply (c) to select $n_{j(1)}$ such that $\sup_{i\in I} \frac{v_{k+1}(i)}{v_k(i)^{n_{j(1)}}} = \infty$. Find $i_1 \in I$ such that

$$\frac{v_{k+1}(i_1)}{v_k(i_1)^{n_{j(1)}}} > 2^{2n_{j(1)}}.$$

Inductively, we find a subsequence $n_{j(1)} < n_{j(2)} < ... < n_{j(s)} < ...$ of $(n_j)_j$ and a sequence (i_s) of pairwise different points in I such that, for each $s \in \mathbb{N}$,

$$\frac{v_{k+s}(i_s)}{v_k(i_s)^{n_{j(s)}}} > (s+1)^{(s+1)n_{j(s)}}.$$

Now we define $x = (x_i)_i$ by $x_{i_s} := 2^{-s}/v_k(i_s), s \in \mathbb{N}$ and $x_i = 0$ otherwise. It is easy to see that $x \in \ell_p(v_k)$. By assumption $\sum_n \alpha_n x^n$ converges in $k_p(V)$. In particular $\{\alpha_n x^n \mid n \in \mathbb{N}\}$ is bounded in $k_p(V)$. Since $k_p(V)$ is a regular inductive limit for $p \neq 0$, we can find $l \in \mathbb{N}$ such that $\{\alpha_{n(j(s)} x^{n(j(s)} \mid s \in \mathbb{N}\}$ is bounded in $\ell_p(v_l)$ (for $p = 0$ we use the fact that $k_0(V)$ is a topological subspace of $k_\infty(V)$ and we work in $k_\infty(V)$, see [10], [9]). We have found $l \in \mathbb{N}$ such that $x^{n_{j(s)}} \in \ell_p(v_l)$ for each s (with $p = \infty$ if $p = 0$). This implies

$$\sum_s \left(\frac{2^{-sn_{j(s)}} v_l(i_s)}{v_k(i_s)^{n_{j(s)}}} \right)^p < \infty$$

for all s. However, for $s > l - k$, we get

$$\frac{2^{-sn_{j(s)}} v_l(i_s)}{v_k(i_s)^{n_{j(s)}}} \geq \frac{2^{-sn_{j(s)}} v_{k+s}(i_s)}{v_k(i_s)^{n_{j(s)}}} > 2^{-sn_{j(s)}}(s+1)^{(s+1)n_{j(s)}},$$

which does not tend to 0 as s goes to ∞; a contradiction. □

Corollary 3.3 *Only the polynomials act on the algebra* $\Lambda^p_\infty(\alpha)'$ *while all entire functions vanishing at zero act on* $\Lambda^p_0(\alpha)'$.

Corollary 3.4 *There is an entire function which is not a polynomial acting on an algebra* $k_p(V)$, $1 \leq p \leq \infty$ *or* $p = 0$, *if and only if for each* k *there is* $l > k$ *such that for al* n *we have* $\sup_{i \in I} \frac{v_l(i)}{v_k(i)^n} < \infty$.

Example 3.5 We define $\exp_1(z) := \exp(z)$ and $\exp_k(z) := \exp(\exp_{k-1}(z))$ for $z \in \mathbb{C},\ k \in \mathbb{N}$, and $v_k(i) := 1/\exp_k(i))$ for each $i \in \mathbb{N}$. The co-echelon space $k_\infty(V)$ is an algebra, it is not a locally m-convex algebra by Theorem 2.8, but there are entire functions which are not polynomials acting on $k_\infty(V)$, since it is easy to show that the sequence (v_k) satisfies the condition in Corollary 3.4. In fact, $f(z) = \exp(z)$ acts on $k_\infty(V)$, as can be directly checked.

Remark 3.6 Observe than the fact that an entire function acts on an individual element x of an algebra $k_p(V)$ does not imply that it acts on $2x$. To see this, consider the weights $v_1(i) = 1/i$ and $v_k(i) = \exp(-\frac{2k-1}{k} i), k \geq 2$, for $i \in \mathbb{N}$. It is easy to see that $f(z) = \exp(z)$ acts on $x = (1, 2, 3, ...) \in \ell_\infty(v_1)$, since $(\sum_{n=0}^\infty \frac{1}{n!} i^n)_i$ converges in $\ell_\infty(v_2)$. However, $f(z)$ does not act on $2x$. Indeed, otherwise there would be k such that $(\exp(2i))_i \in \ell_\infty(v_k)$, but this is impossible.

4 Multiplicative functionals and ideals in ℓ_∞-modules

In the last two sections we assume that $I = \mathbb{N}$, which is no loss of generality, but makes the notation smoother. An algebra A of complex sequences with pointwise multiplication is an *ℓ_∞-module* if for every $x \in A$ and every $y \in \ell_\infty$ also $xy \in A$. Of course, algebras $k_p(V)$ are ℓ_∞-modules. Clearly every ℓ_∞-module is a *solid* algebra, i.e., if a sequence x satisfies $|x| \leq |y|$ for some $y \in A$ then $x \in A$.

For $1 \leq p < \infty$ or $p = 0$, the unit vectors $(e_i)_i$ form a *orthogonal basis* of $k_p(V)$, i.e. it is a Schauder basis such that $e_i e_j = 0$ if $i \neq j$ and $e_i e_i = e_i$ for each i. Thus we can apply in this case the results of [18] and [19]. Algebras with orthogonal Schauder bases are also examples of ℓ_∞-modules.

An ideal $\mathcal{J}$ in an algebra A is called *regular* if the algebra $A/\mathcal{J}$ is unital; see [23, Chapter II]. These ideals are sometimes called modular. It is well known that every regular ideal is contained in a maximal regular ideal.

It is also well known that maximal ideals $\mathcal{J}$ in an algebra A are either regular or *degenerate* in the sense that the product of two arbitrary elements in $A/\mathcal{J}$ equals 0. An ideal $\mathcal{J}$ in A is degenerate if and only if it contains the ideal

$$A^2 := \operatorname{span}\{xy \mid x, y \in A\}.$$

Therefore proper maximal degenerate ideals are linear subspaces of codimension one in A containing A^2. Thus the description of maximal proper ideals requires a description of maximal regular ideals and a description of the ideal A^2. We will do the first part of this task for general ℓ_∞-modules.

We say that a subset E of $\mathbb{N}$ is *unital* if the characteristic function e_E of E belongs to A. This happens if and only if the sectional subalgebra

$$A(E) := \{x \in A \mid x_i = 0 \ \ \forall\, i \notin E\}$$

has a unit. The *solid hull* $\tilde{\mathcal{J}}$ of a ideal $\mathcal{J} \subset A$ is the set of all $z \in A$ such that $|z| \leq |x|$ for some $x \in \mathcal{J}$. An ideal is called *solid* if $\tilde{\mathcal{J}} = \mathcal{J}$.

Let us define for an ideal $\mathcal{J}$ in an ℓ_∞-module A the family of sets

$$\mathcal{F}_{\mathcal{J}} := \{U \subset \mathbb{N} \mid A(U^c) \subset \mathcal{J}\}.$$

The proof of the following result is easy.

Lemma 4.1 *Let A be an ℓ_∞-module. For each proper ideal $\mathcal{J} \subset A$, the family of sets $\mathcal{F}_{\mathcal{J}}$ is a filter.*

On the other hand, if $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ we define the ideal in the ℓ_∞-module A as follows:

$$\mathcal{J}_{\mathcal{U}} := \{x \in A \mid \ \forall\, y \in A \quad \lim_{\mathcal{U}} xy = 0\}.$$

Now, we are ready to present the main result of this section. In [18, Theorem 2 and Lemma 6] (comp. [15, Th. 2.5]) a version of the following theorem is proved for *unital* algebras with orthogonal unconditional Schauder bases. The same proof works for unital ℓ_∞-modules. Our proof is easier than the original one also in those cases.

Theorem 4.2 *The map*

$$\mathcal{J} \mapsto \mathcal{F}_{\mathcal{J}}$$

is a bijection between the family of all proper maximal regular ideals in an ℓ_∞-module A and all ultrafilters on $\mathbb{N}$ containing a unital set. The inverse map is defined as

$$\mathcal{U} \mapsto \mathcal{J}_{\mathcal{U}}.$$

In particular, all maximal proper regular ideals are solid.

Proof is based on a sequence of Propositions and Lemmas.

Proposition 4.3 *Let A be an ℓ_∞-module.*

(a) If an ultrafilter $\mathcal{U}$ contains no unital sets, then $\mathcal{J}_{\mathcal{U}} = A$.

(b) If an ultrafilter $\mathcal{U}$ contains a unital set then $\mathcal{J}_{\mathcal{U}}$ is a proper ideal in A. It is of codimension one if and only if

$$\forall\, x \in A \qquad \lim_{\mathcal{U}} |x| < \infty.$$

Otherwise $\mathcal{J}_{\mathcal{U}}$ is of infinite codimension.

(c) *All ideals $\mathcal{J}_\mathcal{U}$ for free ultrafilters $\mathcal{U}$ are equal to A if and only if there is no infinite unital sets.*

Proof. (a): If $\lim_\mathcal{U} x \neq 0$ for some $x \in A$, then there are $\varepsilon > 0$ and a set $U \in \mathcal{U}$ such that $|x_i| > \varepsilon$ for every $i \in U$. Thus U is a unital set. We have shown that if $\mathcal{U}$ contains no unital set, then $\lim_\mathcal{U} x = 0$ for all $x \in A$. Therefore $\lim_\mathcal{U} yx = 0$ for every $x, y \in A$, and $\mathcal{J}_\mathcal{U} = A$.

(b): Clearly, $\mathcal{J}_\mathcal{U}$ is an ideal. If U is a unital set in $\mathcal{U}$, then $e_U \notin \mathcal{J}_\mathcal{U}$ and $\mathcal{J}_\mathcal{U}$ is proper. Observe that if

$$\forall\, x \in A \qquad \lim_\mathcal{U} |x| < \infty$$

then

$$\mathcal{J}_\mathcal{U} = \{x : \lim_\mathcal{U} x = 0\}.$$

Clearly, $\dim A/\mathcal{J}_\mathcal{U} = 1$.

If there is $x \in A$ such that

$$\lim_\mathcal{U} x = \infty,$$

then there is a decreasing sequence of unital sets $(U_n) \subset \mathcal{U}$ such that $|x|$ is bigger than n on U_n, $U_0 := \mathbb{N}$. Let us define a matrix of positive real numbers $(a_{n,k})$ such that

$$\forall\, n, k \quad 0 < a_{n,k} \leq n, \quad a_{n,k+1} > a_{n,k} > a_{n-1,k}$$

and

$$\forall\, k \qquad a_{n,k+1} - a_{n,k} \to \infty \quad \text{as } n \to \infty.$$

Thus

$$y^{(k)} := (y_i^{(k)}) \in A$$

whenever $y_i^{(k)} = a_{n,k}$ for $i \in U_n \setminus U_{n+1}$ since $|y^{(k)}| \leq |x|$. If $\sum_{k=1}^m \alpha_k y^{(k)} \in \mathcal{J}_\mathcal{U}$ then for $i \in U_n$ we have

$$\left|\sum_{k=1}^{m-1} \alpha_k y_i^{(k)}\right| \leq \sum_{k=1}^{m-1} |\alpha_k| a_{n,k}.$$

Therefore for $i \in U_n \setminus U_{n+1}$

$$\left|\sum_{k=1}^{m} \alpha_k y_i^{(k)}\right| \geq |\alpha_m| a_{n,m} - \sum_{k=1}^{m-1} |\alpha_k| a_{n,m-1}$$

and thus if $\alpha_m \neq 0$, then

$$\lim_\mathcal{U} \sum_{k=1}^{m} \alpha_k y^{(k)} \neq 0.$$

Hence we have proved that $(y^{(k)} + \mathcal{J}_\mathcal{U})_k$ is a linearly independent sequence in $A/\mathcal{J}_\mathcal{U}$.

(c): The conclusion follows from (a) and (b). □

Lemma 4.4 *Let A be an ℓ_∞-module. If a proper maximal ideal $\mathcal{J} \subset A$ is solid or regular, then $\mathcal{F}_\mathcal{J}$ is an ultrafilter.*

Proof. Suppose that there is a set $E \subset \mathbb{N}$ such that E and E^c do not belong to $\mathcal{F}_\mathcal{J}$. Then $\mathcal{J}$ contains neither $A(E)$ nor $A(E^c)$.

If $\mathcal{J}$ is solid then $\mathcal{J} + A(E)$ is a proper ideal containing properly the maximal ideal $\mathcal{J}$. This implies that $\mathcal{J} + A(E) = A$. Therefore, for $y \in A(E^c) \setminus \mathcal{J}$, there are $z \in \mathcal{J}$ and $w \in A(E)$ such that $z + w = y$. In particular $y_i = z_i$ for each $i \in E^c$ and $y_i = 0$ for each $i \in E$. Thus $|y| \leq |z|$, which implies $y \in \mathcal{J}$, as $\mathcal{J}$ is solid. This is a contradiction and $\mathcal{F}_\mathcal{J}$ is an ultrafilter.

If $\mathcal{J}$ is a regular proper maximal ideal, then there is $x \in A$ such that $x + \mathcal{J}$ is the unit of $A/\mathcal{J}$. Then for arbitrary $y \in A(E)$, $y - xy \in \mathcal{J}$. If $\mathcal{J} \supset (A(E))^2$ then $xy = (xe_E)y \in (A(E))^2$ and we have $y \in \mathcal{J}$; a contradiction. We have proved that $\mathcal{J} \not\supset (A(E))^2$. Thus there are $x_1, x_2 \in A(E)$ such that $x_1 \cdot x_2 \notin \mathcal{J}$. Clearly, the ideal $\mathcal{J} + A(E^c)$ properly contains $\mathcal{J}$. By maximality, $\mathcal{J} + A(E^c) = A$, thus

$$x_1 = v + w, \qquad \text{for some } v \in \mathcal{J},\ w \in A(E^c).$$

Now, although e_E is not necessarily an element of A we have

$$ve_E = x_1 \quad \text{and} \quad x_1 x_2 = (ve_E)x_2 = vx_2 \in \mathcal{J};$$

a contradiction. Therefore also in that case $\mathcal{F}_{\mathcal{J}}$ is an ultrafilter. □

Lemma 4.5 *If $\mathcal{J}$ is an proper ideal in an ℓ_∞-module A, such that $\mathcal{F}_{\mathcal{J}}$ is an ultrafilter not containing unital sets, then every element of the algebra $A/\mathcal{J}$ is quasi-invertible. Thus there are no regular ideals containing $\mathcal{J}$.*

Proof. Take $x \in A \setminus \mathcal{J}$. Since $\mathcal{F}_{\mathcal{J}}$ contains no unital sets, $\lim_{\mathcal{F}_{\mathcal{J}}} x = 0$ (see the proof of Prop. 4.3 (a)). Thus there is $U \in \mathcal{F}_{\mathcal{J}}$ such that $|x_i| < 1/2$ for each $i \in U$. The element $y := \frac{x}{x-1} e_U$, defined pointwise, belongs to $A(U)$. We have

$$y + x - yx = \frac{x}{x-1} e_U + xe_U + xe_{U^c} - \frac{x^2}{x-1} e_U = xe_{U^c} \in A(U^c) \subset \mathcal{J},$$

as $U \in \mathcal{F}_{\mathcal{J}}$. This implies that $x + \mathcal{J}$ is quasi-invertible in $A/\mathcal{J}$. □

Lemma 4.6 *Let $\mathcal{J}$ be a proper ideal in an ℓ_∞-module A and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. If $\mathcal{U} \supset \mathcal{F}_{\mathcal{J}}$ then the ideal $\mathcal{J}_{\mathcal{U}}$ contains $\mathcal{J}$. If $\mathcal{U}$ contains a unital set then the converse holds as well.*

Proof. If $\mathcal{J} \subset \mathcal{J}_{\mathcal{U}}$ then $\mathcal{F}_{\mathcal{J}} \subset \mathcal{F}_{\mathcal{J}_{\mathcal{U}}}$. If $\mathcal{U}$ contains a unital set then $\mathcal{F}_{\mathcal{J}_{\mathcal{U}}} = \mathcal{U}$. On the other hand, if $\mathcal{J} \not\subset \mathcal{J}_{\mathcal{U}}$ then there are $x \in \mathcal{J}$, $y \in A$ such that $\lim_{\mathcal{U}} xy \neq 0$. Thus there is a set $U \in \mathcal{U}$, $\varepsilon > 0$, such that $|xy| \geq \varepsilon$ on U. We find $z \in A$, $|z| \leq (1/\varepsilon)y$ such that $e_U = xz \in \mathcal{J}$. Clearly $A(U) \subset \mathcal{J}$ and $U^c \in \mathcal{F}_{\mathcal{J}}$ and $\mathcal{U} \not\supset \mathcal{F}_{\mathcal{J}}$. □

Proof of Theorem 4.2. By Lemmas 4.4 and 4.5, for every proper maximal regular ideal $\mathcal{J}$ the set $\mathcal{F}_{\mathcal{J}}$ is an ultrafilter containing an unital set. On the other hand, for every ultrafilter $\mathcal{U}$ containing an unital set, by Proposition 4.3, $\mathcal{J}_{\mathcal{U}}$ is a proper ideal. Obviously, $\mathcal{F}_{\mathcal{J}_{\mathcal{U}}} \supseteq \mathcal{U}$ and since $\mathcal{U}$ is an ultrafilter $\mathcal{F}_{\mathcal{J}_{\mathcal{U}}} = \mathcal{U}$. If $\mathcal{J} \supseteq \mathcal{J}_{\mathcal{U}}$ is a proper ideal then $\mathcal{F}_{\mathcal{J}} \supseteq \mathcal{F}_{\mathcal{J}_{\mathcal{U}}}$ and thus $\mathcal{F}_{\mathcal{J}} = \mathcal{U}$. By Lemma 4.6, $\mathcal{J}_{\mathcal{U}} \supseteq \mathcal{J}$; we have proved that $\mathcal{J}_{\mathcal{U}}$ is maximal.

Observe that, by Lemma 4.6, $\mathcal{J}_{\mathcal{F}_{\mathcal{J}}} \supseteq \mathcal{J}$. Since $\mathcal{J}$ is a maximal proper ideal thus $\mathcal{J}_{\mathcal{F}_{\mathcal{J}}} = \mathcal{J}$. □

The following result is proved in [28, Corollary 14] (cf. also [11, Lemma 2]):

Corollary 4.7 *Every multiplicative functional m in an ℓ_∞-module is of the form*

$$m(x) = \lim_{\mathcal{U}} x,$$

where $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ containing a unital set.

Proof. Kernels of multiplicative functionals are exactly proper maximal regular ideals of codimension one. By Theorem 4.2 and the proof of Proposition 4.3 (b), they are of the form

$$\mathcal{J}_{\mathcal{U}} = \{x : \lim_{\mathcal{U}} x = 0\}$$

for ultrafilters $\mathcal{U}$ containing a unital set and such that for every $x \in A$ we have $\lim_{\mathcal{U}} |x| < \infty$. □

Clearly, not every ultrafilter generates a multiplicative functional, since it may happen that $\lim_{\mathcal{U}} |x| = \infty$ for some $x \in A$ and some free ultrafilter $\mathcal{U}$.

From Lemma 4.6 and Theorem 4.2 it follows immediately:

Corollary 4.8 *A proper ideal $\mathcal{J}$ in an ℓ_∞-module A is contained in a regular maximal proper ideal if and only if there is a unital set U which intersects all elements of $\mathcal{F}_{\mathcal{J}}$.*

There are proper ideals $\mathcal{J}$ of A with $\mathcal{F}_{\mathcal{J}}$ containing no unital sets.

Example 4.9 Assume that A is a non-unital ℓ_∞-module and take $\mathcal{F} := \{U \subset \mathbb{N} \mid U^c \text{ is unital }\}$, which is clearly a filter. Let $\mathcal{U}$ be a ultrafilter containing $\mathcal{F}$. By construction no element of $U \in \mathcal{U}$ is unital (since otherwise $U^c \in \mathcal{F} \subset \mathcal{U}$). We define

$$\mathcal{J}_0 := \bigcup_{U \in \mathcal{U}} A(U^c).$$

Clearly $\mathcal{J}_0$ is an ideal of the required properties.

Clearly if for the algebra A we have the equality $A = A^2$ then all maximal ideals are regular. We clarify when there are non-solid maximal ideals.

Theorem 4.10 *Let A be an ℓ_∞-module. Let us consider the following conditions:*

(a) $A^2 = A$.

(b) For every proper ideal $\mathcal{J}$ in A its solid hull $\tilde{\mathcal{J}}$ is also proper (i.e., $\tilde{\mathcal{J}} \neq A$).

(c) Every maximal proper ideal is solid.

(d) For every maximal proper ideal $\mathcal{J}$ the filter $\mathcal{F}_{\mathcal{J}}$ is an ultrafilter.

(e) Either $A^2 = A$ or the filter $\mathcal{F}_{A^2}$ is an ultrafilter.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Leftrightarrow$ (e).

If $A = k_p(V)$, $1 \leq p \leq \infty$ or $p = 0$, then all the above conditions are equivalent.

Proof. (a) $\Rightarrow$ (b): Assume that $\tilde{\mathcal{J}} = A$. Let $x \in A \setminus \mathcal{J}$, then $x = \sum_j y_j z_j$ for some $y_j, z_j \in A$. Clearly, there are $w_j \in \mathcal{J}$ such that $|y_j| \leq |w_j|$. Thus there are sequences u_j such that

$$|u_j| \leq |z_j| \qquad \text{and} \qquad y_j z_j = w_j u_j.$$

Clearly $u_j \in A$. We have proved that $x = \sum_j w_j u_j \in \mathcal{J}$; a contradiction.

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (d): Follows from Lemma 4.4.

(d) $\Rightarrow$ (e): If $A^2 \neq A$ and $\mathcal{F}_{A^2}$ is not an ultrafilter then there is a set E such that both $A(E) \neq A(E)^2$ and $A(E^c) \neq A(E^c)^2$. Thus there are vectors $x_+ \in A(E) \setminus A(E)^2$ and $x_- \in A(E^c) \setminus A(E^c)^2$. We will find linear functionals $f_+, f_- \in A'$ such that $f_\pm(A^2) = f_+(A(E^c)) = f_-(A(E)) = 0$ but $f_+(x_+) = f_-(x_-) = 1$. We define

$$\mathcal{J} := \{x \in A : f_+(x) - f_-(x) = 0\}.$$

Clearly, $\mathcal{J} \supset A^2$ and $\dim A/\mathcal{J} = 1$. Thus $\mathcal{J}$ is a maximal proper degenerate ideal. Moreover, both E and E^c does not belong to $\mathcal{F}_{\mathcal{J}}$ since $x_+, x_- \notin \mathcal{J}$. In fact $\mathcal{J}$ is not solid because $x_+ + x_- \in \mathcal{J}$ but $|x_+| \leq |x_+ + x_-|$.

(e) $\Rightarrow$ (d): Every proper maximal ideal $\mathcal{J}$ is either regular, and in this case $\mathcal{F}_{\mathcal{J}}$ is an ultrafilter by Lemma 4.4, or degenerate. In the latter case $\mathcal{F}_{\mathcal{J}} \supset \mathcal{F}_{A^2}$ and we get the conclusion.

(e) $\Rightarrow$ (a) for $A = k_p(V)$: We proceed by contradiction and suppose that $(k_p(V))^2 \neq k_p(V)$. We use Proposition 5.10 proved below and assume first that

$$\exists\, n \;\forall\, k \quad \frac{v_k^2}{v_n} \notin \ell_\infty,$$

then there is an increasing sequence (i_m) such that

$$\frac{v_m(i_m)^2}{v_n(i_m)} \geq m.$$

Clearly, for the sets

$$E_+ := \{i_{2m} : m \in \mathbb{N}\}, \qquad E_- := \{i_{2m+1} : m \in \mathbb{N}\}$$

there are elements:

$$x_+ \in k_p(V, E_+) \setminus (k_p(V, E_+))^2, \qquad x_- \in k_p(V, E_-) \setminus (k_p(V, E_-))^2. \tag{4.1}$$

Analogously, if $k_p(V)$ is not nuclear, $1 \leq p < \infty$, then

$$\exists\ n\ \ \forall\ k \quad \sum_i \frac{v_k}{v_n} = \infty.$$

We construct inductively strictly increasing sequences of numbers (i_m), (j_m) such that $i_m < j_m < i_{m+1}$ and

$$\sum_{i=i_{2m}}^{i=j_{2m}} \frac{v_m(i)}{v_n(i)} \geq m, \qquad \sum_{i=i_{2m+1}}^{i=j_{2m+1}} \frac{v_m(i)}{v_n(i)} \geq m.$$

Clearly, both $k_p(V, E_+)$ and $k_p(V, E_-)$ are not nuclear if

$$E_+ := \{i :\ \exists\ m\ \ i_{2m} \leq i \leq j_{2m}\}, \qquad E_- := \{i :\ \exists\ m\ \ i_{2m+1} \leq i \leq j_{2m+1}\}.$$

Thus also in that case the condition (4.1) holds for suitable x_+, x_-.

In both cases $\mathcal{F}_{A^2}$ is not an ultrafilter since both $A(E_+) \neq A(E_+)^2$ and $A(E_-) \neq A(E_-)^2$ and a fortiori $A(E_+^c) \neq A(E_+^c)^2$. □

We will give an Example 5.18 showing that the implication (d) ⇒ (c) in Theorem 4.10 does not hold in general.

Open problem: Does the implication (c) ⇒ (a) in Theorem 4.10 hold in every ℓ_∞-module?

5 Multiplicative functionals and ideals in algebras $k_p(V)$

In this section we again assume that $I = \mathbb{N}$, which is no loss of generality. An algebra $k_p(V), 1 \leq p \leq \infty$ or $p = 0$, is clearly an ℓ_∞-module. Therefore all the results of the preceding section remain valid. Nevertheless we can get more, especially since $k_p(V)$ is a locally convex space.

Theorem 5.1 *Every multiplicative functional on an algebra $k_p(V), 1 \leq p \leq \infty$ or $p = 0$, is continuous. For $p \neq \infty$ it is a point evaluation at some point in $\mathbb{N}$. For $p = \infty$ it is either a point evaluation at a point in $\mathbb{N}$ or a limit over a free ultrafilter $\mathcal{U}$ such that*

$$\lim_{\mathcal{U}} \frac{1}{v_n} < \infty \qquad \textit{for every } n \in \mathbb{N}.$$

First, we need a lemma.

Lemma 5.2 *If $1 \leq p < \infty$ or $p = 0$ and U is an infinite unital set for the algebra $k_p(V)$ then there is $y = (y_j) \in k_p(V, U)$ such that*

$$\lim_{j \to \infty, j \in U} y_j = \infty.$$

Proof. For $1 \le p < \infty$, by Prop. 2.5, there exists n_0 such that for all $n \ge n_0$ we have $\sum_{i \in U} |v_n(i)|^p < \infty$ (see Prop. 2.5). Now, it is easy to see that there is $y \in k_p(V,U)$ such that

$$\lim_{j \to \infty, j \in U} y_j = \infty, \quad y_j > 0 \quad \text{for } j \in U.$$

$\square$

Proof of Theorem 5.1 Let $m \neq 0$ be a multiplicative functional. By Corollary 4.7, there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that

$$m(x) = \lim_{\mathcal{U}} x.$$

and $k_p(V,U)$ is unital.

For $p \neq \infty$, by Lemma 5.2, we find y such that $m(y) = \lim_{\mathcal{U}} y_j = \infty$; a contradiction. A similar argument works for $p = 0$.

For $p = \infty$, for any $n \in \mathbb{N}$, $\frac{1}{v_n} \in k_\infty(V)$. Therefore $m\left(\frac{1}{v_n}\right) = C_n < \infty$ for every $n \in \mathbb{N}$ which implies that m is continuous on $\ell_\infty(v_n)$ for every $n \in \mathbb{N}$. $\square$

Corollary 5.3 *If an algebra $k_p(V), 1 \le p \le \infty$ or $p = 0$, is locally m-convex, then the set $M(k_p(V))$ of all multiplicative functionals on $k_p(V)$ is equicontinuous.*

Proof. By Theorem 2.8, $k_p(V) \subset \ell_\infty$. The inclusion is necessarily continuous by the closed graph theorem. By Theorem 5.1, every $m \in M(k_p(V))$ is continuous and is of the form $m(x) = \lim_{\mathcal{U}} x$. This implies that $M(k_p(V))$ is a $\sigma(k_p(V)', k_p(V))$-bounded subset of $k_p(V)'$. As $k_p(V)$ is an (LB)-space, hence barrelled, $M(k_p(V))$ is equicontinuous. $\square$

We have already described maximal regular ideals in Theorem 4.2 as $\mathcal{J}_\mathcal{U}$ ideals. We can identify which of them are closed in algebras $k_p(V)$.

Theorem 5.4 *Let $k_p(V)$ be an algebra, $1 \le p \le \infty$ or $p = 0$, and let $\mathcal{U}$ be an ultrafilter containing a unital set. Then exactly one of the following two conditions hold*

(a) The ideal $\mathcal{J}_\mathcal{U}$ is a closed ideal of codimension one; this is so for $p \neq \infty$ if and only if $\mathcal{U}$ is fixed and for $p = \infty$ if and only if $\lim_\mathcal{U}(1/v_n) < \infty$ for every $n \in \mathbb{N}$.

(b) The ideal $\mathcal{J}_\mathcal{U}$ is dense of infinite codimension; this is so for $p \neq \infty$ if and only if $\mathcal{U}$ is free and for $p = \infty$ if and only if there is n such that $\lim_\mathcal{U}(1/v_n) = \infty$.

All ideals $\mathcal{J}_\mathcal{U}$ are closed if and only if $\inf_{i \in \mathbb{N}} v_n(i) > 0$ for every $n \in \mathbb{N}$.

Proof. If $\mathcal{U}$ is fixed then $\mathcal{J}_\mathcal{U}$ is closed of codimension one. Now, for $p \neq \infty$ if $\mathcal{U}$ is not fixed then it contains an infinite unital set. By Lemma 5.2 and Prop. 4.3, $\mathcal{J}_\mathcal{U}$ is of infinite codimension. Since unit vectors belong to $\mathcal{J}_\mathcal{U}$ and they are linearly dense the ideal in question is also dense.

Now, we prove the result for $p = \infty$. Assume that $\lim_\mathcal{U}(1/v_n) < \infty$ for every $n \in \mathbb{N}$. For every $x \in k_\infty(V)$ there is n such that $|x| \le \frac{C}{v_n}$ thus $\lim_\mathcal{U} x < \infty$. Clearly, if $x \in \mathcal{J}_\mathcal{U}$ then $\lim_\mathcal{U} x = 0$. On the other hand, if $\lim_\mathcal{U} x = 0$ then $x \in \mathcal{J}_\mathcal{U}$. Indeed, for every $y \in k_\infty(V)$ we have $\lim_\mathcal{U} y < \infty$ thus $\lim_\mathcal{U} yx = 0$. Therefore

$$\mathcal{J}_\mathcal{U} = \{x : \lim_{\mathcal{U}} x = 0\} = \ker m_\mathcal{U},$$

where $m_\mathcal{U}$ is a continuous multiplicative functional

$$m_\mathcal{U}(x) = \lim_{\mathcal{U}} x \qquad \text{(comp. Theorem 5.1).}$$

Assume now that there is $n \in \mathbb{N}$ such that $\lim_\mathcal{U}(1/v_n) = \infty$. By Prop. 4.3, the considered ideal is of infinite codimension. Let $U \in \mathcal{U}$ be a unital set. It suffices to show that e_U belongs to the closure of $\mathcal{J}_{\mathcal{U}}$. Let us observe that for $W \in \mathcal{U}$ we have $e_{U\setminus W} \in \mathcal{J}_{\mathcal{U}}$. For every $m \geq n$, $\varepsilon > 0$, there is $W \in \mathcal{U}$ such that $v_n(i) \leq \varepsilon$ for $i \in W$. Then

$$|e_U(i) - e_{U\setminus W}(i)|v_m(i) \leq \varepsilon \qquad \text{for } i \in \mathbb{N}.$$

We have proved that $e_U \in \overline{\mathcal{J}_{\mathcal{U}}}$. □

Corollary 5.5 *In dual power series spaces of finite type $\Lambda_0^p(\alpha)'$ all regular maximal ideals are closed of codimension one (i.e., they are kernels of point evaluations in $\mathbb{N}$). In dual power series of infinite type $\Lambda_\infty^\infty(\alpha)'$ all ultrafilters give proper maximal regular ideals.*

Now, it is time to identify the ideal $(k_p(V))^2$. Given a sequence of weights $V = (v_n)_n$, we denote $V^2 := (v_n^2)$. If $k_p(V), 1 \leq p \leq \infty$ or $p = 0$, is an algebra we can apply Proposition 2.1 to conclude that $k_p(V^2) \subset k_p(V)$. In our next result we use the spaces $\ell_q(v), 0 < q < 1$, $x = (x_i)_i \in \ell_q(v)$ if and on if $\sum_i (v(i)|x_i|)^q < \infty$.

Proposition 5.6 *Let $k_p(V), 1 \leq p \leq \infty$ or $p = 0$, be an algebra, and define $q = p/2$ if $p \neq \infty$ and $q = \infty$ if $p = \infty$. Then $(k_p(V))^2 = k_p(V)k_p(V) = k_q(V^2)$.*

Remark 5.7 It follows that all degenerate ideals in algebras $k_p(V)$ are dense for $p \neq \infty$.

Proof. We present the details for $1 \leq p < \infty$; the other two cases being easier. Set $q = p/2$. First observe that if $x, y \in \ell_p(v_n)$, then $xy \in \ell_q(v_n^2)$ by the Cauchy-Schwarz inequality. This implies $k_p(V)k_p(V) \subset k_q(V^2)$. Conversely, if $x \in \ell_q(v_k^2)$ for some k, we can write $x = uz$ with $|u| \leq 1$ on $\mathbb{N}$, $0 \leq z$ on $\mathbb{N}$ and $z \in \ell_q(v_k^2)$. Clearly $a := (z_i^{1/2})_i$ satisfies $a \in \ell_p(v_k)$ and $b := ua \in \ell_p(v_k)$. Thus $x = ab \in k_p(V)k_p(V)$. In particular, $k_p(V)k_p(V) = k_q(V^2)$ is a linear space which must coincide with $(k_p(V))^2$. □

Now, we explain when there is no degenerate ideals in an algebra $k_p(V)$, i.e., when $k_p(V) = (k_p(V))^2$. The following result is a consequence of Grothendieck's factorization theorem.

Lemma 5.8 *For each $0 \leq s$ or $s = \infty$ the equality $k_s(V^2) = k_s(V)$ holds if and only if two following conditions are satisfied:*

(1) For each n there is $k \geq n$ such that $v_k/v_n^2 \in \ell_\infty$,

(2) For each n there is $k \geq n$ such that $v_k^2/v_n \in \ell_\infty$.

Lemma 5.9 *An (LB)-space $k_p(V), 1 \leq p < \infty$, is nuclear if and only if $k_p(V) = k_{p/2}(V)$ algebraically.*

Proof. If $p \geq 2$ this follows from [24, Theorems 27.16 and 28.16]. We then suppose that $1 \leq p < 2$. If $k_p(V)$ is nuclear for each n there is k such that $v_k/v_n \in \ell_1$. Using this fact again, we find l such that $v_l/v_n \in \ell_{1/2}$. This implies $k_\infty(V) = k_{1/2}(V)$ and in particular $k_p(V) = k_{p/2}(V)$ if $1 \leq p < 2$. Conversely, suppose that $k_p(V) = k_{p/2}(V)$ and $1 \leq p < 2$. For each n there is k such that $\ell_p(v_n)$ is contained in $\ell_{p/2}(v_k)$ with continuous inclusion. This implies, for $y := v_k/v_n$, that $xy \in \ell_{p/2}$ for each $x \in \ell_p$. Therefore y is a multiplier from ℓ_p into $\ell_{p/2}$. Hence, by [22, Prop. 3], $y = v_k/v_n \in \ell_p$ and $k_p(V)$ is nuclear by [24, Theorems 27.16 and 28.16]. □

If $k_p(V)$, $1 \leq p \leq \infty$ or $p = 0$, is an algebra, Proposition 2.1 ensures that condition (1) in Lemma 5.8 is satisfied.

Proposition 5.10 *Let $k_p(V), 1 \leq p \leq \infty$ or $p = 0$, be an algebra. Then*

(a) $(k_p(V))^2 = k_p(V), 1 \leq p < \infty$, if and only if $k_p(V)$ is nuclear and for each n there is $k \geq n$ such that $v_k^2/v_n \in \ell_\infty$.

(b) $(k_p(V))^2 = k_p(V), p = 0$ *or* $p = \infty$*, if and only if for each* n *there is* $k \geq n$ *such that* $v_k^2/v_n \in \ell_\infty$*.*

In particular, $(\Lambda_r^p(\alpha)')^2 = \Lambda_r^p(\alpha)'$ *holds always for* $p = 0$ *or* $p = \infty$ *and it holds exactly in the nuclear case for* $1 \leq p < \infty$*.*

Proof. (a): Suppose that $1 \leq p < \infty$ and set $q := p/2$. Since condition (1) in Lemma 5.8 is satisfied and $q < p$, we have $(k_p(V))^2 = k_p(V)k_p(V) = k_q(V^2) \subset k_q(V) \subset k_p(V)$, by Proposition 5.6. Thus $(k_p(V))^2 = k_p(V)$ if and only if $k_q(V) = k_p(V)$ and $k_q(V^2) \subset k_q(V)$. This is equivalent to the nuclearity of $k_p(V)$ by Lemma 5.9 and condition (2) in Lemma 5.8.

(b): We assume $p = 0$. In this case, $(k_0(V))^2 = k_0(V^2)$, by Proposition 5.6, and $k_0(V^2) = k_0(V)$ if and only if condition (2) in Lemma 5.8 is satisfied. □

Corollary 5.11 *In a dual power series algebra* $\Lambda_r^p(\alpha)'$ *for* $p = \infty$ *or* $p = 0$ *all maximal proper ideals are regular but for* $1 \leq p < \infty$ *all maximal proper ideals are regular if and only if the space is nuclear.*

For arbitrary ideals we can say much less.

Definition 5.12 *Let* $\mathcal{J}$ *be an ideal. We define*

$$V(\mathcal{J}) := \{j \in \mathbb{N} \mid \forall x = (x_n)_n \in \mathcal{J} : x_j = 0\}.$$

An ideal $\mathcal{J}$ contains e_j if and only if j does not belong to $V(\mathcal{J})$. Indeed, clearly if $e_j \in \mathcal{J}$, then $j \notin V(\mathcal{J})$. Conversely, assume that $j \notin V(\mathcal{J})$. There is $x \in \mathcal{J}$ with $x_j \neq 0$. Then $e_j = (\frac{1}{x_j}x)e_j \in \mathcal{J}$.

Since $k_p(V)$, $1 \leq p < \infty$ or $p = 0$ has an unconditional orthogonal basis, [19, Theorem 2.2.] implies immediately:

Proposition 5.13 *If* $\mathcal{J}$ *is a closed ideal in an algebra* $k_p(V)$*,* $1 \leq p < \infty$ *or* $p = 0$*, then*

$$\mathcal{J} = \{x \in k_p(V) : \ x_j = 0 \ \forall \ j \in V(\mathcal{J})\}.$$

By Corollary 5.1 and the above we get immediately:

Corollary 5.14 *Every proper closed ideal in an algebra* $k_p(V), 1 \leq p < \infty$ *or* $p = 0$*, is an intersection of closed proper maximal ideals (kernels of continuous multiplicative functionals).*

Proposition 5.15 *Let us fix a set* $M \subseteq \mathbb{N}$*. Among ideals* $\mathcal{J}$ *such that* $V(\mathcal{J}) = M \subset \mathbb{N}$ *in an algebra* $k_p(V)$*,* $1 \leq p \leq \infty$ *or* $p = 0$*, there is the smallest one*

$$\mathcal{J}_{min}(M) := \{x \in k_p(V) : \ x_j = 0 \ \forall \ j \in V(\mathcal{J}) \ \text{ and only finitely many } \ x_j \neq 0\}$$

and the maximal one

$$\mathcal{J}_{max}(M) := \overline{\mathcal{J}_{min}(M)} := \{x \in k_p(V) : \ x_j \neq 0 \ \forall \ j \in V(\mathcal{J})\}.$$

There are plenty of ideals between these two, for instance let $M = \emptyset$. Indeed, let $\mathcal{F}$ be a free filter of subsets of $\mathbb{N}$, then

$$\mathcal{J}(\mathcal{F}) = \{x \in k_p(V) : \ \exists U \in \mathcal{F} \text{ such that } \forall \ j \in U \ x_j = 0\}.$$

Let $p > 1$ and assume $\inf_i v_n(i) > 0$ for all n. Then $k_1(V)$ is an ideal in $k_p(V)$ since $k_p(V)$ consists of bounded sequences. For $p = 1$ and $\inf_i v_n(i) > 0$ for all n, $k_1(V)$ contains an ideal $k_r(V)$ for any $0 < r < 1$.

These examples show that non-closed ideals need not be contained in a maximal proper closed ideal in the non-unital case.

Example 5.16 In the algebra $\Lambda_0^\infty(\alpha)'$ the ideal φ of finitely non-zero sequences is not contained in any maximal proper ideal. This follows from Corollaries 4.8, 5.5 and 5.11.

We clarify now when all ideals are solid. From Theorem 5.1 and Corollary 5.14 every proper closed ideal in $k_p(V)$, $1 \le p < \infty$ or $p = 0$, is solid.

Proposition 5.17 *An algebra $k_p(V), 1 \le p \le \infty$ or $p = 0$, is unital if and only if every ideal in $k_p(V)$ is solid.*

Proof. If the algebra is unital, there is n such that $e \in \ell_p(v_n)$. Let $\mathcal{J}$ be an ideal, and suppose that $x \in \mathcal{J}, y \in k_p(V)$ and $|y| \le |x|$ on $\mathbb{N}$. Define $z = (z_i)_i$ by $z_i := y_i/x_i$ if $x_i \ne 0$ and $z_i := 0$ otherwise. Since $|z| \le e$, we have $z \in k_p(V)$. Since $\mathcal{J}$ is an ideal, $y = zx \in \mathcal{J}$.

Conversely, suppose that $k_p(V)$ is not unital. We can apply Propositions 2.5, 2.6, Corollary 2.7 and a diagonal procedure to find a strictly increasing sequence $(i_j)_j$ of natural numbers such that $\lim_{j\to\infty} v_m(i_j) = \infty$ for each m. Now select a sequence $(\beta_j)_j$ of numbers greater than 1 tending to infinity such that for each m there is $D_m > 0$ with $\beta_j \le D_m v_m(i_j)$ for each j. Take any $x \in k_p(V)$ such that $x_{i_j} \ne 0$ for each j and $x_i = 0$ otherwise. The set

$$\langle x \rangle := \{\lambda x + yx \mid \lambda \in \mathbb{C},\ y \in k_p(V)\}$$

is an ideal of $k_p(V)$, as $(\lambda x + yx)z = x(\lambda z + yz) \in \langle x \rangle$ for each $z \in k_p(V)$ and each $\lambda x + yx \in \langle x \rangle$. We show that $\langle x \rangle$ is not solid, from where it also follows that it is a proper ideal. Define $\lambda_j := 1 - \beta_j^{-1/2}, j \in \mathbb{N}$. Then $\lim_{j\to\infty} \lambda_j = 1$ and $0 < \lambda_j < 1$ for each j. We set $z_{i_j} := \lambda_j x_{i_j}, j \in \mathbb{N}$ and $z_i = 0$ otherwise. Clearly $|z| \le |x|$ and $x \in \langle x \rangle$. However, $z \notin \langle x \rangle$. Indeed, suppose that there are $\lambda \in \mathbb{C}$ and $y \in k_p(V)$ such tat $z = \lambda x + yx$. Since $x_{i_j} \ne 0$ for each j, we get $y_{i_j} = \lambda_j - \lambda$ for each j. We apply that $y \in k_p(V)$ to find m such that $M := \sup_j v_m(i_j)|\lambda_j - \lambda| < \infty$. Hence

$$|\lambda_j - \lambda| \le \frac{M}{v_m(i_j)} \le \frac{MD_m}{\beta_j}.$$

Thus

$$|\lambda - 1| \le |\lambda_j - \lambda| + |\lambda_j - 1| \le \frac{MD_m}{\beta_j} + \frac{1}{\beta_j^{1/2}},$$

for each j. As $\lim_{j\to\infty} \beta_j = \infty$, we conclude $\lambda = 1$. Thus, for each j, we have

$$M \ge v_m(i_j)(1 - \lambda_j) \ge \beta_j \frac{1}{\beta_j^{1/2}} = \beta_j^{1/2};$$

a contradiction because $\lim_{j\to\infty} \beta_j = \infty$. □

Example 5.18 We show that the implication (d) ⇒ (c) in Theorem 4.10 does not hold in general.

Let

$$V = (v_n), v_n(j) := e^{-nj} \qquad \text{and} \qquad W = (w_n), w_n(j) := j/(\log j)^n.$$

Clearly, $k_\infty(V)$, $k_\infty(W)$ are algebras. By Proposition 5.10, $(k_\infty(V))^2 = k_\infty(V)$ while $(k_\infty(W))^2 \ne k_\infty(W)$. Let $\mathcal{U}$ be a free ultrafilter and let:

$$A := \{x : \ \exists\, U \in \mathcal{U} : xe_U \in k_\infty(W,U), xe_{U^c} \in k_\infty(V,U^c)\}.$$

It is not difficult to see that A is an ℓ_∞-module, $\mathcal{F}_{A^2} = \mathcal{U}$ but from the proof of Prop. 5.17 the ideal $\langle x \rangle$ is not solid for $x = \frac{1}{w_0}$. Even more, the vector z constructed in this proof with $|z| \le |x|$, $z \notin \langle x \rangle$ satisfies

$$\alpha x + \beta z \notin A^2 = \{x : \ \exists\, U \in \mathcal{U}\ xe_U \in (k_\infty(W,U))^2, xe_{U^c} \in k_\infty(V,U^c)\}$$

for any non zero pair of scalars (α, β). Thus there is a one codimensional subspace $\mathcal{J}$ in A such that $\mathcal{J} \supset x + A^2$ but $z \notin \mathcal{J}$. The ideal $\mathcal{J}$ has the required properties.

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